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ON A HILBERT MODULE OVER AN OPERATOR ALGEBRA AND ITS APPLICATION TO HARMONIC ANALYSIS

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1. Introduction.

We study a left A-module with an A-valued inner product where A is an operator algebra. Such a space has been investigated by many authors: Kaplansky [7], Saworotnow [14], Paschke [12], Rieffel [13], Ozawa [11], Itoh [5], Kakihara and Terasaki [6] and others.

Let A be a von Neumann algebra. Then a Hilbert A-module is defined to be a left A-module with an A-valued inner product respecting the module action, called a Gramian, which is complete with respect to (w.r.t.) the norm induced from the Gramian. Our main object is harmonic analysis on a topological group in the Hilbert A-module context. Especially, a Stone type and a Bochner type theorems are formulated and proved.

Basic definitions of a Hilbert A-module are given in section 2. In section 3, A-valued positive definite kernels are considered in connection with reproducing kernel Hilbert A-modules which are analogous to Aronszajn's reproducing kernel Hilbert spaces [1]. Section 4 deals with Gramian unitary representations of a topological group and Gramian *-representations of a L^1-group algebra on a Hilbert A-module. Results stated in sections 3 and 4 hold when A is a (unital) C^*-algebra. In section 5, we prove our main result which is a Stone type theorem for a continuous, in an appropriate sense, Gramian unitary representation of a locally compact abelian group. As a corollary we give a proof to a Bochner type theorem for a weakly continuous A-valued positive definite function. Section 6 is devoted to Hilbert A-module valued processes over a locally compact abelian group. Such a formulation of processes is closely related to Banach space valued stochastic processes (cf. Cobanjan and Weron [2], Weron [19] and Miamee [8]).

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2. Hilbert A-modules.

Throughout this paper A stands for a von Neumann algebra with the norm

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$\|\cdot\|$. We denote the *action* of A on a left A-module X by $(a, x) \to a \cdot x$, $a \in A$, $x \in X$. We assume that all such modules treated below have a vector space structure compatible with that of A in the sense that $\alpha(a \cdot x) = (\alpha a) \cdot x = a \cdot (\alpha x)$ for $x \in X$, $a \in A$ and a complex number α.

2.1. DEFINITION. A *(left) pre-Hilbert A-module* is a left A-module for which there is a map $[\cdot, \cdot]: X \times X \to A$ such that for $x, y, z \in X$ and $a \in A$ (1) $[x, x] \geqq 0$, and $[x, x] = 0$ iff $x = 0$; (2) $[x+y, z] = [x, z] + [y, z]$; (3) $[a \cdot x, y] = a[x, y]$; (4) $[x, y]^* = [y, x]$. The map $[\cdot, \cdot]$ is called a *Gramian* on X. We sometimes denote it explicitly by $[\cdot, \cdot]_X$.

If X is a right A-module, then we can define (right) pre-Hilbert A-module structure for X in a similar manner as above except that the condition (3) is replaced by (3′) $[x \cdot a, y] = [x, y]a$. Since there is no essential difference between right and left A-modules, we restrict our attention to left A-modules.

In a pre-Hilbert A-module X define $\|x\|_X = \|[x, x]\|^{1/2}$, $x \in X$. Then by [12, 2.3 Proposition], $\|\cdot\|_X$ becomes a norm on X and we have for $x, y \in X$ and $a \in A$

$$\|a \cdot x\|_X \leqq \|a\| \cdot \|x\|_X, \qquad \|[x, y]\| \leqq \|x\|_X \cdot \|y\|_X. \tag{2.1}$$

2.2. DEFINITION A pre-Hilbert A-module X which is complete w.r.t. the norm $\|\cdot\|_X$ is called a *Hilbert A-module*.

Examples of (right) Hilbert A-modules are seen in [12] where A is a C^*-algebra.

2.3. DEFINITION. Let X be a Hilbert A-module. We define the *Gramian orthogonal complement* of a subset Y of X by $Y^\# = \{x \in X;\ [x, y] = 0,\ y \in Y\}$. A subset Y is called a *submodule* if it is a left A-module and is closed w.r.t. $\|\cdot\|_X$. In this case Y is itself a Hilbert A-module. Denote by $\mathfrak{S}(Y)$ the submodule generated by a subset Y. It is seen that for each subset Y its Gramian orthogonal complement $Y^\#$ is a submodule and the relation $\mathfrak{S}(Y) \subset (Y^\#)^\#$ holds.

2.4. DEFINITION. Let X and Y be two Hilbert A-modules with Gramians $[\cdot, \cdot]_X$ and $[\cdot, \cdot]_Y$, respectively. $B(X, Y)$ denotes the Banach space of all bounded linear operators from X into Y. $\mathfrak{A}(X, Y)$ denotes the set of all operators $S \in B(X, Y)$ for which there is an operator $T \in B(Y, X)$ such that $[Sx, y]_Y = [x, Ty]_X$, $x \in X$, $y \in Y$. It is seen that T is unique if it exists, so that we denote it by S^* and call it the *Gramian adjoint* of S. An operator $U \in B(X, Y)$ is said to be *Gramian unitary* if it is onto and satisfies that $[Ux, Ux']_Y = [x, x']_X$, $x, x' \in X$. It can be seen that each Gramian unitary operator $U \in B(X, Y)$ belongs to $\mathfrak{A}(X, Y)$ and satisfies $U^*U = I_X$, the identity operator on X. We write $B(X) = B(X, X)$ and $\mathfrak{A}(X) = \mathfrak{A}(X, X)$. An operator $P \in B(X)$ is called a *Gramian projection* if $P \in \mathfrak{A}(X)$ and $P^2 = P^* = P$. Two Hilbert A-modules X and Y are said to be *isomorphic*, in symbols $X \cong Y$, if there is a Gramian unitary operator in

$\mathfrak{A}(X, Y)$.

