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A THEOREM ON THE SPREAD RELATION

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0. Introduction.

Let $u=u_1-u_2$ be nonconstant, where u_1 and u_2 are subharmonic in the plane $\mathbf{C}$. For such a function u, we will write

$$N(r, u)=\frac{1}{2\pi}\int_{-\pi}^{+\pi} u(re^{i\theta})d\theta .$$

Then the Nevanlinna characteristic of $u=u_1-u_2$ is defined by

$$T(r)\equiv T(r, u)=N(r, u^+)+N(r, u_2) .$$

For $b\in(-\infty, +\infty)$ we define

$$\sigma_b(r, u)=|\{\theta ; u(re^{i\theta})>b\}| .$$

(Here, and throughout this note, $|E|$ denotes the one-dimensional Lebesgue measure of the set E. Also, θ is understood to vary between $-\pi$ and $+\pi$.)

In [4], Baernstein proved the following result.

THEOREM A. *Suppose $u=u_1-u_2$ is nonconstant, where u_1 and u_2 are subharmonic in $\mathbf{C}$. Let δ and λ be numbers satisfying*

$$\lambda>0, \qquad 0<\delta\leqq 1, \qquad \frac{4}{\lambda}\sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}\leqq 2\pi .$$

Assume there exist $r_0\geqq 0$ and $b\in(-\infty, +\infty)$ such that $r\geqq r_0$ implies

$$N(r, u_2)\leqq(1-\delta)T(r, u)+O(1)$$

and

$$\sigma_b(r, u)<\frac{4}{\lambda}\sin^{-1}\left(\frac{\delta}{2}\right)^{1/2} . \tag{1}$$

Then

$$\lim_{r\to\infty}\frac{T(r, u)}{r^\lambda}=\alpha$$

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exists, and is positive or infinite.

Theorem A may be regarded as an analogue of Kjellberg's definitive form [6] of the $\cos\pi\rho$ theorem.

In this note we consider the above result under somewhat weaker assumptions.

THEOREM. *Let u, δ, and λ be as in Theorem A except for the condition* (1). *Assume instead of* (1) *that there exists $b\in(-\infty, +\infty)$ such that*

$$m_l\left\{r>1\,;\ \sigma_b(r, u)\geqq\frac{4}{\lambda}\sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}\right\}<+\infty .$$

where m_lE denotes the logarithmic measure of the set E. Then

$$\lim_{r\to\infty}\frac{T(r, u)}{r^{\lambda}}=\alpha$$

exists, and is positive or infinite.

From this, we immediately deduce the fellowing

COROLLARY. *Let $u=u_1-u_2$, where u_1 and u_2 are subharmonic in* **C**, *and suppose u has lower order $\mu\in(0, \infty)$. If $\delta(\infty)\equiv\delta(\infty, u)>0$, then for any fixed $b\in(-\infty, +\infty)$ and $\varepsilon>0$,*

$$m_l\left\{r>1\,;\ \sigma_b(r, u)>\min\left(2\pi,\ \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\infty)}{2}\right)^{1/2}\right)-\varepsilon\right\}=+\infty .$$

In the above corollary, the quantities μ and $\delta(\infty)$ ore defined by

$$\mu=\varliminf_{r\to\infty}\frac{\log T(r, u)}{\log r},\qquad \delta(\infty)=1-\varlimsup_{r\to\infty}\frac{N(r, u_2)}{T(r, u)} .$$

We remark that this corollary can be deduced also from Theorem 1 in [8].

Without loss of generality, we may prove our theorem under the following additional conditions:

(i) u_1 and u_2 are harmonic in a neighborhood of 0,

(ii) $b=0$,

(iii) $u_1(z)\geqq u_2(z)$ for all z, $u_1(0)=u_2(0)=0$.

For the details, see [4, p89].

