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ON THE GROWTH OF ENTIRE FUNCTIONS OF ORDER LESS THAN 1/2

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0. Introduction. Let $f(z)$ be meromorphic in the plane. Throughout this paper we shall assume familiarity with the standard notation of the Nevanlinna theory,

$$T(r, f),\ N(r, f),\ m(r, f),\ \delta(a, f),\ \cdots.$$

We define

$$M(r, f)=\max_{|z|=r} |f(z)|, \qquad m^*(r, f)=\min_{|z|=r} |f(z)|.$$

In [2], Anderson proved the following result.

THEOREM A. *Let $f(z)$ be meromorphic in the plane and such that for some ρ, $0<\rho<1$, either*

$$\pi\rho N(r, 0, f)\leqq \sin \pi\rho \log M(r, f)+\pi\rho \cos \pi\rho N(r, f)$$

or

$$\sin \pi\rho \log m^*(r, f)\leqq \pi\rho \cos \pi\rho N(r, 0, f)-\pi\rho N(r, f) \tag{1}$$

for all large r. Then

$$\beta=\varliminf_{r\to\infty} \frac{T(r, f)}{r^\rho}>0.$$

If, further, $\beta<\infty$ then

$$\alpha=\varlimsup_{r\to\infty} \frac{T(r, f)}{r^\rho}<\infty.$$

The inequality (1) and its conclusion have been used to show that for a meromorphic function of lower order $\lambda<1/2$,

$$\varlimsup_{r\to\infty} \frac{\log^+ m^*(r, f)}{T(r, f)} \geqq \frac{\pi\lambda}{\sin \pi\lambda}(\cos \pi\lambda-1+\delta(\infty, f)). \tag{2}$$

Later, Edrei [6] obtained this estimate by making use of the notion of the local form of the Phragmén-Lindelöf indicator. The estimate (2) is best possible. (For example, see [6, p 151].)

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If $f(z)$ is an entire function of order $\rho<1/2$, (2) implies that

$$\varlimsup_{r\to\infty}\frac{\log m^*(r, f)}{m(r, f)}\geqq\pi\rho\cot\pi\rho\,. \tag{3}$$

An important consequence of Theorem A is that if $f(z)$ is an entire function of order ρ $(0<\rho<1/2)$ and minimal type, then

$$\log m^*(r, f)>\pi\rho\cot\pi\rho m(r, f) \tag{4}$$

holds for a sequence of $r=r_n\uparrow\infty$.

The main purpose of this paper is to refine the estimate (3) for entire functions of order ρ $(0<\rho<1/2)$ and mean type.

THEOREM 1. *Let $h(r)$ be positive and continuous for $r\geqq r_0$ and, for each $s>0$,*

$$\frac{h(sr)}{h(r)}\longrightarrow 1\qquad(r\to\infty)\,.$$

Suppose that $h(r)\to 0$ $(r\to\infty)$ and that

$$\int^{\infty}\frac{h(t)}{t}\,dt=\infty\,.$$

If $f(z)$ is an entire function of order ρ $(0<\rho<1/2)$ and mean type, then

$$\log m^*(r, f)>\pi\rho\cot\pi\rho(1-h(r))m(r, f)$$

on a sequence of $r\to\infty$.

This result is regarded as an analogue of the Barry's one [4, Theorem 2] for the $\cos\pi\rho$ theorem. It is worth while to be pointed out that in his above theorem the assumption that $h'(r)>-O(r^{-1})$ $(r\to\infty)$ can be dropped. The proof is essentially contained in the proof of our theorem.

For an entire (or a meromorphic) function $f(z)$, we define

$$m_2(r, f)=\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}\{\log|f(re^{i\theta})|\}^2d\theta\right]^{1/2}.$$

In [9], we showed that if $f(z)$ is an entire function of order ρ $(<1/2)$ then

$$\varlimsup_{r\to\infty}\frac{\log m^*(r, f)}{m_2(r, f)}\geqq\frac{\cos\pi\rho}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}\equiv A(\rho)\,. \tag{5}$$

(The estimate (5) is best possible.) The method of the proof of Theorem 1 yields the following results.

THEOREM 2. *Let $f(z)$ be an entire function of order ρ $(0<\rho<1/2)$ and minimal type. Then*

$$\log m^*(r, f)>A(\rho)m_2(r, f)$$

for a sequence of $r\to\infty$.

THEOREM 3. *Let* $h(r)$ *be given as in Theorem* 1. *If* $f(z)$ *is an entire function of order* ρ $(0<\rho<1/2)$ *and mean type, then*

$$\log m^*(r, f)>A(\rho)(1-h(r))\cdot m_2(r, f)$$

on a sequence of $r\to\infty$.

1. Proof of Theorem 1.

1.1. Preliminary discussion.

Let $f(z)$ be an entire function of order ρ $(0<\rho<1/2)$ and mean type. Since $\rho<1$, we know that

$$f(z)=cz^p\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right), \tag{1.1}$$

where a_n's are the nonzero zeros of $f(z)$ arranged in order of increasing magnitude. Set

$$f_1(z)=|c|z^p\prod_{n=1}^{\infty}\left(1+\frac{z}{|a_n|}\right). \tag{1.2}$$

Then we have (cf. [5, 3.2])

$$m^*(r, f)\geqq m^*(r, f_1)=|f_1(-r)| \qquad (r\geqq 0). \tag{1.3}$$

And also

$$m(r, f)\leqq m(r, f_1) \qquad (r\geqq 0). \tag{1.4}$$

This is due to Gol'dberg [7]. By (1.2),

$$\begin{aligned}
\log M(r, f_1)&=\log f_1(r)\\
&=r\int_0^{\infty}\frac{n(t)-n(0)}{t(t+r)}dt+p\log r+\log|c|\\
&\leqq\int_0^{r}\frac{n(t)-n(0)}{t}dt+r\int_r^{\infty}\frac{n(t)}{t^2}dt+O(\log r)\\
&\leqq N(r)+r\int_r^{\infty}\frac{dN(t)}{t}+O(\log r) \qquad (1.5)\\
&=r\int_r^{\infty}\frac{N(t)}{t^2}dt+O(\log r)\\
&\leqq r\int_r^{\infty}\frac{\log M(t, f)}{t^2}dt+O(\log r).
\end{aligned}$$