For $a \in A$ define $\pi(a)$ by $\pi(a)x = a \cdot x$, $x \in X$, X being a Hilbert A-module. Then, by (2.1), $\pi(a) \in B(X)$. A kind of functionals on a Hilbert A-module is defined in the following (cf. [7, 12, 14]).

2.5. DEFINITION. Let X be a Hilbert A-module. Denote by X^* the set of all bounded *module maps* $\tau: X \to A$. That is, τ satisfies $\tau(a \cdot x + b \cdot y) = a\tau(x) + b\tau(y)$, $x, y \in X$, $a, b \in A$, and there is some $\alpha > 0$ such that $\|\tau(x)\| \leqq \alpha \|x\|_X$, $x \in X$. Each $x \in X$ gives rise to a map $\hat{x} \in X^*$ defined by $\hat{x}(y) = [y, x]$, $y \in X$. X is said to be *self-dual* if $X^* = \hat{X}(=\{\hat{x}\,;\, x \in X\})$.

2.6. *Remark* ([12]). Let X be a Hilbert A-module. Then X^* becomes a self-dual Hilbert A-module in which X can be embedded as a submodule. Moreover, each operator in $\mathfrak{A}(X)$ can be uniquely extended to an operator in $\mathfrak{A}(X^*)$. If X is self-dual, then we have $\mathfrak{A}(X) = \{S \in B(X)\,;\, S\pi(a) = \pi(a)S,\ a \in A\}$. That is, $\mathfrak{A}(X)$ consists of all bounded module maps from X into itself. Furthermore, there is a collection $\{p_\iota\,;\, \iota \in \mathfrak{J}\}$ of (not necessarily distinct) nonzero projections in A such that $X \cong \mathrm{UDS}\{Ap_\iota\,;\, \iota \in \mathfrak{J}\}$, the ultraweak direct sum of self-dual Hilbert A-modules Ap_ι, $\iota \in \mathfrak{J}$. For each $\iota \in \mathfrak{J}$ the Gramian on Ap_ι is defined by $[ap_\iota, bp_\iota]_\iota = ap_\iota b^*$, $a, b \in A$. As a consequence of this decomposition, for any self-dual submodule Y of X, we have that $X = Y \oplus Y^\#$, the direct sum, and that there is a Gramian projection of X onto Y.

3. Positive definite kernels.

We consider A-valued positive definite kernels on $\Omega \times \Omega$, Ω being a set, and construct reproducing kernel Hilbert A-modules.

3.1. DEFINITION. An A-valued function Γ on $\Omega \times \Omega$ is called a *positive definite kernel* (*PDK*) if for every finite $\{\omega_1, \cdots, \omega_n\} \subset \Omega$ and $\{a_1, \cdots, a_n\} \subset A$ it holds that $\sum_{i,j} a_i \Gamma(\omega_i, \omega_j) a_j^* \geqq 0$. Every PDK Γ on $\Omega \times \Omega$ satisfies that $\Gamma(\omega, \omega') = \Gamma(\omega', \omega)^*$, $\omega, \omega' \in \Omega$.

For each A-valued PDK Γ on $\Omega \times \Omega$ we can associate a Hilbert A-module $\Omega \otimes_\Gamma A$ by the method similar to that of Umegaki [17]. To this end, let $F(\Omega\,;\, A)$ be the set of all A-valued functions on Ω with finite supports. For $f, g \in F(\Omega\,;\, A)$ and $a \in A$ define $(a \cdot f)(\cdot) = af(\cdot)$, $[f, g]_\Gamma = \sum_{\omega, \omega'} f(\omega)\Gamma(\omega, \omega')g(\omega')^*$ and $\|f\|_\Gamma = \|[f, f]_\Gamma\|^{1/2}$. Then $[\cdot, \cdot]_\Gamma$ satisfies conditions of a Gramian except that $[f, f]_\Gamma = 0$ implies $f = 0$. Put $N_\Gamma = \{f \in F(\Omega\,;\, A)\,;\, [f, f]_\Gamma = 0\}$ and let $\Omega \otimes_\Gamma A$ be the completion of the quotient space $F(\Omega\,;\, A)/N_\Gamma$ w.r.t. the norm $\|\cdot\|_\Gamma$. Then $\Omega \otimes_\Gamma A$ is a Hilbert A-module. Moreover, it is closely related to the reproducing kernel Hilbert A-module of Γ defined below.

3.2. DEFINITION. Let Γ be an A-valued PDK on $\Omega\times\Omega$ and X be a Hilbert A-module consisting of A-valued functions on Ω. Then X is said to be the *reproducing kernel* (*RK*) *Hilbert A-module* of Γ if

(1) for each $\omega\in\Omega$, $\Gamma(\omega,\ \cdot)\in X$;

(2) for each $\omega\in\Omega$ and $x\in X$, $x(\omega)=[x(\cdot),\ \Gamma(\omega,\ \cdot)]$.

The PDK Γ is called the *reproducing kernel* (*RK*) for X.