1. Lemmas.

LEMMA 1. ([3]) *Let $u=u_1-u_2$ be nonconstant, where u_1 and u_2 a.e sub-harmonic in* **C**. *Put*

$$u^*(re^{i\theta})=\sup_E\frac{1}{2\pi}\int_E u(re^{i\omega})d\omega\qquad (r>0,\ 0\leqq\theta\leqq\pi),$$

where the sup is taken over all sets $E \subset [-\pi, +\pi]$ *with* $|E|=2\theta$, *and define*

$$u^{\#}(re^{i\theta})=u^{*}(re^{i\theta})+N(r, u_2),$$

Then $u^{\#}(z)$ *is subharmonic in the upper half plane.*

LEMMA 2. ([7], cf. [5, § 5]) *Let* n *be a positive integer. Let* $\Gamma=\bigcup_{i=1}^{n}[-r'_i, -r_i]$, *where* $0 \leqq r_i < r'_i \leqq r_{i+1} \leqq 1$, $1 \leqq i \leqq n$, *and* $r_{n+1}=1$. *Put* $\Gamma^{+}=\{r; -r \in \Gamma\}$. *Let* u *be subharmonic in the unit disk* Δ, *and put* $m^{*}(r, u)=\inf_{|z|=r} u(z)$, $M(r, u)=\max_{|z|=r} u(z)$, *for* $0<r<1$. *For given* $\lambda \in (0, 1)$, *consider subharmonic functions in* Δ *which satisfy*

$$m^{*}(r, u) \leqq \cos \pi\lambda M(r, u) \qquad (r \in \Gamma^{+}-\{0, 1\}), \tag{1.1}$$

$$u(z) \leqq 1 \qquad (z \in \Delta). \tag{1.2}$$

For such a fixed Γ *and* λ, *there exists a function* $U(z) \equiv U(z, \Gamma, \lambda)$ *which has the following properties:*

- (i) U *is bounded, continuous and subharmonic in* Δ,
- (ii) U *is harmonic in* $\Delta-\Gamma$,
- (iii) $\lim_{z \to e^{i\theta}} U(z)=1$ $(|\theta|<\pi)$,
- (iv) $U(-r)=\cos \pi\lambda U(r)$ $(r \in \Gamma^{+}-\{1\})$,
- (v) *if* u *is subharmonic in* Δ, *and if* u *satisfies* (1.1) *and* (1.2), *then* $M(r, u) \leqq U(r)$ *for* $0<r<1$,
- (vi) U *is the unique function which satisfies* (i)-(v),
- (vii) *if* $r \in [0, 1)$, *then*

$$U(r) \leqq C(\lambda) \exp[-\lambda m_l(\Gamma^{+} \cap [r, 1])],$$

where $C(\lambda)$ *is a positive constant which depends only on* λ.

LEMMA 3. ([1]) *Let* v *be subharmonic in* $\mathbf{C}$, *and suppose that* $0<\sigma<1$. *If*

$$a=\varliminf_{r \to \infty} \frac{M(r, v)}{r^{\sigma}}<+\infty \qquad (M(r, v) \equiv \max_{|z|=r} v(z))$$

and

$$\varlimsup_{r_1, r_2 \to \infty} \int_{r_1}^{r_2} \frac{m^{*}(r, v)-\cos \pi\sigma M(r, v)}{r^{1+\sigma}} dr \leqq 0 \qquad (m^{*}(r, v)=\inf_{|z|=r} v(z)),$$

then

$$\lim_{r \to \infty} \frac{M(r, v)}{r^{\sigma}}=a.$$

2. Preliminaries.

2.1. A function $h(z)$. Set β and $\prec$ as follows:

$$\beta=\frac{2}{\lambda}\sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}, \qquad \gamma=\frac{\beta}{\pi}.$$

Then

$$\alpha\equiv\gamma\lambda=\frac{2}{\pi}\sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}\leqq\frac{1}{2}. \tag{2.1}$$