Since $f(z)$ is of order ρ (<1) and mean type, (1.4) and (1.5) imply that $f_1(z)$ is also of order ρ and mean type. By (1.3) and (1.4),

$$\frac{\log m^*(r, f)}{m(r, f)} \geqq \frac{\log m^*(r, f_1)}{m(r, f_1)},$$

provided that $\log m^*(r, f_1) \geqq 0$. Hence, we may prove our theorem for $f=f_1$.

1.2. An inequality on a class of entire functions of order$<1/2$.

Let $f_1(z)$ be a nonconstant entire function of order$<1/2$, where $f_1(z)$ is of the form (1.2). Assume that, corresponding to $f_1(z)$, there exists a function $H(z)$ defined in the whole plane satisfying the following conditions.

(2.1) $H(z)$ is a one-valued positive continuous function in the whole plane, and is harmonic in $|\arg z|<\pi$.

(2.2) $B(r)\equiv\max\limits_{|z|=r} H(z)$ is of order less than $1/2$.

(2.3) $\log f_1(r)=o(H(-r)) \quad (r\to\infty)$.

Under the conditions (2.1)-(2.3), Barry's argument in [3, Lemma 5] implies that given $\varepsilon>0$, there are two numbers $a(\varepsilon)$, $r(\varepsilon)$ such that

$$\log|f_1(-r)|-\frac{H(-r)}{H(re^{i\theta})}\log|f_1(re^{i\theta})|\geqq a(\varepsilon)\left\{1-\frac{H(-r)}{H(re^{i\theta})}\right\} \tag{2.4}$$
$$(|\theta|\leqq\pi,\ r\equiv r(\varepsilon)\to\infty,\ a(\varepsilon)\to\infty \text{ as } \varepsilon\to 0).$$

1.3. Some properties of the Legendre's polynomials $P_n(x)$ $(n=0, 1, 2, \cdots)$.

In the proof of our theorem, we need some properties of the Legendre's polynomials $P_n(x)$ $(n=0, 1, 2, \cdots)$ defined by

$$P_n(x)=\frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}(x^n-1)^n.$$

For the sake of convenience, we write down some properties.

$$(1-2t\cdot\cos\theta+t^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(\cos\theta)\cdot t^n \qquad (|t|<1). \tag{3.1}$$

$$|P_n(\cos\theta)|\leqq 1 \qquad (n=0, 1, 2, \cdots). \tag{3.2}$$

$$|P_n(\cos\theta)|\leqq 2(n\pi\sin\theta)^{-1/2} \qquad (0<\theta<\pi,\ n=1, 2, \cdots). \tag{3.3}$$

$$|P_n(-\cos\theta)|=|P_n(\cos\theta)| \qquad (n=0, 1, 2, \cdots). \tag{3.4}$$

1.4. A harmonic function $H(z)$.

We set

$$h(t)=(t/r_0)\cdot h(r_0) \qquad 0\leqq t\leqq r_0,$$

and let

(4.1) $$L(r)=\exp\left\{\delta\int_1^r \frac{h(t)}{t}\,dt\right\},$$

where $\delta>0$. Since $h(t)\to 0$ $(t\to\infty)$, $L(r)$ varies slowly. Hence

(4.2) $$\Lambda(r)\equiv r^\rho L(r)$$

has order ρ. Further, by our assumption on $h(t)$, $L(r)\uparrow\infty$ $(r\to\infty)$. Hence, if $f_1(z)$ is of order ρ $(<1/2)$ and mean type, then

(4.3) $$\log M(r, f_1)=o(\Lambda(r)) \qquad (r\to\infty).$$

Now, we put

(4.4) $$H(re^{i\theta})=\frac{1}{\pi}\int_0^\infty \frac{r^{1/2}(r+s)\Lambda(s)\cos(\theta/2)}{s^{1/2}(r^2+s^2+2rs\cos\theta)}\,ds,$$

which provides a solution of the Dirichlet problem with boundary values

(4.5) $$H(-r)=\Lambda(r) \qquad (r\geqq 0).$$

In view of (4.1)-(4.5), $H(z)$ satisfies the conditions (2.1)-(2.3), and thus, (2.4) holds.

1.5. An estimate of $H(re^{i\theta})$.

We write

$$H(re^{i\theta})=I_1(r,\theta)+I_2(r,\theta),$$

where

$$I_1(r,\theta)=\frac{1}{\pi}\int_0^\infty \frac{r^{1/2}(r+s)s^\rho L(r)\cos(\theta/2)}{s^{1/2}(r^2+s^2+2rs\cos\theta)}\,ds,$$

$$I_2(r,\theta)=\frac{1}{\pi}\int_0^\infty \frac{r^{1/2}(r+s)s^\rho[L(s)-L(r)]\cos(\theta/2)}{s^{1/2}(r^2+s^2+2rs\cos\theta)}\,ds.$$

First, we compute $I_1(r,\theta)$. Putting $s=rt$, we have

$$I_1(r,\theta)=\Lambda(r)\cos\left(\frac{\theta}{2}\right)\left[\frac{1}{\pi}\int_0^\infty \frac{t^{\rho+1/2}}{t^2+2t\cos\theta+1}\,dt+\frac{1}{\pi}\int_0^\infty \frac{t^{\rho-1/2}}{t^2+2t\cos\theta+1}\,dt\right].$$