3.3. PROPOSITION. *For each A-valued PDK Γ on $\Omega\times\Omega$ there is a unique, up to isomorphism, Hilbert A-module X_Γ admitting Γ as a RK. Moreover, the relation $X_\Gamma\cong\Omega\otimes_\Gamma A$ holds.*

Proof. The proof mimics that of [9, 2.5. Lemma] and we only give the outline. Let X_0 be the set of all A-valued functions on Ω of the form

$$x(\cdot)=\sum_{i=1}^{n}a_i\Gamma(\omega_i,\ \cdot),\quad a_i\in A,\ \omega_i\in\Omega,\ 1\leqq i\leqq n$$

with n finite. Define for $x(\cdot)=\sum a_i\Gamma(\omega_i,\ \cdot)$, $y(\cdot)=\sum b_j\Gamma(\omega_j',\ \cdot)\in X_0$ and $a\in A$

$$(a\cdot x)(\cdot)=\sum_i aa_i\Gamma(\omega_i,\ \cdot),\qquad [x,\ y]_0=\sum_{i,j}a_i\Gamma(\omega_i,\ \omega_j')b_j^*.$$

Then X_0 becomes a pre-Hilbert A-module with an action and a Gramian defined as above. Moreover, for $x\in X_0$ and $\omega\in\Omega$ the reproducing property $x(\omega)=[x(\cdot),\ \Gamma(\omega,\ \cdot)]_0$ holds. Hence we have $\|x(\omega)\|\leqq\|x\|_0\cdot\|\Gamma(\omega,\ \cdot)\|_0$ where $\|y\|_0=\|[y,\ y]_0\|^{1/2}$, $y\in X_0$.

Let $\{x_n\}$ be a Cauchy sequence in X_0 w.r.t. the norm $\|\cdot\|_0$. It follows from the above inequality that for every $\omega\in\Omega$ there exists some $x(\omega)\in A$ such that $\|x_n(\omega)-x(\omega)\|\to 0$. Denote by X_Γ the set of all A-valued functions x on Ω obtained in this way. For $x,\ y\in X_\Gamma$ define $[x,\ y]=\lim_{n\to\infty}[x_n,\ y_n]_0$ where $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X_0 determining x and y, respectively. Then we can check that X_Γ is actually a Hilbert A-module with the Gramian $[\cdot,\ \cdot]$. Furthermore, the reproducing property of Γ can also be checked and, therefore, Γ is a RK for X_Γ. The uniqueness of X_Γ and the isomorphism $X_\Gamma\cong\Omega\otimes_\Gamma A$ are readily verified.

4. Gramian unitary reresentations and Gramian *-representations.

We first consider Gramian unitary representations of a topological group on a Hilbert A-module and their relation to A-valued positive definite functions on the group.

4.1. DEFINITION. Let G be a topological group and X be a Hilbert A-module. An A-valued function Γ on G is said to be *positive definite* (*PD*) if for every finite $\{a_1,\ \cdots,\ a_n\}\subset A$ and $\{s_1,\ \cdots,\ s_n\}\subset G$ it holds that $\sum_{i,j}a_i\Gamma(s_j^{-1}s_i)a_j^*\geqq 0$. Putting $\tilde{\Gamma}(s,\ t)=\Gamma(t^{-1}s)$, $s,\ t\in G$, Γ is *PD* iff $\tilde{\Gamma}$ is a PDK on $G\times G$. Γ is said to be *continuous* if it is norm continuous on G. A *Gramian unitary representation* (*GUR*) of G on X is a homomorphism $s \to U(s)$ from G into $\mathfrak{A}(X)$ for which $U(s)$ is Gramian unitary for every $s \in G$. A GUR $s \to U(s)$ is said to be *continuous* if for every $x \in X$ the function $s \to U(s)x$ is norm continuous on G. A vector $x_0 \in X$ is said to be *cyclic* for a GUR $s \to U(s)$ if $\mathfrak{S}\{U(s)x_0;\ s \in G\} = X$.

Then we can prove the following.

4.2. PROPOSITION. *Let G be a topological group and Γ: $G \to A$ be PD. Then there exist a Hilbert A-module X, a GUR $s \to U(s)$ of G on X and a cyclic vector $x_0 \in X$ such that $\Gamma(s) = [U(s)x_0, x_0]$, $s \in G$. It also holds*

$$\|\Gamma(s)\| \leqq \|\Gamma(e)\|, \qquad \|\Gamma(s) - \Gamma(t)\|^2 \leqq 2\|\Gamma(e) - \Gamma(s^{-1}t)\| \cdot \|\Gamma(e)\| \tag{4.1}$$

for $s, t \in G$ where e is the identity of G. Furthermore, Γ is continuous if and only if so is $s \to U(s)$.

Proof. Put $\tilde{\Gamma}(s, t) = \Gamma(t^{-1}s)$, $s, t \in G$ and let X be the *RK* Hilbert A-module of $\tilde{\Gamma}$ with the Gramian $[\cdot, \cdot]$ (cf. 3.3. Proposition). Then we have $\Gamma(s) = \tilde{\Gamma}(s, e) = [\tilde{\Gamma}(s, \cdot), \tilde{\Gamma}(e, \cdot)]$, $s \in G$. Let X_0 be the set of all A-valued functions on G of the form $\sum_{i=1}^{n} a_i \Gamma(s_i, \cdot)$, $a_i \in A$, $s_i \in G$, $1 \leqq i \leqq n$ with n finite. For $s \in G$ define $U(s)$ on X_0 by $U(s) \sum a_i \Gamma(s_i, \cdot) = \sum a_i \Gamma(ss_i, \cdot)$. Then it is easy to see that for $x, y \in X_0$ the equality $[U(s)x, U(s)y] = [x, y]$ holds. Hence $U(s)$ can be uniquely extended to a Gramian unitary operator on X since X_0 is dense in X. Thus $s \to U(s)$ is a GUR of G on X. Putting $x_0 = \tilde{\Gamma}(e, \cdot) \in X$, it is readily seen that x_0 is a cyclic vector for $s \to U(s)$ and that the equality $\Gamma(s) = [U(s)x_0, x_0]$ holds for $s \in G$. Two inequalities in (4.1) follow from this equality as in the case of scalar valued PD functions (cf. [18]). The last assertion is not hard to check.