By assumption, there exists a positive number A such that

$$N(r,\ u_2)\leqq(1-\delta)T(r,\ u)+A \qquad (r\geqq 0).$$

Since $u(z)$ is nonconstant, $T(r,\ u)\equiv T(r)$ is unbounded, and so there exists a number $r_0>0$ such that

$$T(r_0^\gamma)>\pi A_1\equiv\pi A/(1-\cos\pi\alpha).$$

Now, Fix $R>2\max(1,\ r_0)$ and define

$$B(t)=\begin{cases} T(t^\gamma) & (0\leqq t\leqq R) \\ T_1(R^\gamma-)\log\left(\dfrac{t}{R}\right)+T(R^\gamma) & (R\leqq t<\infty), \end{cases}$$

where $T_1(t^\gamma)$ denotes the logarithmic derivative of the function $t\rightarrow T(t^\gamma)$. Then $B(t)$ is a convex increasing function of $\log t$, and the Poisson integral

$$h(z)=\frac{1}{\pi}\int_0^\infty\frac{r\sin\theta}{t^2+r^2+2tr\cos\theta}B(t)dt \qquad (z=re^{i\theta})$$

is harmonic in the slit plane $|\arg z|<\pi$, is zero on the positive axis and tends to $B(r)$ as $\theta\rightarrow\pi-$. Further,

$$h_\theta(re^{i\theta})=\frac{1}{\pi}\int_0^\infty\log\left|1+\frac{re^{i\theta}}{t}\right|dB_1(t) \qquad (|\theta|<\pi) \tag{2.2}$$

and

$$h_\theta(-r)\equiv\lim_{\theta\rightarrow\pi-}\frac{B(r)-h(re^{i\theta})}{\pi-\theta}=\lim_{\theta\rightarrow\pi-}h_\theta(re^{i\theta})=\frac{1}{\pi}\int_0^\infty\log\left|1-\frac{r}{t}\right|dB_1(t) \tag{2.3}$$

hold, where $B_1(t)$ is the logarithmic derivative of logarithmically convex non-decreasing function $B(t)$, which were established in § 3 of [2].

By (2.2), (iii) in § 0 and (12) of [2]

$$\begin{aligned}\pi h_\theta\left(\frac{R}{2}\right)&=\int_0^R\log\left(1+\frac{R}{2t}\right)dB_1(t)\\ &=B_1(R-)\log\frac{3}{2}+\int_0^R\frac{R/2}{t+R/2}dB(t)\\ &=B_1(R-)\log\frac{3}{2}+B(R)\frac{1}{3}+\int_0^R\frac{R/2}{(t+R/2)^2}B(t)dt\end{aligned}$$

$$\leqq B_1(R-)+B(R)\leqq B(Re)+B(R)\leqq 2B(Re)$$

$$\leqq 2\{T_1(R^\gamma-)+T(R^\gamma)\}\leqq 2\{T(R^\gamma e^\gamma)+T(R^\gamma)\}\leqq 4T(R^\gamma e^\gamma), \quad \text{i.e.}$$

$$h_\theta\left(\frac{R}{2}\right)\leqq\frac{4}{\pi}T(R^\gamma e^\gamma). \tag{2.4}$$

Also, for $0<r<R$,

$$T(r^\gamma)=B(r)=h(-r)=\int_0^\pi h_\theta(r)d\theta<\pi h_\theta(r), \quad \text{i.e.}$$

$$h_\theta(r)>\frac{1}{\pi}T(r^\gamma). \tag{2.5}$$

2.2. A function $h_1(z)$. Let $\Delta_R=\{z\,;\,|z|<R\}$ and let $h_1(z)$ be the bounded harmonic function in Δ_R defined by

$$h_1(re^{i\theta})=\frac{1}{2\pi}\int_0^\pi T(R^\gamma)\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}dt$$

$$+\frac{1}{2\pi}\int_\pi^{2\pi}(-T(R^\gamma))\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}dt.$$