Residue calculation shows that

$$\frac{1}{\pi}\int_0^\infty \frac{t^\alpha\sin\theta}{t^2+2t\cos\theta+1}\,dt=\frac{\sin\theta\alpha}{\sin\pi\alpha} \qquad (-1<\alpha<1).$$

Hence

(5.1) $$I_1(r,\theta)=\Lambda(r)\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{\sin\theta}\frac{1}{\cos\pi\rho}\{\sin\theta(\rho+1/2)-\sin\theta(\rho-1/2)\}$$

$$=\Lambda(r)\frac{\cos\theta\rho}{\cos\pi\rho}.$$

Next, we estimate $I_2(r,\theta)$.

$$
\begin{aligned}
I_2(r,\theta)&=\frac{r^{1/2}}{\pi}\cos\left(\frac{\theta}{2}\right)\cdot\int_0^\infty\frac{(r+s)s^\rho[L(s)-L(r)]}{s^{1/2}(r^2+s^2+2rs\cos\theta)}ds\\
&=\frac{r^{1/2}}{\pi}\cos\left(\frac{\theta}{2}\right)\left[\int_0^r+\int_r^\infty\right]\frac{(r+s)s^\rho[L(s)-L(r)]}{s^{1/2}(r^2+s^2+2rs\cos\theta)}ds \qquad (5.2)\\
&\equiv\frac{r^{1/2}}{\pi}\cos\left(\frac{\theta}{2}\right)[A_\theta(r)+B_\theta(r)],\ say.
\end{aligned}
$$

By (3.1)

$$(t^2+2t\cdot\cos\theta+1)^{-1/2}=\sum_{n=0}^\infty P_n(-\cos\theta)\cdot t^n \qquad (|t|<1)\,.$$

Hence

$$
\begin{aligned}
&A_\theta(r)\\
&=\int_0^r\frac{[s^{\rho-1/2}r+s^{\rho+1/2}][L(s)-L(r)]}{r^2}\sum_{m,n=0}^\infty P_m(-\cos\theta)P_n(-\cos\theta)\left(\frac{s}{r}\right)^{m+n}ds\\
&=\sum_{m,n=0}^\infty P_m(-\cos\theta)P_n(-\cos\theta)\left[\frac{1}{r}\int_0^r s^{\rho-1/2+m+n}r^{-(m+n)}[L(s)-L(r)]ds\right.\\
&\quad\left.+\frac{1}{r^2}\int_0^r s^{\rho+1/2+m+n}r^{-(m+n)}[L(s)-L(r)]ds\right]\\
&=\sum_{m,n=0}^\infty P_m(-\cos\theta)P_n(-\cos\theta)\left[-\frac{1}{m+n+1/2+\rho}r^{-(m+n+1)}\int_0^r s^{m+n+\rho-1/2}sL'(s)ds\right.\\
&\quad\left.-\frac{1}{m+n+3/2+\rho}r^{-(m+n+2)}\int_0^r s^{m+n+\rho+1/2}sL'(s)ds\right] \qquad (5.3)\\
&=r^{\rho-1/2}\sum_{m,n=0}^\infty P_m(-\cos\theta)P_n(-\cos\theta)\left[-\frac{1}{m+n+1/2+\rho}\int_0^1 t^{m+n+\rho-1/2}rtL'(rt)dt\right.\\
&\quad\left.-\frac{1}{m+n+3/2+\rho}\int_0^1 t^{m+n+\rho+1/2}rtL'(rt)dt\right]\\
&=-r^{\rho-1/2}\int_0^1 rtL'(rt)\sum_{m,n=0}^\infty\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+1/2+\rho}t^{m+n+\rho-1/2}dt\\
&\quad-r^{\rho-1/2}\int_0^1 rtL'(rt)\sum_{m,n=0}^\infty\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+3/2+\rho}t^{m+n+\rho+1/2}dt\,.
\end{aligned}
$$

The inversions in the order of integration and summation are legitimate because

$$
(5.4)\qquad
\begin{cases}
\displaystyle\sum_{m,n=0}^\infty\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+1/2+\rho}\int_0^1|rtL'(rt)t^{m+n+\rho-1/2}|\,dt<+\infty\,,\\
\displaystyle\sum_{m,n=0}^\infty\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+3/2+\rho}\int_0^1|rtL'(rt)t^{m+n+\rho+1/2}|\,dt<+\infty\,.
\end{cases}
$$

Here, we prove the first estimate of (5.4). Since $h(t)$ is continuous in $[0, \infty)$ and $h(t)\to 0$ as $t\to\infty$, there exists an $M>0$ such that