In the remainder of this section let G be a locally compact group with a left Haar measure ds and consider the space $L^1(G;\ Z_A)$ of all Z_A-valued Bochner integrable functions on G w.r.t. ds where Z_A is the center of A, i.e., $Z_A = \{a \in A;\ ab = ba,\ b \in A\}$. $L^1(G;\ Z_A)$ is a Banach *-algebra whose multiplication, involution and norm are respectively defined by $(fg)(t) = \int_G f(s)g(s^{-1}t)\,ds$, $f^*(t) = \Delta(t)^{-1} f(t^{-1})^*$ and $\|f\|_1 = \int_G \|f(s)\|\,ds$ for each $f, g \in L^1(G;\ Z_A)$ and $t \in G$ where Δ is the modular function of G. Define $(a \cdot f)(\cdot) = af(\cdot)$, $a \in A$, $f \in L^1(G;\ Z_A)$ and denote by $\mathfrak{L}^1(G;\ Z_A)$ the left A-module generated by $L^1(G;\ Z_A)$. Now we consider Gramian *-representations of $L^1(G;\ Z_A)$ on a Hilbert A-module in connection with GURs of G.

4.3. DEFINITION. Let X be a Hilbert A-module. Then a map $f \to T(f)$ from $\mathfrak{L}^1(G;\ Z_A)$ into $B(X)$ is called a *Gramian *-representation* (*G*R*) of $L^1(G;\ Z_A)$ on

X if the restriction of T to $L^1(G; Z_A)$ is $\mathfrak{A}(X)$-valued and if $T(a\cdot f+b\cdot g)=\pi(a)T(f)+\pi(b)T(g)$, $T(f^*)=T(f)^*$ and $T(fg)=T(f)T(g)$ for each f, $g\in L^1(G; Z_A)$ and a, $b\in A$ where $\pi(a)x=a\cdot x$, $x\in X$. A G*R $f\to T(f)$ is said to be *nondegenerate* if $\mathfrak{S}\{T(f)x;\ f\in L^1(G; Z_A),\ x\in X\}=X$.

Given a continuous GUR $s\to U(s)$ of G on a Hilbert A-module X, define $T(f)$ for $f\in L^1(G; Z_A)$ by

$$T(f)x=\int_G U(s)(f(s)\cdot x)ds\,,\qquad x\in X\,, \tag{4.2}$$

where the right hand side is a well-defined Bochner integral. If X is self-dual, then we can show that $f\to T(f)$ is a G*R of $L^1(G; Z_A)$ on X.

Let $\mathfrak{B}_G$ be the Borel σ-algebra of G and $M(G; Z_A)$ be the set of all Z_A-valued countably additive (CA) measures, in the norm, on $\mathfrak{B}_G$ of bounded variations. For μ, $\nu\in M(G; Z_A)$ and a, $b\in A$ define $(a\cdot\mu+b\cdot\nu)(B)=a\mu(B)+b\nu(B)$, $\mu^*(B)=\mu(B^{-1})^*$ and $(\mu\nu)(B)=\mu\times\nu(B')(B'=\{(s, t);\ st\in B\})$ for $B\in\mathfrak{B}_G$, and $\|\mu\|=$ the total variation of μ. Then $M(G; Z_A)$ becomes a Banach *-algebra. $\mathfrak{M}(G; Z_A)$ denotes the left A-module generated by $M(G; Z_A)$. By a *Gramian *-representation* of $M(G; Z_A)$ on a Hilbert A-module X we mean a map $\mu\to T(\mu)$ from $\mathfrak{M}(G; Z_A)$ into $B(X)$ whose restriction to $M(G; Z_A)$ is $\mathfrak{A}(X)$-valued and which satisfies that $T(a\cdot\mu+b\cdot\nu)=\pi(a)T(\mu)+\pi(b)T(\nu)$, $T(\mu^*)=T(\mu)^*$ and $T(\mu\nu)=T(\mu)T(\nu)$ for μ, $\nu\in M(G; Z_A)$ and a, $b\in A$. $L^1(G; Z_A)$ is a Banach *-subalgebra of $M(G; Z_A)$ by identifying $f\in L^1(G; Z_A)$ with $f(s)ds\in M(G; Z_A)$. By similar proofs of [3, 13.3.1. and 13.3.4. Propositions] we can show the following.

4.4. PROPOSITION. *Let X be a self-dual Hilbert A-module. Given a continuous GUR $s\to U(s)$ of G on X, define for $\mu\in M(G; Z_A)$*

$$T(\mu)x=\int_G U(s)\pi(\mu(ds))x\,,\qquad x\in X\,.$$

*Then $T(\mu)$ is a well-defined operator on X and $\mu\to T(\mu)$ is a G*R of $M(G; Z_A)$ on X whose restriction to $L^1(G; Z_A)$ is nondegenerate.*

*If $f\to T(f)$ is a nondegenerate G*R of $L^1(G; Z_A)$ on X, then there is a unique continuous GUR $s\to U(s)$ of G on X such that* (4.2) *holds.*

5. A Stone type and a Bochner type theorems.

In this section we assume that G is a locally compact abelian group. Denote by A_* the predual of A and by A_*^+ its positive part. For a Hilbert A-module X we define the *Gramian σ-weak topology* on $\mathfrak{A}(X)$ (or $B(X)$) to be the topology determined by the family of seminorms

$$S\in\mathfrak{A}(X)\,(\text{or } B(X))\to|\rho([Sx, y])|\,,\qquad x,\ y\in X,\ \rho\in A_*^+\,.$$

We prove a Stone type spectral representation theorem for a Gramian σ-weakly continuous GUR of G on some self-dual Hilbert A-module. As a consequence we give a proof to a Bochner type integral representation theorem of an A-valued weakly continuous PD function on G. For the scalar valued case we refer to Nakamura and Umegaki [10] and Umegaki [18].

