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BOUND OF THE DEFICIENCIES OF ALGEBROID FUNCTIONS WITH NEGATIVE ZEROS

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1. Nevanlinna [3] showed that

$$\varlimsup_{r\to\infty}\frac{N(r,a)+N(r,b)}{T(r,f)}\geqq k(\lambda)>0$$

for any meromorphic function f of finite nonintegral order λ and any two values a, b.

Here k is a function of λ alone, and a problem of some forty year's standing is that of finding the exact value of $k(\lambda)$. Nevanlinna himself conjectured that the best possible choice of $k(\lambda)$ is

$$k(\lambda)=\begin{cases}\dfrac{|\sin\pi\lambda|}{q+|\sin\pi\lambda|} & (q\leqq\lambda\leqq q+1/2),\\[2ex] \dfrac{|\sin\pi\lambda|}{q+1} & (q+1/2<\lambda\leqq q+1),\end{cases}$$

where q is a nonnegative integer.

For the best known bounds on $k(\lambda)$ when $\lambda>1$, see the results of Edrei and Fuchs in [1].

Recently, Hellerstein and Williamson [2] have obtained a complete answer when they restricted themselves to the class of entire functions with negative zeros.

Their results is the following:

THEOREM A *Let $f(z)$ be an entire function of genus q, order λ and lower order μ, having only negative zeros. Then for any ρ satisfying*

$$\mu\leqq\rho\leqq\lambda$$

we have

$$\varlimsup_{r\to\infty}\frac{N(r,0)}{T(r,f)}\geqq\begin{cases}\dfrac{|\sin\pi\rho|}{q+|\sin\pi\rho|} & (q\leqq\rho\leqq q+1/2),\\[2ex] \dfrac{|\sin\pi\rho|}{q+1} & (q+1/2<\rho\leqq q+1).\end{cases}$$

These bounds are best possible.

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THEOREM B *Let $f(z)$ be an entire function of genus q, order λ and lower order μ, having only negative zeros. Then for any ρ satisfying*

$$\mu \leqq \rho \leqq \lambda$$

we have

$$\varliminf_{r\to\infty} \frac{N(r, 0)}{T(r, f)} \leqq \begin{cases} \dfrac{|\sin \pi\rho|}{q+|\sin \pi\rho|} & (q \leqq \rho \leqq q+1/2), \\ \dfrac{|\sin \pi\rho|}{q+1} & (q+1/2 < \rho \leqq q+1). \end{cases}$$

In particular, if either λ or μ is a positive integer,

$$\varliminf_{r\to\infty} \frac{N(r, 0)}{T(r, f)} = 0.$$

These bounds are best possible.

The purpose of this paper is to extend thses theorems to n-valued entire algebroid functions with negative zeros.

Let $f(z)$ be an n-valued transcendental entire algebroid function defined by an irreducible equation

$$f^n + A_1(z) f^{n-1} + \cdots + A_{n-1}(z) f + A_n(z) = 0, \tag{1.1}$$

where $A_1, \cdots, A_n$ are entire functions without common zeros.

To formulate our theorems, we define the genus q of an entire algebroid function $f(z)$, as follows:

Let

$$A_j(z) = z^{l_j} e^{P_j(z)} \Pi_j(z), \tag{1.2}$$

where $\Pi_j(z)$ is the canonical product formed by the zeros of $A_j(z)$, l_j is a non-negative integer. Let q_j be the genus of $A_j(z)$ and d_j the degree of $P_j(z)$. Put $q = \max_j q_j$, $d = \max_j d_j$. Let s_j be the genus of $\Pi_j(z)$. Put $s = \max_j s_j$. By the definition of genus

$$q_j = \max(d_j, s_j).$$

Thus

$$q = \max(d, s).$$

Then q is called the genus of the entire algebroid function $f(z)$.

We shall prove the following extension of Hellerstein and Williamson's theorems:

THEOREM 1. *Let $f(z)$ be an n-valued transcendental algebroid entire function of genus q, order λ and lower order μ. Assume that $f(z) = a_j$, $j = 1, \cdots, n$, have their roots only on the negative real axis.*

Then there is at least one a_ν among different finite numbers a_j, satisfying

$$\varlimsup_{r\to\infty}\frac{nN(r\,;a_\nu,f)}{T(r,f)}\geqq\begin{cases}\dfrac{|\sin\pi\rho|}{q+|\sin\pi\rho|} & (q\leqq\rho\leqq q+1/2)\,,\\ \dfrac{|\sin\pi\rho|}{q+1} & (q+1/2<\rho\leqq q+1)\end{cases}$$

for any ρ with $\mu\leqq\rho\leqq\lambda$.