$$(0<)h(t)\leqq M \qquad (t\geqq 0). \tag{5.5}$$

The positivity of $h(t)$ implies that $L(r)$ is increasing. Hence

$$0<rtL'(rt)=\delta h(rt)L(rt)<\delta ML(r). \tag{5.6}$$

From (5.5), (5.6), and (3.2)-(3.4) it follows that

$$\begin{aligned}
&\sum_{m,n=1}^{\infty}\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+1/2+\rho}\int_0^1|rtL'(rt)t^{m+n+\rho-1/2}|\,dt\\
&\leqq\delta ML(r)\sum_{m,n=1}^{\infty}\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{(m+n+1/2+\rho)^2}\\
&=\delta ML(r)\{\sum_{m\geqq n\geqq 1}+\sum_{n>m\geqq 1}\}\\
&\leqq\delta ML(r)\Big\{\sum_{m\geqq n\geqq 1}\frac{1}{(m+n+1/2+\rho)^2}\frac{2}{\sqrt{m\pi}\sin\theta}\\
&\quad+\sum_{n>m\geqq 1}\frac{1}{(m+n+1/2+\rho)^2}\frac{2}{\sqrt{n\pi}\sin\theta}\Big\}\\
&\leqq\delta ML(r)\frac{2}{\sqrt{\pi}\sin\theta}\Big\{\sum_{m=1}^{\infty}\frac{m}{(m+3/2+\rho)^2}\frac{1}{\sqrt{m}}\\
&\quad+\sum_{n=2}^{\infty}\frac{n-1}{(n+3/2+\rho)^2}\frac{1}{\sqrt{n}}\Big\}\\
&\leqq\frac{2\delta M}{\sqrt{\pi}\sin\theta}L(r)\Big\{\sum_{m=1}^{\infty}m^{-3/2}+\sum_{n=1}^{\infty}n^{-3/2}\Big\}<+\infty.
\end{aligned}$$

Also

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{|P_m(-\cos\theta)|}{m+1/2+\rho}\int_0^1|rtL'(rt)t^{m+\rho-1/2}|\,dt\\
&\qquad\leqq\delta ML(r)\sum_{m=1}^{\infty}\frac{1}{(m+1/2+\rho)^2}<+\infty.
\end{aligned}$$

Therefore, the first estimate of (5.4) holds.

For $B_\theta(r)$, we have

$$\begin{aligned}
B_\theta(r)&=\int_r^\infty\frac{[rs^{\rho-1/2}+s^{\rho+1/2}][L(s)-L(r)]}{s^2}\sum_{m,n=0}^{\infty}P_m(-\cos\theta)P_n(-\cos\theta)\Big(\frac{r}{s}\Big)^{m+n}ds\\
&=\sum_{m,n=0}^{\infty}P_m(-\cos\theta)P_n(-\cos\theta)\Big[\int_r^\infty s^{\rho-5/2-(m+n)}r^{1+m+n}[L(s)-L(r)]ds\\
&\quad+\int_r^\infty s^{\rho-3/2-(m+n)}r^{m+n}[L(s)-L(r)]ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m,n=0}^{\infty} p_m(-\cos\theta)P_n(-\cos\theta)\Big[\frac{1}{m+n+3/2-\rho} r^{1+m+n}\int_r^{\infty} s^{\rho-5/2-(m+n)} sL'(s)ds \\
&\quad + \frac{1}{m+n+1/2-\rho} r^{m+n}\int_r^{\infty} s^{\rho-3/2-(m+n)} sL'(s)ds\Big] \qquad (5.7)\\
&= r^{\rho-1/2}\sum_{m,n=0}^{\infty} P_m(-\cos\theta)P_n(-\cos\theta)\Big[\frac{1}{m+n+3/2-\rho}\int_1^{\infty} t^{\rho-5/2-(m+n)} rtL'(rt)dt \\
&\quad + \frac{1}{m+n+1/2-\rho}\int_1^{\infty} t^{\rho-3/2-(m+n)} rtL'(rt)dt\Big] \\
&= r^{\rho-1/2}\int_1^{\infty} rtL'(rt)\sum_{m,n=0}^{\infty}\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+3/2-\rho} t^{\rho-5/2-(m+n)}dt \\
&\quad + r^{\rho-1/2}\int_1^{\infty} rtL'(rt)\sum_{m,n=0}^{\infty}\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+1/2-\rho} t^{\rho-3/2-(m+n)}dt\,.
\end{aligned}
$$

In order to prove

$$
(5.8)\qquad \begin{cases} \displaystyle\sum_{m,n=0}^{\infty}\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+3/2-\rho}\int_1^{\infty}|rtL'(rt)t^{\rho-5/2-(m+n)}|\,dt<+\infty\,, \\ \displaystyle\sum_{m,n=0}^{\infty}\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+1/2-\rho}\int_1^{\infty}|rtL'(rt)t^{\rho-3/2-(m+n)}|\,dt<+\infty\,, \end{cases}
$$

we may use (5.5), (3.2)-(3.4), and the fact that

$$
\frac{L(rt)}{L(r)} = \exp\Big[\delta\int_r^{rt}\frac{h(t)}{t}dt\Big] < t^{\delta M}\,.
$$

(If we choose δ such that $\delta M<1/2-\rho$, (5.8) holds.) Substituting (5.3) and (5.7) into (5.2), we have

$$
\begin{aligned}
|I_2(r,\theta)| &= \frac{r^{\rho}\cos(\theta/2)}{\pi}\Big|\Big\{-\int_0^1 rtL'(rt)\sum_{m,n=0}^{\infty}\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+1/2+\rho}t^{m+n+\rho-1/2}dt \\
&\quad -\int_0^1 rtL'(rt)\sum_{m,n=0}^{\infty}\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+3/2+\rho}t^{m+n+\rho+1/2}dt \\
&\quad +\int_1^{\infty} rtL'(rt)\sum_{m,n=0}^{\infty}\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+3/2-\rho}t^{\rho-5/2-(m+n)}dt \\
&\quad +\int_1^{\infty} rtL'(rt)\sum_{m,n=0}^{\infty}\frac{P_m(-\cos\theta)P_n(-\cos\theta)}{m+n+1/2-\rho}t^{\rho-3/2-(m+n)}dt\Big\}\Big| \\
(5.9)\qquad &< \frac{r^{\rho}\cos(\theta/2)}{\pi}\Big\{\int_0^1 rtL'(rt)\sum_{m,n=0}^{\infty}\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+1/2+\rho}t^{m+n+\rho-1/2}dt \\
&\quad +\int_0^1 rtL'(rt)\sum_{m,n=0}^{\infty}\frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+3/2+\rho}t^{m+n+\rho+1/2}dt
\end{aligned}
$$