Before we proceed we need some preparations. Let $\mathfrak{B}_{\hat{G}}$ be the Borel σ-algebra of the dual group $\hat{G}$ of G and X be a Hilbert A-module.

5.1. DEFINITION. A map $P: \mathfrak{B}_{\hat{G}} \to \mathfrak{A}(X)$ is called a *Gramian spectral measure* on $\hat{G}$ if P is Gramian projection valued and if, for each $\rho \in A_*^+$ and $x, y \in X$, $\rho([P(\cdot)x, y])$ is a regular measure on $\hat{G}$.

Take $\rho \in A_*^+$ and define a semi-inner product on X by $(x, y)_\rho = \rho([x, y])$, $x, y \in X$. Put $N_\rho = \{x \in X; (x, x)_\rho = 0\}$ and define X_ρ to be the completion of the quotient space X/N_ρ w.r.t. $(\cdot, \cdot)_\rho$. Then X_ρ is a Hilbert space where we denote the inner product and the norm by $(\cdot, \cdot)_\rho$ and $\|\cdot\|_\rho$, respectively. Write $x_\rho = x + N_\rho \in X/N_\rho$ for $x \in X$. Note that the inequality $\|x_\rho\|_\rho \leqq \|\rho\|^{1/2} \cdot \|x\|_X$ holds for $x \in X$. Let $s \to U(s)$ be a Gramian σ-weakly continuous GUR of G on X. For each $s \in G$ define an operator $U_\rho(s)$ on X/N_ρ by $U_\rho(s)x_\rho = (U(s)x)_\rho$, $x \in X$. Then $U_\rho(s)$ is well-defined, maps X/N_ρ onto itself and is isometry on X/N_ρ. Hence $U_\rho(s)$ can be uniquely extended to a unitary operator, still denoted by $U_\rho(s)$, on X_ρ. Moreover, $s \to U_\rho(s)$ is a weakly continuous unitary representation of G on the Hilbert space X_ρ by the Gramian σ-weak continuity of $s \to U(s)$. By Stone's theorem there is a regular spectral measure P_ρ on $\hat{G}$ such that $U_\rho(s) = \int_{\hat{G}} \overline{\langle s, \chi \rangle} P_\rho(d\chi)$, $s \in G$ where $\langle \cdot, \cdot \rangle$ is the duality pair of G and $\hat{G}$ (cf. [18, Theorem 7.1]).

Now let $x, y \in X$ and $B \in \mathfrak{B}_{\hat{G}}$ be fixed and consider the functional Λ on A_*^+ defined by

$$\Lambda(\rho) = (P_\rho(B)x_\rho, y_\rho)_\rho, \qquad \rho \in A_*^+. \tag{5.1}$$

We first show that Λ can be uniquely extended to a bounded linear functional on A_*.

5.2. LEMMA. *The functional Λ on A_*^+ defined by* (5.1) *is uniquely extended to a bounded linear functional on A_*.*

Proof. It suffices to prove that if $\rho_1, \cdots, \rho_n \in A_*^+$ and complex numbers $\alpha_1, \cdots, \alpha_n$ are such that $\sum_{j=1}^n \alpha_j \rho_j = 0$, then $\sum_{j=1}^n \alpha_j \Lambda(\rho_j) = 0$. Put $m_j(\cdot) = (P_{\rho_j}(\cdot)x_{\rho_j}, y_{\rho_j})_{\rho_j}$, $1 \leqq j \leqq n$ and define $m = |m_1| + \cdots + |m_n|$ where $|m_j|$ is the variation of m_j. Then m is a finite positive regular measure on $\hat{G}$ and the linear span of G, regarded as the dual group of $\hat{G}$, is dense in $L^1(\hat{G}, m)$. It follows that for any $\varepsilon > 0$ there exist some $s_1, \cdots, s_l \in G$ and complex numbers $\beta_1, \cdots, \beta_l$ such that

$$\int_{\hat{G}}\left|1_B(\chi)-\sum_{k=1}^{l}\beta_k\overline{\langle s_k, \chi\rangle}\right|m(d\chi)<\frac{\varepsilon}{n}(\max_{1\leq j\leq n}|\alpha_j|)^{-1}$$

where 1_B is the characteristic function of B. Hence we have

$$\left|\sum_j \alpha_j \Lambda(\rho_j)-\sum_j \alpha_j\int_{\hat{G}}\sum_k \beta_k\overline{\langle s_k, \chi\rangle}m_j(d\chi)\right|$$

$$\leqq\sum_j\left|\alpha_j\int_{\hat{G}}\{1_B(\chi)-\sum_k \beta_k\overline{\langle s_k, \chi\rangle}\}m_j(d\chi)\right|$$

$$\leqq\sum_j|\alpha_j|\int_{\hat{G}}|1_B(\chi)-\sum_k\beta_k\overline{\langle s_k, \chi\rangle}|m(d\chi)<\varepsilon .$$

On the other hand, it follows from the assumption that

$$\sum_j \alpha_j\int_{\hat{G}}\sum_k \beta_k\overline{\langle s_k, \chi\rangle}m_j(d\chi)=\sum_j \alpha_j\sum_k\beta_k(U_{\rho_j}(s_k)x_{\rho_j}, y_{\rho_j})_{\rho_j}$$

$$=\sum_j\alpha_j\sum_k\beta_k\rho_j([U(s_k)x, y])=\sum_j\alpha_j\rho_j(\sum_k\beta_k[U(s_k)x, y])=0 .$$

Consequently, $|\sum\alpha_j\Lambda(\rho_j)|<\varepsilon$. Since ε is arbitrary, we have $\sum\alpha_j\Lambda(\rho_j)=0$, as desired. The boundedness of Λ on A_* is easily verified.

