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ON THE GROWTH OF SUBHARMONIC FUNCTIONS

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1. Introduction. Let $u(z)$ be a subharmonic function in the complex plane $\boldsymbol{C}$. We denote the order and lower order of $u(z)$ by ρ and μ, respectively. Let $M(r, u)$ and $m^*(r, u)$ denote the maximum and infimum of $u(z)$ on $|z|=r$, respectively. The classical $\cos \pi\rho$ theorem asserts that, given $\varepsilon>0$, the inequality

$$m^*(r, u)>(\cos \pi\rho-\varepsilon)M(r, u) \tag{1}$$

holds for a sequence $r=r_n\to\infty$, provided that $\rho<1$. Kjellberg [3] proved a striking improvement of this theorem.

THEOREM A. *If $\lambda\in(0, 1)$, then*

$$m^*(r, u)>\cos \pi\lambda\cdot M(r, u)$$

on an unbounded sequence of r, unless

$$r^{-1}M(r, u)\longrightarrow\alpha \qquad (r\to\infty),$$

where α is positive or ∞.

An important consequence of Theorem A is that if $\mu<1$ then the inequality (1) holds with ρ replaced by μ on an unbounded sequence of r. Another important consequence of Theorem A is the following fact.

"If $u(z)$ is subharmonic of order ρ $(0<\rho<1)$ and minimal type, then

$$m^*(r, u)>\cos \pi\rho\, M(r, u)$$

on a sequence of $r\to\infty$."

Such a fact does not always hold for subharmonic functions of order ρ and mean type. Barry [2] proved the following result.

THEOREM B. *Let $h(r)$ be positive and continuous for $r\geqq r_0$, and for each $s>0$,*

$$\frac{h(sr)}{h(r)}\longrightarrow 1 \qquad (r\to\infty).$$

Suppose that $h(r)\to 0$ $(r\to\infty)$ and $h'(r)>-O(r^{-1})$ $(r\to\infty)$. If $u(z)$ is subharmonic of order ρ $(0<\rho<1/2)$ and mean type, and

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$$\int^{\infty} h(t)\frac{dt}{t}=\infty ,$$

then

$$m^*(r, u)>\cos \pi\rho\{1-h(r)\}M(r, u)$$

on a sequence of $r\to\infty$.

If

$$\int^{\infty} h(t)\frac{dt}{t}<\infty ,$$

there is a subharmonic function of order ρ $(0<\rho<1)$ *and mean type for which*

$$m^*(r, u)<\{\cos \pi\rho-h(r)\}M(r, u) \qquad (r\geqq r_0) .$$

Baernstein [1] generalized Theorem A as follows.

THEOREM C. *Let* $u(z)$ *be a nonconstant subharmonic function in* $\boldsymbol{C}$. *Let* β *and* λ *be numbers with* $0<\lambda<\infty$, $0<\beta\leqq\pi$, $\beta\lambda<\pi$. *Then either* (*a*) *there exist arbitrarily large values of* r *for which the set of* θ *such that* $u(re^{i\theta})>\cos\beta\lambda\cdot M(r, u)$ *contains an interval of length at least* 2β, *or else* (*b*) $\lim_{r\to\infty} r^{-\lambda}M(r, u)$ *exists, and is positive or* ∞.

For $\beta=\pi$, this is Theorem A.
In this note we shall prove the following result.

THEOREM. *Let* $u(z)$ *be subharmonic of order* ρ $(0<\rho<\infty)$ *and mean type in* $\boldsymbol{C}$. *Let* β *be a number satisfying* $0<\beta\leqq\pi$ *and* $\beta\rho<\pi/2$. *Suppose that* $h(r)$ *is positive and continuous for* $r\geqq r_0$ *and, for each* $s>0$,

$$\frac{h(sr)}{h(r)} \longrightarrow 1 \qquad (r\to\infty) .$$

Further assume that $h(r)\to 0$ $(r\to\infty)$, $h'(r)>-O(r^{-1})$ $(r\to\infty)$ *and*

$$\int^{\infty} h(t)\frac{dt}{t}=\infty .$$

Then there exist arbitrarily large values of r *for which the set of* θ *such that* $u(re^{i\theta})>\cos\beta\rho\{1-h(r)\}M(r, u)$ *contains an interval of length at least* 2β.

For $\beta=\pi$, this is the first half of Theorem B.