$$+\int_1^\infty rtL'(rt)\sum_{m,n=0}^\infty \frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+3/2-\rho}t^{\rho-5/2-(m+n)}dt$$

$$+\int_1^\infty rtL'(rt)\sum_{m,n=0}^\infty \frac{|P_m(-\cos\theta)P_n(-\cos\theta)|}{m+n+1/2-\rho}t^{\rho-3/2-(m+n)}dt\Big\}$$

$$\equiv \frac{r^\rho\cos(\theta/2)}{\pi}\{J_1+J_2+J_3+J_4\},\ say.$$

Further,

$$J_1(r,\theta)<\int_0^1 rtL'(rt)\sum_{m\geqq n\geqq 1}\frac{2t^{m+n+\rho-1/2}}{(m+n+1/2+\rho)\sqrt{m\pi}\sin\theta}dt$$

$$+\int_0^1 rtL'(rt)\sum_{n>m\geqq 1}\frac{2t^{m+n+\rho-1/2}}{(m+n+1/2+\rho)\sqrt{n\pi}\sin\theta}dt$$

$$+\int_0^1 rtL'(rt)\sum_{m=0}^\infty\frac{2t^{m+\rho-1/2}}{m+1/2+\rho}dt,\quad \text{etc.}$$

Here, we use a result of Aljančić, Bojanić, and Tomić [1, p 82] to obtain

$$J_1(r,\theta)\leqq\frac{2}{\sqrt{\pi\sin\theta}}(1+o(1))\delta h(r)L(r)\Big\{\int_0^1\sum_{m\geqq n\geqq 1}\frac{t^{m+n+\rho-1/2}}{(m+n+1/2+\rho)\sqrt{m}}dt$$

$$+\int_0^1\sum_{n>m\geqq 1}\frac{t^{m+n+\rho-1/2}}{(m+n+1/2+\rho)\sqrt{n}}dt+\sqrt{\pi}\int_0^1\sum_{m=0}^\infty\frac{t^{m+\rho-1/2}}{m+1/2+\rho}dt\Big\},\quad \text{etc.}$$

where the $o(1)$ tends to zero uniformly as $r\to\infty$ in $\theta\in(0,\pi)$. Hence by (5.9)

$$|I_2(r,\theta)|\leqq\frac{\delta h(r)\Lambda(r)}{\sqrt{\sin\theta}}\cos\Big(\frac{\theta}{2}\Big)\cdot C(\rho)\qquad(0<\theta<\pi,\ r\geqq 0),\tag{5.10}$$

where $C(\rho)$ is a positive constant depending only on ρ.

1.6. The final proof.

Let $f_1(z)$ be an entire function of order ρ $(0<\rho<1/2)$ and mean type. Then as we have shown in 1.4, $H(z)$ defined by (4.4) satisfies (2.4). By (5.10) and (4.5)

$$H(re^{i\theta})\geqq\Big\{\frac{\cos\theta\rho}{\cos\pi\rho}-\delta h(r)C(\rho)\frac{\cos(\theta/2)}{\sqrt{\sin\theta}}\Big\}H(-r)\qquad(0<\theta<\pi,\ r\geqq 0).\tag{6.1}$$

Since $h(r)\to 0$ as $r\to\infty$, for given $\eta>0$ (small) there exists a $R_0\equiv R_0(\eta)$ such that $r\geqq R_0$, $\eta\leqq\theta\leqq\pi-\eta$ imply

$$g(r,\theta)\equiv\frac{\cos\theta\rho}{\cos\pi\rho}-\delta h(r)C(\rho)\frac{\cos(\theta/2)}{\sqrt{\sin\theta}}\geqq 1.$$

Hence

$$H(re^{i\theta})\geqq H(-r)\qquad(\eta\leqq|\theta|\leqq\pi-\eta,\ r\geqq R_0).\tag{6.2}$$

Net, we consider $g(r,\theta)$ for $\pi-\eta<\theta<\pi$. Put $\theta=\pi-\xi$ $(0<\xi<\eta)$. Then

$$\begin{aligned} g(r,\ \theta)\equiv G(r,\ \xi)&=\frac{\cos(\pi-\xi)\rho}{\cos\pi\rho}-\delta h(r)C(\rho)\sqrt{\frac{\cos((\pi-\xi)/2)}{2\sin((\pi-\xi)/2)}} \\ &=\cos\rho\xi+\tan\pi\rho\sin\rho\xi-\delta h(r)C(\rho)\sqrt{\frac{\sin(\xi/2)}{2\cos(\xi/2)}} \\ &\geqq 1-\frac{(\rho\xi)^2}{2}+\tan\pi\rho\Big(\rho\xi-\frac{(\rho\xi)^3}{6}\Big)-\delta h(r)C(\rho)\sqrt{\frac{\xi/2}{2(1-\xi^2/8)}} \\ &>1+(\rho\tan\pi\rho)\xi-\Big(\frac{\rho^2}{2}\xi^2+\frac{\rho^3\tan\pi\rho}{6}\xi^3\Big)-\delta h(r)C(\rho)\sqrt{\xi}. \end{aligned}$$

If we choose $\eta\equiv\eta(\rho)>0$ small enough, we have

$$g(r,\ \theta)>1+\frac{\rho\tan\pi\rho}{2}\xi-\delta h(r)C(\rho)\sqrt{\xi} \qquad (0<\xi\leqq\eta(\rho)),$$

so that

$$g(r,\ \theta)\geqq 1 \qquad \Big(\xi\geqq\frac{4\delta^2h(r)^2C(\rho)^2}{\rho^2\tan^2\pi\rho}\Big). \tag{6.3}$$