It follows from 5.2. Lemma that there is a unique element $P_{x,y}(B)\in A$ such that $\Lambda(\theta)=\theta(P_{x,y}(B))$, $\theta\in A_*$ and, in particular, $(P_\rho(B)x_\rho, y_\rho)_\rho=\rho(P_{x,y}(B))$, $\rho\in A_*^+$. If B varies over $\mathfrak{B}_{\hat{G}}$, the function $P_{x,y}(\cdot)$ defines an A-valued σ-weakly CA measure on $\hat{G}$. Then we have the following.

5.3. LEMMA. (1) *For each x, $y\in X$ the relation*

$$[U(s)x, y]=\int_{\hat{G}}\overline{\langle s, \chi\rangle}P_{x,y}(d\chi), \qquad s\in G \tag{5.2}$$

holds where the integral is in the σ-weak topology of A.

(2) *For each x, y, $z\in X$ and $a\in A$ the equalities $P_{a\cdot x,y}(\cdot)=aP_{x,y}(\cdot)$, $P_{x+y,z}(\cdot)=P_{x,z}(\cdot)+P_{y,z}(\cdot)$ and $P_{x,y}(\cdot)=P_{y,x}(\cdot)^*$ hold.*

(3) *For each $B\in\mathfrak{B}_{\hat{G}}$ and $y\in X$ the function $x\to P_{x,y}(B)$ from X into A is a bounded module map, i.e., $P_{\cdot,y}(B)\in X^*$.*

Proof. (1) Let x, $y\in X$. For every $\rho\in A_*$ it holds that

$$\rho([U(s)x, y])=(U_\rho(s)x_\rho, y_\rho)_\rho=\int_{\hat{G}}\overline{\langle s, \chi\rangle}(P_\rho(d\chi)x_\rho, y_\rho)_\rho$$

$$=\int_{\hat{G}}\overline{\langle s, \chi\rangle}\rho(P_{x,y}(d\chi))=\rho(\int_{\hat{G}}\overline{\langle s, \chi\rangle}P_{x,y}(d\chi)) .$$

This is enough to prove (5.2).

(2) Let x, $y\in X$ and $a\in A$, and take $\rho\in A_*^+$. By $[U(s)(a\cdot x), y]=a[U(s)x, y]$

we have $\int_{\hat{G}}\overline{\langle s, \chi\rangle}\rho(P_{a\cdot x, y}(d\chi))=\int_{\hat{G}}\overline{\langle s, \chi\rangle}\rho(aP_{x, y}(d\chi))$ for $s\in G$. Since $\rho(P_{a\cdot x, y}(\cdot))$ and $\rho(aP_{x, y}(\cdot))$ are regular, they coincide. This is enough to show that $P_{a\cdot x, y}(\cdot)=aP_{x, y}(\cdot)$. Other equalities can be checked in a similar manner.

(3) Let $B\in\mathfrak{B}_{\hat{G}}$ and $y\in X$. It follows from (2) that $x\rightarrow P_{x, y}(B)$ is a module map. To see the boundedness, let $\rho\in A_*^+$. Then we have that $|\rho(P_{x, y}(B))|=|(P_\rho(B)x_\rho, y_\rho)_\rho|\leqq\|x_\rho\|_\rho\cdot\|y_\rho\|_\rho\leqq\|\rho\|\cdot\|x\|_X\cdot\|y\|_X$ for $x\in X$. Thus $\|P_{x, y}(B)\|\leqq 4\|x\|_X\cdot\|y\|_X$, $x\in X$. Therefore $P_{\cdot, y}(B)$ is bounded.

Assume that X is self-dual. Then it follows from 5.3. Lemma (3) that for each $y\in X$ and $B\in\mathfrak{B}_{\hat{G}}$ there is a unique $z\in X$ such that $P_{x, y}(B)=[x, z]$, $x\in X$. Define $z=P(B)y$. Then $P(B)$ is a well-defined operator in $B(X)$ and $P(\cdot)$ is a $B(X)$-valued Gramian σ-weakly CA measure on $\hat{G}$ such that $U(s)=\int_{\hat{G}}\overline{\langle s, \chi\rangle}P(d\chi)$, $s\in G$ where the integral is in the Gramian σ-weak topology. All we have to do is to show that $P(\cdot)$ is a Gramian spectral measure.

5.4. LEMMA. *$P(\cdot)$ is a Gramian spectral measure on $\hat{G}$.*

Proof. Let $B\in\mathfrak{B}_{\hat{G}}$ be fixed. It follows from 5.3. Lemma (2) that $[x, P(B)y]=P_{x, y}(B)=P_{y, x}(B)^*=[y, P(B)x]^*=[P(B)x, y]$ for $x, y\in X$. Hence $P(B)\in\mathfrak{A}(X)$ with $P(B)^*=P(B)$. Now we show that $P(B)^2=P(B)$. First we see that $(x_\rho, (P(B)y)_\rho)_\rho=\rho([x, P(B)y])=\rho(P_{x, y}(B))=(x_\rho, P_\rho(B)y_\rho)_\rho$ for $x, y\in X$ and $\rho\in A_*^+$. Hence $(P(B)y)_\rho=P_\rho(B)y_\rho$, $y\in X$, $\rho\in A_*^+$. Consequently we have $(P(B)y)_\rho=P_\rho(B)^2y_\rho=P_\rho(B)(P_\rho(B)y_\rho)=P_\rho(B)(P(B)y)_\rho=(P(B)^2y)_\rho$ for $y\in X$ and $\rho\in A_*^+$. Therefore $P(B)^2=P(B)$, as desired. It is clear that $\rho([P(\cdot)x, y])$ is a regular measure on $\hat{G}$ for each $x, y\in X$ and $\rho\in A_*^+$. Thus $P(\cdot)$ is a Gramian spectral measure.