Then

$$(h_1)_\theta(re^{i\theta})=\frac{1}{2\pi}\int_0^\pi T(R^\gamma)\frac{\partial}{\partial\theta}\left(\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}\right)dt$$

$$+\frac{1}{2\pi}\int_\pi^{2\pi}(-T(R^\gamma))\frac{\partial}{\partial\theta}\left(\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}\right)dt$$

$$=\frac{1}{2\pi}\int_0^\pi(-T(R^\gamma))\frac{\partial}{\partial t}\left(\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}\right)dt \tag{2.6}$$

$$+\frac{1}{2\pi}\int_\pi^{2\pi}T(R^\gamma)\frac{\partial}{\partial t}\left(\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}\right)dt$$

$$=\frac{T(R^\gamma)}{\pi}\left[\frac{R^2-r^2}{R^2+r^2-2Rr\cos\theta}-\frac{R^2-r^2}{R^2+r^2+2Rr\cos\theta}\right] \qquad (r<R).$$

Hence $(h_1)_\theta(z)$ is also harmonic in Δ_R and

$$(h_1)_\theta\left(\frac{R}{2}\right)=\frac{8T(R^\gamma)}{3\pi}. \tag{2.7}$$

2.3. A function $H(z)$. Consider the harmonic function $H(z)$ in $\Delta_R^+=\{z\,;\,z\in\Delta_R,\ \operatorname{Im} z>0\}$ defined by

$$H(re^{i\theta})=h(re^{i\theta})+\cos\pi\alpha h(re^{i(\pi-\theta)})+h_1(re^{i\theta})+A(\pi-\theta).$$

The boundary values of H satisfy

$$
(2.8)\qquad \begin{cases} H(-r)=h(-r)=T(r^{\gamma}) & (0\leqq r<R),\\ H(r)=\cos \pi\alpha h(-r)+A\pi & (0<r<R),\\ H(Re^{i\theta})\geqq h_1(Re^{i\theta})=T(R^{\gamma}) & (0\leqq\theta\leqq\pi). \end{cases}
$$

Now, set

$$
v(z)=u^{\#}(z^{\gamma}).
$$

Then by (2.8)

$$
(2.9)\qquad \begin{cases} v(-r)=u^{\#}(r^{\gamma}e^{i\beta})\leqq T(r^{\gamma})=H(-r) & (0\leqq r<R),\\ v(r)=u^{\#}(r^{\gamma})=N(r^{\gamma},\ u_2)\leqq(1-\delta)T(r^{\gamma})+A\leqq H(r) & (0<r<R),\\ v(Re^{i\theta})=u^{\#}(R^{\gamma}e^{i\gamma\theta})\leqq T(R^{\gamma})\leqq H(Re^{i\theta}) & (0\leqq\theta\leqq\pi). \end{cases}
$$

Hence, by Lemma 1 and (2.9)

$$
(2.10)\qquad v(z)\leqq H(z) \qquad (z\in\Delta_R^{+}).
$$

3. Proof of Theorem.

Set

$$
G_{\lambda}=\{r>1\,;\ \sigma_0(r,\ u)\geqq 2\beta\}
$$

and

$$
F_{\lambda}=(0,\ \infty)-G_{\lambda}.
$$

Suppose $r^{\gamma}\in F_{\lambda}$. Since $u\geqq 0$ everywhere, we easily deduce that

$$
(3.1)\qquad v(-r)=u^{\#}(r^{\gamma}e^{i\beta})=T(r^{\gamma}).
$$

It follows from (3.1), (2.9) and (2.10) that for $r^{\gamma}\in F_{\lambda}\cap(0,\ R^{\gamma})$

$$
(3.2)\qquad H_{\theta}(-r)\equiv\lim_{\theta\to\pi-}\frac{H(-r)-H(re^{i\theta})}{\pi-\theta}\leqq v_{\theta}(-r)=\gamma u_{\theta}^{\#}(r^{\gamma}e^{i\beta}).
$$