From (6.1) and (6.3), it follows that

$$H(re^{i\theta})\geqq H(-r) \qquad \Big(\eta(\rho)\leqq|\theta|\leqq\pi-\frac{4\delta^2h(r)^2C(\rho)^2}{\rho^2\tan^2\pi\rho}\Big). \tag{6.4}$$

It remains to consider $H(re^{i\theta})$ for $|\theta|\leqq\eta$. By (4.4)

$$H(re^{i\theta})\geqq\cos\frac{\eta}{2}H(r) \qquad (|\theta|\leqq\eta). \tag{6.5}$$

An estimate for $H(r)$ has been done by Barry [4, pp 53, 54], which gives

$$H(r)\geqq\Big\{\frac{1}{\cos\pi\rho}-\delta h(r)C_1(\rho)\Big\}A(r) \qquad (r\geqq 0,\ C_1(\rho)>0). \tag{6.6}$$

Combining (6.5) with (6.6), we have

$$H(re^{i\theta})\geqq\cos\Big(\frac{\eta}{2}\Big)\Big(\frac{1}{\cos\pi\rho}-\delta h(r)C_1(\rho)\Big)H(-r)\geqq H(-r) \tag{6.7}$$
$$(\eta<2\pi\rho,\ r\geqq R_1=R_1(\eta)).$$

In view of (6.2), (6.4) and (6.7), we have

$$H(re^{i\theta})\geqq H(-r) \qquad \Big(|\theta|\leqq\pi-\frac{4\delta^2h(r)^2C(\rho)^2}{\rho^2\tan^2\pi\rho},\ r\geqq R_2>0\Big). \tag{6.8}$$

Now, we use (2.4). Taking (6.8) into consideration, we deduce that

$$0<\frac{\log|f_1(re^{i\theta})|}{\log|f_1(-r)|}\leqq\frac{H(re^{i\theta})}{H(-r)} \qquad \Big(r=r_n\uparrow\infty,\ |\theta|\leqq\pi-\frac{4\delta^2h(r)^2C(\rho)^2}{\rho^2\tan^2\pi\rho}\Big). \tag{6.9}$$

Hence by (5.10) and (6.9)

$$\frac{1}{\log|f_1(-r)|}\frac{1}{2\pi}\int_{-\pi+4\delta^2h(r)^2C(\rho)^2/\rho^2\tan^2\pi\rho}^{\pi-4\delta^2h(r)^2C(\rho)^2/\rho^2\tan^2\pi\rho}\log|f_1(re^{i\theta})|\,d\theta$$

$$\leqq\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left\{\frac{\cos\theta\rho}{\cos\pi\rho}+\delta h(r)C(\rho)\frac{\cos(\theta/2)}{\sqrt{\sin\theta}}\right\}d\theta$$

$$<\frac{\tan\pi\rho}{\pi\rho}+\sqrt{2}\,\delta h(r)C(\rho)\qquad(r=r_n\uparrow\infty).$$

Since $\log|f_1(re^{i\theta})|$ is decreasing for $|\theta|$ $(\leqq\pi)$, we have

$$\frac{1}{\log|f_1(-r)|}\frac{1}{2\pi}\int_{-\pi}^{+\pi}\log|f_1(re^{i\theta})|\,d\theta$$

$$<\left\{\frac{\tan\pi\rho}{\pi\rho}+\sqrt{2}\,\delta h(r)C(\rho)\right\}\left\{1+\frac{4\delta^2h(r)^2C(\rho)^2/\rho^2\tan^2\pi\rho}{\pi-4\delta^2h(r)^2C(\rho)^2/\rho^2\tan^2\pi\rho}\right\}\ (r=r_n\uparrow\infty).$$

Therefore

$$\frac{\log m^*(r,f_1)}{m(r,f_1)}\geqq\frac{1-4\delta^2h(r)^2C(\rho)^2/\rho^2\tan^2\pi\rho}{\tan\pi\rho/\pi\rho+\sqrt{2}\,\delta h(r)C(\rho)}\ (r=r_n\uparrow\infty)$$

$$>\pi\rho\cot\pi\rho\left\{1-\frac{\pi\rho\sqrt{2}\,\delta}{\tan\pi\rho}C(\rho)h(r)\right\}\left\{1-\frac{4\delta^2h(r)^2C(\rho)^2}{\rho^2\tan^2\pi\rho}\right\}$$

$$>\pi\rho\cot\pi\rho(1-h(r))\qquad(r=r_n\uparrow\infty),$$

if $\delta>0$ is sufficiently small. This completes the proof of Theorem 1.

1.7. A complementary note.

In Theorem 1, the assumption

$$\int^{\infty}\frac{h(t)}{t}\,dt=\infty$$

is essential. In this section, we prove the following result.

"Let $h(r)$ be positive and continuous for $r\geqq r_0$ and, for each $s>0$,

$$\frac{h(sr)}{h(r)}\longrightarrow 1\qquad(r\to\infty).$$

Suppose that $h(r)\to 0$ $(r\to\infty)$ and that

$$\int^{\infty}\frac{h(t)}{t}\,dt<\infty. \tag{7.1}$$

Then, there exists an entire function $f(z)$ of order ρ $(0<\rho<1/2)$ and mean type for which

$$\log m^*(r,f)<\pi\rho\cot\pi\rho(1-h(r))\cdot m(r,f)\qquad(r\geqq r_1)."$$

Proof. We refer to Barry's argument in [4, pp 55-58]. Define $L(r)$ by (4.1).