We summarize these discussions in the following theorem.

5.5. THEOREM. *Let X be a self-dual Hilbert A-module and $s\rightarrow U(s)$ be a Gramian σ-weakly continuous GUR of G on X. Then there is a Gramian spectral measure P on $\hat{G}$ such that*

$$U(s)=\int_{\hat{G}}\overline{\langle s, \chi\rangle}P(d\chi), \qquad s\in G$$

where the integral is in the Gramian σ-weak topology.

Now we can prove a Bochner type theorem as follows.

5.6. COROLLARY. *For an A-valued weakly continuous PD function Γ on G there is an A-valued σ-weakly CA measure F on $\hat{G}$ such that*

$$\Gamma(s)=\int_{\hat{G}}\overline{\langle s, \chi\rangle}F(d\chi), \qquad s\in G$$

where the integral is in the σ-weak topology of A.

Proof. It follows from 4.2. Proposition that there exist a Hilbert A-module X_Γ, a GUR $s \to U_0(s)$ of G on X_Γ and a cyclic vector $x_0 \in X_\Gamma$ such that $\Gamma(s) = [U_0(s)x_0, x_0]_\Gamma$, $s \in G$ where $[\cdot, \cdot]_\Gamma$ is the Gramian on X_Γ. Again by 4.2. Proposition Γ is σ-weakly continuous since weak and σ-weak topologies coincide on bounded subsets of A. Hence we can see that $s \to U_0(s)$ is Gramian σ-weakly continuous. Then $s \to U_0(s)$ can be uniquely extended to a Gramian σ-weakly continuous GUR $s \to U(s)$ of G on the self-dual Hilbert A-module X_Γ^*. Consequently, by 5.5. Theorem, there is a Gramian spectral measure P on $\hat{G}$ such that $U(s) = \int_{\hat{G}} \overline{\langle s, \chi \rangle} P(d\chi)$, $s \in G$. Putting $F(\cdot) = [P(\cdot)x_0, x_0]$ where $[\cdot, \cdot]$ is the Gramian on X_Γ^*, we have that F is an A-valued σ-weakly CA measure on $\hat{G}$ and that, for $s \in G$,

$$\Gamma(s) = [U(s)x_0, x_0] = \left[\int_{\hat{G}} \overline{\langle s, \chi \rangle} P(d\chi)x_0, x_0\right] = \int_{\hat{G}} \overline{\langle s, \chi \rangle} [P(d\chi)x_0, x_0]$$

$$= \int_{\hat{G}} \overline{\langle s, \chi \rangle} F(d\chi) .$$

6. Hilbert A-module valued processes.

Let G be a locally compact abelian group and X be a Hilbert A-module. We consider X-valued processes over G.

6.1. DEFINITION. (1) An *X-valued process* $\{x(t)\}$ *over* G is a map $t \to x(t)$ from G into X.

(2) The *covariance function* Γ of a process $\{x(t)\}$ is defined by $\Gamma(s, t) = [x(s), x(t)]$, $s, t \in G$. Γ is an A-valued PDK on $G \times G$.

(3) A process $\{x(t)\}$ is said to be *stationary* if its covariance function $\Gamma(s, t)$ depends only on st^{-1} and, putting $\Gamma(s, t) = \Gamma(st^{-1})$, if Γ is an A-valued weakly continuous function on G.

(4) For a process $\tilde{x} = \{x(t)\}$ the *time domain* $\mathfrak{H}(\tilde{x})$ and an *observation space* $\mathfrak{H}(\tilde{x}; D)$ of a subset D of G are defined as submodules by $\mathfrak{H}(\tilde{x}) = \mathfrak{S}\{x(t); t \in G\}$ and $\mathfrak{H}(\tilde{x}; D) = \mathfrak{S}\{x(t); t \in D\}$, respectively.

(5) Let $\tilde{x} = \{x(t)\}$ be an X-valued process and $\tilde{y} = \{y(t)\}$ be a Y-valued process, Y being a Hilbert A-module. Then $\tilde{x}$ and $\tilde{y}$ are said to be *equivalent* if there exists a Gramian unitary operator $U: \mathfrak{H}(\tilde{x}) \to \mathfrak{H}(\tilde{y})$ such that $Ux(t) = y(t)$, $t \in G$.

Then the following is easily proved.

6.2. PROPOSITION. (1) *For any A-valued PDK Γ on $G \times G$ there is some Hilbert A-module valued process with the covariance function Γ.*

(2) *Let $\tilde{x}$ be an X-valued process with the covariance function Γ. Then we have, for each subset D of G,* $\mathfrak{H}(\tilde{x}\,;D)\cong D\otimes_{\Gamma}A$ *and, in particular,* $\mathfrak{H}(\tilde{x})\cong G\otimes_{\Gamma}A$ *where* $D\otimes_{\Gamma}A$ *was constructed in section* 3.