2. Proof of Theorem. Since we are interested in results valid for large values of r, we may assume without loss of generality that $u(z)$ is harmonic in a neighborhood of $z=0$. Let β be a number satisfying $0<\beta\leqq\pi$. Put

$$u(r, \beta, \phi)=\int_{-\beta}^{+\beta} u(re^{i(\omega+\phi)})d\omega \qquad (r>0,\ \phi: \text{real}) .$$

For $re^{i\beta}$ fixed, $u(r, \beta, \phi)$ is a continuous (periodic) function of ϕ (cf. [4, Lemma 3]). Therefore each fixed $re^{i\beta}$, there exists a ϕ_0 $(-\pi \leqq \phi_0 < \pi)$ satisfying

$$N(r, \beta, u) \equiv \sup_{\phi} u(r, \beta, \phi) = u(r, \beta, \phi_0).$$

Here we set

$$\mu(r, u) = \inf\{u(re^{i\omega}); \omega \in [\phi_0 - \beta, \phi_0 + \beta]\}.$$

In order to prove our theorem, it is sufficient to show that

$$\mu(r, u) > \cos \beta\rho \{1 - h(r)\} M(r, u) \tag{2}$$

for a sequence $r = r_n \to \infty$.

If $v(z) = u(z) - u(0)$, then

$$\mu(r, u) = \mu(r, v) + u(0), \qquad M(r, u) = M(r, v) + u(0).$$

By Theorem C, we may assume that

$$\lim_{r\to\infty} r^{-\rho} M(r, u) = \alpha \qquad (\alpha;\ \text{a positive constant}).$$

Now assume that our assertion is proved for $v(z)$, that is,

$$\mu(r, v) > \cos \beta\rho \{1 - (h(r)/2)\} M(r, v)$$

for a sequence of $r = r_n \to \infty$. Then for $r = r_n$

$$\begin{aligned}
\mu(r, u) &> \cos \beta\rho \{1 - (h(r)/2)\} M(r, u) + u(0)[1 - \cos \beta\rho \{1 - (h(r)/2)\}] \\
&> \cos \beta\rho \{1 - (h(r)/2)\} M(r, u) - |u(0)| \qquad (n \geqq n_0) \\
&> \cos \beta\rho \{1 - (h(r)/2) - O(r^{-\rho})\} M(r, u) \\
&> \cos \beta\rho \{1 - h(r)\} M(r, u) \qquad (n \geqq n_1),
\end{aligned}$$

since $h(r)$ is slowly varying. Thus we may assume that $u(0) = 0$.

Set

$$B(t) = N(t^{\gamma}, \beta, u),$$

where $\gamma = \beta/\pi$. Since $u(z)$ is of order ρ and mean type, we have

$$B(t) \leqq 2\beta M(t^{\gamma}, u) = O(t^{\gamma\rho}) \qquad (t \to \infty). \tag{3}$$

Since $\gamma\rho = \beta\rho/\pi < 1/2$, the Poisson integral

$$b(re^{i\theta}) = \frac{1}{\pi}\int_0^{\infty} B(t) \frac{r \sin \theta}{t^2 + r^2 + 2tr \cos \theta}\, dt \tag{4}$$

is harmonic in the slit plane $|\arg z| < \pi$, is zero on the positive axis and tends to $B(r)$ as $\theta \to \pi-$. By Proposition 1 in [1], $B(t)$ is a nondecreasing convex function of $\log t$ $(0 < t < \infty)$. Differentiating (4) with respect to θ, we have

$$(5)\qquad b_\theta(re^{i\theta})=\frac{1}{\pi}\int_0^\infty \log\left|1+\frac{re^{i\theta}}{t}\right| dB_1(t)\qquad (|\theta|<\pi),$$

$$(6)\qquad b_\theta(-r)\equiv\lim_{\theta\to\pi-}\frac{B(r)-b(re^{i\theta})}{\pi-\theta}=\lim_{\theta\to\pi-} b_\theta(re^{i\theta})=\frac{1}{\pi}\int_0^\infty \log\left|1-\frac{r}{t}\right| dB_1(t),$$

where $B_1(t)$ denotes the logarithmic derivative of $B(t)$. Since $B(t)$ is a nondecreasing convex function of $\log t$, $B_1(t)$ exists a.e., and is a nonnegative nondecreasing function of t. It follows from (5) and (6) that $b_\theta(z)$ is subharmonic in $\boldsymbol{C}$ and that

$$(7)\qquad m^*(r, b_\theta)=b_\theta(-r), \qquad M(r, b_\theta)=b_\theta(r).$$

Using (3) and the fact that $B(0)=0$, we easily see

$$B_1(t)=O(t^{\gamma\rho})\qquad (t\to\infty),$$

$$\lim_{t\to 0}\left(\log\frac{1}{t}\right)B_1(t)=0.$$

Hence

$$b_\theta(r)=\frac{1}{\pi}\int_0^\infty \log\left(1+\frac{r}{t}\right)dB_1(t)$$

$$=\frac{1}{\pi}\left[\log\left(1+\frac{r}{t}\right)B_1(t)\right]_0^\infty-\frac{1}{\pi}\int_0^\infty\frac{-\frac{r}{t^2}}{1+\frac{r}{t}}B_1(t)dt$$