(Existence of the limit follows from (2.3).) Let $\tilde{u}(re^{i\theta})$ denote the symmetric decreasing rearrangement of $u(re^{i\theta})$ (cf. [3, § 3]). Then

$$
(3.3)\qquad u_{\theta}^{\#}(r^{\gamma}e^{i\beta})=\tilde{u}(r^{\gamma}e^{i(\beta-)})=0 \qquad (r^{\gamma}\in F_{\lambda}).
$$

Hence, by (3.2) and (3.3), $H_{\theta}(-r)\leqq 0$, i.e.

$$
h_{\theta}(-r)+(h_1)_{\theta}(-r)\leqq\cos\pi\alpha h_{\theta}(r)+A \qquad (r^{\gamma}\in F_{\lambda}\cap(0,\ R^{\gamma})).
$$

If $\alpha<1/2$, then by (2.6)

$$
(3.4)\qquad h_{\theta}(-r)+(h_1)_{\theta}(-r)-A_1<\cos\pi\alpha[h_{\theta}(r)+(h_1)_{\theta}(r)-A_1]
$$

$$
(r^{\gamma}\in F_{\lambda}\cap(0,\ R^{\gamma})).
$$

If $\alpha=1/2$, then by assumption $N(r, u_2)$ is bounded. This implies that u_2 is harmonic, in which case $N(r, u_2)=u_2(0)=0$. Hence (2.10) holds with $A=0$. Then, arguing as above, we obtain

$$h_\theta(-r)+(h_1)_\theta(-r)\leqq 0 \qquad (r^\gamma\in F_\lambda\cap(0, R^\gamma)).$$

This shows that (3.4) is true also for $\alpha=1/2$ with $A_1=A$ (an arbitrary positive number).

Here, we consider the functioin $k(z)$ defined by

$$k(z)=\frac{h_\theta(Rz/2)+(h_1)_\theta(Rz/2)-A_1}{h_\theta(R/2)+(h_1)_\theta(R/2)-A_1} \qquad (z\in\Delta).$$

(Note that the denominator is positive by (2.5) and the choice of R.) In view of (2.2), (2.3) and (2.6), $k(z)$ is subharmonic in Δ and

$$\text{(3.5)} \qquad \begin{cases} m^*(r, k)=\dfrac{h_\theta(-Rr/2)+(h_1)_\theta(-Rr/2)-A_1}{h_\theta(R/2)+(h_1)_\theta(R/2)-A_1}, \\[2ex] M(r, k)=\dfrac{h_\theta(Rr/2)+(h_1)_\theta(Rr/2)-A_1}{h_\theta(R/2)+(h_1)_\theta(R/2)-A_1}. \end{cases}$$

Combining (3.4) and (3.5), we have

$$\text{(3.6)} \qquad m^*(r, k)<\cos\pi\alpha M(r, k) \qquad ((Rr/2)^\gamma\in F_\lambda\cap(0, (R/2)^\gamma)).$$

As is easily verified, $m^*(r, k)-\cos\pi\alpha M(r, k)$ is upper semicontinuous. Hence

$$E_\alpha=\{r\in(0, 1);\, m^*(r, k)-\cos\pi\alpha M(r, k)<0\}$$

is open. It is clear that

$$m^*(r, k)\leqq\cos\pi\alpha M(r, k) \qquad (r\in E_\alpha)$$

and

$$k(z)\leqq 1 \qquad (z\in\Delta).$$

Now, E_α is open and so $E_\alpha=\bigcup_{n=1}^\infty(s_n, t_n)$, where $0\leqq s_n<t_n\leqq 1$. Here we allow repetition of intervals. Let

$$T_j=\bigcup_{n=1}^{j}\left[s_n+\frac{(t_n-s_n)}{3j},\, t_n-\frac{(t_n-s_n)}{3j}\right] \qquad (j=1, 2, 3, \cdots).$$