Let $f(z)$ be an entire function of genus zero, all of whose zeros are negative and such that $f(0)=1$ and $n(r, 0, f)=[r^\rho L(r)]$. Then

$$\begin{aligned}\log m^*(r, f)<r^\rho L(r)\Big[\pi\cos\pi\rho\cdot\operatorname{cosec}\pi\rho-\delta h(r)\sum_{n=0}^{\infty}\{(n+\rho)^{-2}\\ +(n+1-\rho)^{-2}\}+O\{\log r/r^\rho L(r)\}+o(h(r))\Big]\qquad(r\to\infty),\end{aligned}\tag{7.2}$$

and

$$\log M(r, f)\sim r^\rho L(r)\cdot\pi\operatorname{cosec}\pi\rho\qquad(r\to\infty).\tag{7.3}$$

By (4.1), (7.1) and (7.3), $f(z)$ is of order ρ and mean type. Now, we estimate $N(r, 0, f)$. Evidently,

$$\begin{aligned}N(r, 0, f)&=\int_0^r\frac{[t^\rho L(t)]}{t}dt\\ &=L(r)\int_0^r t^{\rho-1}dt+\int_0^r t^{\rho-1}\{L(t)-L(r)\}\,dt+K_1,\end{aligned}$$

where

$$|K_1|\leqq\int_1^r\frac{dt}{t}=\log r.$$

Also

$$\begin{aligned}\int_0^r t^{\rho-1}\{L(t)-L(r)\}\,dt&=\Big[\frac{t^\rho}{\rho}\{L(t)-L(r)\}\Big]_0^r\\ &\quad-\frac{1}{\rho}\int_0^r t^\rho L'(t)dt=-\frac{\delta}{\rho}\int_0^r h(t)L(t)t^{\rho-1}dt\\ &\sim-\frac{\delta}{\rho^2}h(r)L(r)r^\rho\qquad(r\to\infty),\end{aligned}$$

since $h(r)L(r)$ is slowly varying. Hence

$$N(r, 0, f)>\frac{r^\rho L(r)}{\rho}\Big[1-\frac{\delta(1+o(1))}{\rho}h(r)-O\{\log r/r^\rho L(r)\}\Big]\qquad(r\to\infty).\tag{7.4}$$

It is well known that

$$\sum_{n=0}^{\infty}\{(n+\rho)^{-2}+(n+1-\rho)^{-2}\}=\sum_{n=-\infty}^{+\infty}(\rho-n)^{-2}=\frac{\pi^2}{\sin^2\pi\rho}.\tag{7.5}$$

It follows from (7.2), (7.4) and (7.5) that

$$\frac{\log m^*(r, f)}{m(r, f)}\leqq\frac{\log m^*(r, f)}{N(r, 0, f)}<\frac{\rho\begin{Bmatrix}\pi\cot\pi\rho-\delta h(r)\pi^2/\sin^2\pi\rho\\ +o(h(r))+O(\log r/r^\rho L(r))\end{Bmatrix}}{1-(\delta(1+o(1))/\rho)h(r)-O(\log r/r^\rho L(r))}$$

$$< \frac{\rho\{\cot\pi\rho-\delta(1-o(1))(\pi^2/\sin^2\pi\rho)h(r)\}}{1-(\delta(1+o(1))/\rho)h(r)}$$

$$=\pi\rho\cot\pi\rho\left\{\frac{1-\delta(1-o(1))\pi(\sin\pi\rho\cos\pi\rho)^{-1}h(r)}{1-\delta(1+o(1))\rho^{-1}h(r)}\right\}$$

$$<\pi\rho\cot\pi\rho\left\{1-\left[\frac{\delta(1-o(1))2\pi\rho(\sin 2\pi\rho)^{-1}-\delta(1+o(1))}{1-\delta(1+o(1))\rho^{-1}h(r)}\right]\frac{1}{\rho}h(r)\right\}$$

$$<\pi\rho\cot\pi\rho\left\{1-\delta\left(\frac{2\pi\rho}{\sin 2\pi\rho}-1\right)\frac{1}{\rho}\frac{h(r)}{2}\right\} \qquad (r\to\infty)$$

$$<\pi\rho\cot\pi\rho\{1-h(r)\},$$

if we choose $\delta>2\rho(2\pi\rho/\sin 2\pi\rho-1)^{-1}$. This completes the proof.

The method of this section can be used also when we prove the following results.

(i) Let $h(r)$ be given as in Theorem 1. Then there exists an entire function $f(z)$ of order ρ $(0<\rho<1/2)$ and minimal type for which

$$\log m^*(r, f)<\pi\rho\cot\pi\rho\cdot(1+h(r))\cdot m(r, f) \qquad (r\geqq r_1).$$

Compare this with the estimate (4).

(ii) Let $h(r)$ be given as in Theorem 1. Then there exists an entire function $f(z)$ of order ρ $(0<\rho<1/2)$ and maximal type for which

$$\log m^*(r, f)<\pi\rho\cot\pi\rho\cdot(1-h(r))\cdot m(r, f) \qquad (r\geqq r_1).$$

This shows that the conclusion of Theorem 1 does not hold in general for entire functions of order ρ $(0<\rho<1/2)$ and maximal type.

2. Proof of Theorem 2.

Given $f(z)$, we associate $f_1(z)$ as (1.2). Then Miles and Shea proved in [8] that

$$m_2(r, f_1)\geqq m_2(r, f). \tag{8.1}$$

Since $f(z)$ is of order ρ $(<1/2)$ and minimal type, (1.4) and (1.5) imply that $f_1(z)$ is also of order ρ and minimal type. By (1.3) and (8.1),

$$\frac{\log m^*(r, f)}{m_2(r, f)}\geqq\frac{\log m(r, f_1)}{m_2(r, f_1)},$$

provided that $\log m^*(r, f_1)\geqq 0$. Hence we may prove Theorem 2 for $f=f_1$.