$$(8)\qquad =\frac{r}{\pi}\int_0^\infty\frac{1}{t+r}\frac{B_1(t)}{t}dt=\frac{r}{\pi}\int_0^\infty\frac{dB(t)}{t+r}$$

$$=\frac{r}{\pi}\left[\frac{B(t)}{t+r}\right]_0^\infty-\frac{r}{\pi}\int_0^\infty\frac{-B(t)}{(t+r)^2}dt$$

$$=\frac{r}{\pi}\int_0^\infty\frac{B(t)}{(t+r)^2}dt=O(r^{\gamma\rho})\qquad (r\to\infty).$$

In view of (7) and (8), we have

$$(9)\qquad M(r, b_\theta)=O(r^{\gamma\rho})\qquad (r\to\infty).$$

Define D by $D=\{z\,;\,0<\arg z<\beta\}$. Let $H(z)$ be the harmonic function in D defined by $H(z)=b(z^{1/\gamma})$. Taking the estimate (3) into consideration, Baernstein's reasoning in [1, pp 192-195] gives

$$(10)\qquad \begin{cases} H_\theta(r)\geqq 2M(r, u) \\ H_\theta(r)+H_\theta(re^{i\beta})\leqq 2[\mu(r, u)+M(r, u)] \qquad (0<r<\infty). \end{cases}$$

It follows from (10) and (7) that if $H_\theta(r)+H_\theta(re^{i\beta})\geqq 0$ then

$$\frac{\mu(r, u)}{M(r, u)} \geqq \frac{H_\theta(re^{i\beta})}{H_\theta(r)} = \frac{b_\theta(-r^{1/\gamma})}{b_\theta(r^{1/\gamma})} = \frac{m^*(r^{1/\gamma}, b_\theta)}{M(r^{1/\gamma}, b_\theta)} . \tag{11}$$

By (9) the order of $b_\theta(z)$ is less than or equal to $\gamma\rho(<1/2)$. Assume first that $b_\theta(z)$ has order less than $\gamma\rho$. The classical $\cos \pi\rho$ theorem gives the estimate

$$\frac{m^*(r, b_\theta)}{M(r, b_\theta)} > \cos \pi(\gamma\rho) = \cos \beta\rho \qquad (r = r_n \to \infty) . \tag{12}$$

Combining (11) and (12), we have

$$\frac{\mu(r, u)}{M(r, u)} > \cos \beta\rho \qquad (r = r_n^\gamma \to \infty) . \tag{13}$$

Assume next that $b_\theta(z)$ is of order $\gamma\rho$ and minimal type. In this case, we use Theorem A to obtain the estimate (12), so that (13) follows.

It remains to consider the case that $b_\theta(z)$ is of order $\gamma\rho$ and mean type. Define $h_1(t)$ by $h_1(t)=h(t^\gamma)$. Then $h_1(t)$ is positive and continuous for $t \geqq r_0^{1/\gamma}$, and for each $s>0$,

$$\frac{h_1(st)}{h_1(t)} = \frac{h(s^\gamma t^\gamma)}{h(t^\gamma)} \longrightarrow 1 \qquad (t \to \infty) .$$

Further $h_1(t) \to 0$ $(t \to \infty)$, $h_1'(t) = \gamma t^{\gamma-1} h'(t^\gamma) > -O(t^{-1})$ $(t \to \infty)$, and

$$\int_{r_0^{1/\gamma}}^\infty h_1(t) \frac{dt}{t} = \int_{r_0^{1/\gamma}}^\infty h(t^\gamma) \frac{dt}{t} = \frac{1}{\gamma} \int_{r_0}^\infty h(t) \frac{dt}{t} = \infty .$$

Hence by Theorem B, the inequality

$$\frac{m^*(r, b_\theta)}{M(r, b_\theta)} > \cos \pi(\gamma\rho)\{1-h_1(r)\} = \cos \beta\rho\{1-h_1(r)\} \tag{14}$$

holds on a sequence $r=r_n \to \infty$. Combining (14) and (11), we have

$$\frac{\mu(r, u)}{M(r, u)} > \cos \beta\rho\{1-h(r)\} \qquad (r = r_n^\gamma \to \infty) .$$

This completes the proof of our theorem.

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