(3) *Let $\tilde{x}$ be an X-valued process and $\tilde{y}$ be a Y-valued process, Y being a Hilbert A-module. Then $\tilde{x}$ and $\tilde{y}$ are equivalent if and only if their covariance functions are identical.*

(4) *Stationarity is invariant within equivalence. More precisely, let $\tilde{x}$ and $\tilde{y}$ be as in* (3) *above. If they are equivalent and $\tilde{x}$ is stationary, then $\tilde{y}$ is also stationary.*

(5) *Let $\{x(t)\}$ be an X-valued stationary process with the covariance function Γ. Then there exist an X^*-valued CA orthogonally scattered measure ξ and an A-valued CA measure F on $\hat{G}$ such that*

$$x(t)=\int_{\hat{G}}\overline{\langle t,\,\chi\rangle}\xi(d\chi)\,,\qquad \Gamma(t)=\int_{\hat{G}}\overline{\langle t,\,\chi\rangle}F(d\chi)\,,\qquad t\in G$$

where the orthogonal scatteredness of ξ means that $[\xi(A),\,\xi(B)]=0$ for every disjoint pair $A,\,B\in\mathfrak{B}_{\hat{G}}$.

Let $(\Omega,\,\mathfrak{B},\,\mu)$ be a probability measure space and E be a Banach space with the dual space E^*. An E-valued function x on Ω is said to be *of weak second order* if it is weakly measurable and $f^*(x(\cdot))\in L^2(\Omega,\,\mu)$ for every $f^*\in E^*$. For each such function x there is an operator $T_x: E^*\rightarrow L^2(\Omega,\,\mu)$ such that $(T_xf^*)(\cdot)=f^*(x(\cdot))$, $f^*\in E^*$. If E is separable, then $T_x^*\colon L^2(\Omega,\,\mu)\rightarrow E\subset E^{**}$ (cf. [19, 2.2. Proposition]). Putting $H=L^2(\Omega,\,\mu)$ and $L=E^*$, we define an *E-valued process over G of weak second order* to be a $B(L,\,H)$-valued process over G where $B(L,\,H)$ is the Banach space of all bounded linear operators from L into H. The case where L is a Hilbert space was studied by Gangolli [4]. In this case $B(L,\,H)$ is a (right) Hilbert $B(L)$-module as was noted by Gangolli. Susiu and Valsescu [16] considered in this view point (see also Saworotnow [15]). The case where L is an arbitrary Banach space was studied by several authors such as Cobanjan and Weron [2], Weron [19] and Miamee [8] (cf. [9]).

Let $\{x(t)\}$ be an E-valued process of weak second order, i.e., $\{x(t)\}$ is a $B(E^*,\,H)$-valued process. When E is separable or reflexive, the *adjoint process* $\{x(t)^*\}$, which is $B(H,\,E^{**})$-valued, becomes a $B(H,\,E)$-valued process. The space $B(H,\,E)$ is a (right) Hilbert $B(H)$-module if we define a module action and a Gramian by $x\cdot a=xa$ and $[x,\,y]=y^*x$ for $x,\,y\in B(H,\,E)$ and $a\in B(H)$, respectively. Hence our theory is available in this respect.

REFERENCES

[1] ARONSZAJN, N., Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950), 337-404.

[2] COBANJAN, S.A. AND WERON, A., Banach space valued stationary processes and

their linear prediction, Dissertationes Math. **125** (1975), 1-45.

[3] DIXMIER, J., C^*-algebras, North-Holland, Amsterdam, 1977.

[4] GANGOLLI, R., Wide-sense stationary sequences of distributions on Hilbert space and the factorization of operator valued functions, J. Math. Mech. **12** (1963), 893-910.

[5] ITOH, S., A note on dilations in modules over C^*-algebras, J. London Math. Soc. (2) **22** (1980), 117-126.

[6] KAKIHARA, Y. AND TERASAKI, T., Hilbert $B(H)$-modules with applications I, Res. Rep. Inst. Inf. Sci. Tech. (Tokyo Denki Univ.) **5** (1979), 23-32.

[7] KAPLANSKY, I., Modules over operator algebras, Amer. J. Math. **75** (1953), 839-858.

[8] MIAMEE, A.G., On $B(X, K)$-valued stationary stochastic processes, Indiana Univ. Math. J. **25** (1976), 921-932.

[9] MIAMEE, A.G. AND SALEHI, H., On the square root of a positive $B(\mathfrak{Z}, \mathfrak{Z}^*)$-valued function, J. Multivar. Anal. **7** (1977), 535-550.

[10] NAKAMURA, M. AND UMEGAKI, H., A remark on theorems of Stone and Bochner, Proc. Japan Acad. **27** (1951), 506-507.

[11] OZAWA, M., Hilbert $B(H)$-modules and stationary processes, Kodai Math. J. **3** (1980), 26-39.

[12] PASCHKE, W.L., Inner product modules over B^*-algebras, Trans. Amer. Math. Soc. **182** (1973), 443-468.

[13] RIEFFEL, M.A., Induced representations of C^*-algebras, Advances Math. **13** (1974), 176-257.

[14] SAWOROTNOW, P.P., A generalized Hilbert space, Duke Math. J. **35** (1968), 191-197.

[15] SAWOROTNOW, P.P., Abstract stationary processes, Proc. Amer. Math. Soc. **40** (1973), 585-589.

[16] SUSIU, I. AND VALSESCU, I., A linear filtering problem in complete correlated actions, J. Multivar. Anal. **9** (1979), 599-613.

[17] UMEGAKI, H., Positive definite function and direct product Hilbert space, Tôhoku Math. J. **7** (1955), 206-211.

[18] UMEGAKI, H., An operator method for harmonic analysis on locally compact abelian group, MRC Tech. Summary Rep. #**725**, Madison, Wisconsin, 1967.

[19] WERON, A., Prediction theory in Banach space, Lecture Notes in Math. No. **472**, pp. 207-228, Springer-Verlag, New York, 1975.

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