Then by Lemma 2

$$M(r, k)\leqq C(\alpha)\exp[-\alpha m_l(T_j\cap[r, 1])] \qquad (j=1, 2, 3, \cdots).$$

Since $m_l(T_j\cap[r, 1])\to m_l(E_\alpha\cap[r, 1])$ $(j\to\infty)$, we obtain for $0<r<1$

$$\text{(3.7)} \qquad \frac{h_\theta(Rr/2)+(h_1)_\theta(Rr/2)-A_1}{h_\theta(R/2)+(h_1)_\theta(R/2)-A_1}\leqq C(\alpha)\exp[-\alpha m_l(E_\alpha\cap[r, 1])].$$

Putting $\tilde{F}_\lambda=\{r\in(0,1); (Rr/2)^\gamma\in F_\lambda\}$, we deduce (3.6) that $\tilde{F}_\lambda\subset E_\alpha$, and so by (3.7)

$$\frac{h_\theta(Rr/2)+(h_1)_\theta(Rr/2)-A_1}{h_\theta(R/2)+(h_1)_\theta(R/2)-A_1}\leqq C(\alpha)\exp[-\alpha m_l(\tilde{F}_\lambda\cap[r,1])] \qquad (0<r<1).$$

Hence for $0<r<R/2$

$$\begin{aligned}(3.8)\qquad &\frac{h_\theta(r)+(h_1)_\theta(r)-A_1}{h_\theta(R/2)+(h_1)_\theta(R/2)-A_1}\\ &\leqq C(\alpha)\exp\left[-\frac{\alpha}{\gamma}m_l(F_\lambda\cap[r^\gamma,(R/2)^\gamma])\right]\\ &=C(\alpha)\exp\left[-\frac{\alpha}{\gamma}\log\left(\frac{R/2}{r}\right)^\gamma+\frac{\alpha}{\gamma}m_l(G_\lambda\cap[r^\gamma,(R/2)^\gamma])\right]\\ &=C(\alpha)\left(\frac{r}{R/2}\right)^\alpha\exp[\lambda m_l(G_\lambda\cap[r^\gamma,(R/2)^\gamma])]\\ &\leqq C(\alpha)\exp[\lambda m_l G_\lambda]\frac{r^\alpha}{(R/2)^\alpha}\equiv B(\alpha)\frac{r^\alpha}{(R/2)^\alpha}<+\infty.\end{aligned}$$

It follows from (2.4)-(2.7), and (3.8) that

$$\frac{T(r^\gamma)/\pi-A_1}{r^\alpha}<B(\alpha)\frac{4T(R^\gamma e^\gamma)/\pi+8T(R^\gamma)/3\pi}{(R/2)^\alpha}.$$

This result may be written

$$\frac{T(r^\gamma)}{r^\alpha}<K_1\frac{T(R^\gamma e^\gamma)}{(Re)^\alpha}+K_2r^{-\alpha},$$

where K_1 and K_2 are positive and depend only on δ and λ. Replace r^γ by r and $R^\gamma e^\gamma$ by R. Then we have

$$\frac{T(r)}{r^\lambda}<K_1\frac{T(R)}{R^\lambda}+K_2r^{-\lambda} \qquad \left(0<r<\frac{R}{(2e)^\gamma}\right).$$

From this, it is easy to see that

$$\lim_{r\to\infty}\frac{T(r)}{r^\lambda}=+\infty,$$

or

$$(3.9)\qquad 0<\varliminf_{r\to\infty}\frac{T(r)}{r^\lambda}\leqq\varlimsup_{r\to\infty}\frac{T(r)}{r^\lambda}=\varlimsup_{r\to\infty}\frac{T(r^\gamma)}{r^\alpha}<+\infty.$$