Now, define $H(re^{i\theta})$ by (4.4) with $\Lambda(r)=r^\rho$. A simple computation gives

$$H(re^{i\theta})=\frac{\cos\theta\rho}{\cos\pi\rho}r^\rho \qquad (r\geqq 0,\ |\theta|\leqq\pi). \tag{8.2}$$

Hence $H(re^{i\theta})$ satisfies (2.1)-(2.3), so that (2.4) holds. By (8.2), $H(re^{i\theta})>H(-r)$ $(|\theta|<\pi)$. It follows from this and (2.4) that

$$0<\frac{\log|f_1(re^{i\theta})|}{\log|f_1(-r)|}<\frac{H(re^{i\theta})}{H(-r)}=\frac{\cos\theta\rho}{\cos\pi\rho}\quad(|\theta|<\pi),$$

for a sequence of $r=r_n\uparrow\infty$. Therefore

$$\frac{m_2(r_n, f_1)}{\log m^*(r_n, f_1)}<\frac{1}{\cos\pi\rho}\left\{\frac{1}{\pi}\int_0^\pi\cos^2\theta\rho\,d\theta\right\}^{1/2}$$

$$=\frac{1}{\cos\pi\rho}\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}\,.$$

This proves Theorem 2.

3. Proof of Theorem 3.

By a similar argument as in the proof of Theorem 2, we may prove Theorem 3 for $f=f_1$. Define $H(re^{i\theta})$ by (4.4). For our proof, the estimate (5.10) is not suitable because

$$\int_0^\pi\frac{\cos^2(\theta/2)}{\sin\theta}d\theta=\infty\,.$$

However, we can obtain the estimate

$$|I_2(r,\theta)|\leqq\delta h(r)A(r)\cos\left(\frac{\theta}{2}\right)C(\rho)\qquad(|\theta|\leqq\pi/2,\ r\geqq 0)\tag{9.1}$$

instead of (5.10). To prove this we may note that in (5.2)

$$I_2(r,\theta)<\frac{r^{1/2}}{\pi}\cos\left(\frac{\theta}{2}\right)B_\theta(r)$$

$$<\frac{r^{1/2}}{\pi}\cos\left(\frac{\theta}{2}\right)B_{\pi/2}(r)\qquad(|\theta|<\pi/2)\,.$$

In view of (6.9), (5.10) and (9.1), we have for $r=r_n\uparrow\infty$,

$$\frac{1}{(\log|f_1(-r)|)^2}\frac{1}{2\pi}\int_{-\pi+4\delta^2h^2(r)C(\rho)^2/\rho^2\tan^2\pi\rho}^{\pi-4\delta^2h^2(r)C(\rho)^2/\rho^2\tan^2\pi\rho}\{\log|f_1(re^{i\theta})|\}^2d\theta$$

$$<\frac{1}{\pi}\int_0^{\pi/2}\left\{\frac{\cos\theta\rho}{\cos\pi\rho}+\delta h(r)\cos\frac{\theta}{2}C(\rho)\right\}^2d\theta$$

$$+\frac{1}{\pi}\int_{\pi/2}^{\pi}\left\{\frac{\cos\theta\rho}{\cos\pi\rho}+\delta h(r)\cos\frac{\theta}{2}C(\rho)\frac{1}{\sqrt{\sin\theta}}\right\}^2d\theta$$

$$<\frac{1}{\cos^2\pi\rho}\left[\frac{1}{2}+\frac{\sin 2\pi\rho}{4\pi\rho}\right]+\frac{3\delta h(r)C(\rho)}{\cos\pi\rho}+\left(\frac{1}{2}+\frac{\pi}{16}\right)\delta^2h^2(r)C(\rho)^2$$

Since $\{\log|f_1(r_ne^{i\theta})|\}^2$ is decreasing for $|\theta|(\leqq\pi)$, we have

$$\frac{m_2^2(r, f_1)}{(\log m^*(r, f_1))^2} < \left\{\frac{1}{\cos^2 \pi\rho}\left[\frac{1}{2}+\frac{\sin 2\pi\rho}{4\pi\rho}\right]+\frac{3\delta h(r)C(\rho)}{\cos \pi\rho}+\left(\frac{1}{2}+\frac{\pi}{16}\right)\delta^2 h^2(r)C(\rho)^2\right\}$$

$$\times\left\{1+\frac{4\delta^2 h(r)^2 C(\rho)^2/\rho^2 \tan^2 \pi\rho}{\pi-4\delta^2 h(r)^2 C(\rho)^2/\rho^2 \tan^2 \pi\rho}\right\} \quad (r=r_n).$$

Thus

$$\left\{\frac{\log m^*(r, f_1)}{m_2(r, f_1)}\right\}^2 > (A(\rho))^2\left\{1-\frac{3\delta h(r)C(\rho)}{\cos \pi\rho}A(\rho)^2-\left(\frac{1}{2}+\frac{\pi}{16}\right)\delta^2 h(r)^2 C(\rho)^2 A(\rho)^2\right\}$$

$$\times\left\{1-\frac{4\delta^2 h(r)^2 C(\rho)^2}{\rho^2 \tan^2 \pi\rho}\right\}$$

$$> (A(\rho))^2\left\{1-\frac{2A(\rho)\sqrt{\delta C(\rho)}}{\sqrt{\cos \pi\rho}}h(r)\right\}^2$$

$$> (A(\rho))^2(1-h(r))^2 \qquad (r=r_n \uparrow \infty),$$

if $\delta>0$ is sufficiently small. Since $\log m^*(r_n, f_1)>0$, we obtain

$$\frac{\log m^*(r, f_1)}{m_2(r, f_1)} > A(\rho)(1-h(r)) \qquad (r=r_n \uparrow \infty).$$

This proves Theorem 3.

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