In what follows, we assume (3.9). Since $\alpha\leqq 1/2$, the Poisson integral

(3.10) $$I(z)=\frac{1}{\pi}\int_0^\infty T(t^\gamma)\frac{r\sin\theta}{t^2+r^2+2tr\cos\theta}\,dt$$

is a positive harmonic function in the upper half plane, with boundary values $I(-r)=T(r^\gamma)$, $I(r)=0$ for $r\geqq 0$. Then, arguing as in 2.3., we have

$$v(re^{i\theta})\leqq I(re^{i\theta})+\cos\pi\alpha I(re^{i(\pi-\theta)})+A(\pi-\theta)$$

on the real axis. Since $v(re^{i\theta})\leqq T(r^\gamma)=O(r^\alpha)<o(r)$, the above inequality holds throughout the upper halp plane. Also equality holds for $r^\gamma\in F^\lambda$ and $\theta=\pi$. Hence

$$I_\theta(-r)-\cos\pi\alpha I_\theta(r)-A\leqq v_\theta(-r)=0 \qquad (r^\gamma\in F_\lambda),$$

so that

(3.11) $$I_\theta(-r)-A_1\leqq\cos\pi\alpha[I_\theta(r)-A_1] \qquad (r^\gamma\in F_\lambda).$$

By (3.10)

(3.12) $$\pi I_\theta(r)=\int_0^\infty\log\left(1+\frac{r}{t}\right)dT_1(t^\gamma)=\int_0^\infty\frac{r}{t+r}\,dT(t^\gamma)=\int_0^\infty\frac{rT(t^\gamma)}{(t+r)^2}\,dt,$$

and so by (3.9)

(3.13) $$0<\varliminf_{r\to\infty}\frac{I_\theta(r)}{r^\alpha}\leqq\varlimsup_{r\to\infty}\frac{I_\theta(r)}{r^\alpha}<+\infty.$$

If we put $J(z)=I_\theta(z)-A_1$ and

$$E'_\alpha=\{r\,;\,m^*(r,J)-\cos\pi\alpha M(r,J)>0\},$$

then by (3.11) and (3.13)

$$\int_1^\infty\frac{[m^*(r,J)-\cos\pi\alpha M(r,J)]^+}{r^{1+\alpha}}\,dr=\int_{E'_\alpha\cap(1,\infty)}\frac{m^*(r,J)-\cos\pi\alpha M(r,J)}{r^{1+\alpha}}\,dr$$

$$<(1-\cos\pi\alpha)\int_{E'_\alpha\cap(1,\infty)}\frac{M(r,J)}{r^{1+\alpha}}\,dr$$

$$=(1-\cos\pi\alpha)\int_{E'_\alpha\cap(1,\infty)}\frac{O(r^\alpha)}{r^{1+\alpha}}\,dr=O(m_lE'_\alpha)<+\infty.$$

Hence

(3.14) $$\varlimsup_{r_1,r_2\to\infty}\int_{r_1}^{r_2}\frac{[m^*(r,J)-\cos\pi\alpha M(r,J)]}{r^{1+\alpha}}\,dr\leqq 0.$$

Using Lemma 3, we deduce from (3.13) and (3.14) that

$$\lim_{r\to\infty}\frac{M(r,J)}{r^\alpha}=a \qquad (0<a<+\infty),\ \text{i.e.}$$

(3.15) $$\lim_{r\to\infty}\frac{I_\theta(r)}{r^\alpha}=a.$$

By (3.12)

$$I_\theta(r)=\frac{1}{\pi}\int_0^\infty T(t^\gamma)\frac{r}{(t+r)^2}\,dt=g*K(r)\,, \tag{3.16}$$

where $g(t)=T(t^\gamma)$, $K(t)=\dfrac{1}{\pi}\dfrac{t}{(1+t)^2}$. Using Lemma 4 of [4], we deduce from (3.15) and (3.16) that

$$\lim_{r\to\infty}\frac{T(r)}{r^\lambda}=\lim_{r\to\infty}\frac{T(r^\gamma)}{r^\alpha}=\frac{\sin\pi\alpha}{\alpha}a\,.$$

This completes the proof of our theorem.

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