THEOREM 2. *Let $f(z)$ be an n-valued transcendental algebroid entire function of genus q, order λ and lower μ. Assume that $f(z)=a_j$, $j=1,\cdots,n$, have their roots only on the negative real axis.*

Then there is at least one a_ν among different finite numbers a_j, satisfying

$$\varliminf_{r\to\infty}\frac{N(r\,;a_\nu,f)}{T(r,f)}\leqq\begin{cases}\dfrac{|\sin\pi\rho|}{q+|\sin\pi\rho|} & (q\leqq\rho\leqq q+1/2)\,,\\ \dfrac{|\sin\pi\rho|}{q+1} & (q+1/2<\rho\leqq q+1)\end{cases}$$

for any ρ with $\mu\leqq\rho\leqq\lambda$. These bounds are best possible.

2. Preliminaries. We put

$$A(z)=\max(1,\ |A_1|,\cdots,\ |A_n|)\,,$$
$$g(z)=\max(1,\ |g_1|,\ \cdots,\ |g_n|)\,,$$
$$g_\nu(z)=F(z,\ a_\nu)\,,\qquad \nu=1,\ \cdots n\,,$$

where $F(z,f)=0$ is the defining equation of f. We put

$$\mu(r,\ A)=\frac{1}{2n\pi}\int_0^{2\pi}\log A(re^{i\theta})d\theta\,.$$

Then Valiron [6] showed that

$$T(r,\ f)=\mu(r,\ A)+O(1)\,. \tag{2.1}$$

Further Ozawa [4] showed that

$$\mu(r,\ g)=\mu(r,\ A)+O(1)\,. \tag{2.2}$$

Hence from (2, 1) and (2, 2) we have

$$T(r,\ f)=\mu(r,\ g)+O(1)=\frac{1}{n}m(r,\ g)+O(1)=\frac{1}{n}T(r,\ g)+O(1)\,. \tag{2.3}$$

Evidently we have

$$\begin{aligned}T(r,\ g_\nu)&=m(r,\ g_\nu)\\&=\frac{1}{2\pi}\int_0^{2\pi}\log^+|g_\nu(re^{i\theta})|\,d\theta\leqq\frac{1}{2\pi}\int_0^{2\pi}\log^+\sum_{\nu=1}^{n}|g_\nu(re^{i\theta})|\,d\theta\end{aligned}$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \log g(re^{i\theta})d\theta + \log n = T(r, g) + \log n$$

and hence

$$\max_\nu T(r, g_\nu) \leqq T(r, g) + \log n .$$

On the other hand we get

$$\sum_{\nu=1}^n T(r, g_\nu) = \sum_{\nu=1}^n \frac{1}{2\pi}\int_0^{2\pi} \log^+ |g_\nu(re^{i\theta})| d\theta \geqq \frac{1}{2\pi}\int_0^{2\pi} \max_\nu \log^+ |g_\nu(re^{i\theta})| d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ (\max_\nu |g_\nu(re^{i\theta})|) d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log g(re^{i\theta}) d\theta$$

$$= T(r, g) .$$

Hence

$$\max_\nu T(r, g_\nu) \leqq T(r, g) + O(1) \leqq \sum_{\nu=1}^n T(r, g_\nu) .$$

From (2.3) we have

$$\max_\nu T(r, g_\nu) \leqq nT(r, f) = n \max_\nu T(r, g_\nu) . \tag{2.4}$$

Next since

$$g_\nu(z) = \sum_{j=0}^n a_\nu^{n-j} A_j(z) , \qquad (A_0(z) \equiv 1), \qquad \nu = 1, \cdots , n \tag{2.5}$$

are entire functions, let

$$g_\nu(z) = z^{m_\nu} e^{Q_\nu(z)} G_\nu(z) ,$$

where $G_\nu(z)$ is the canonical product formed by the zeros of $F(z, a_\nu)$, m_ν is a nonnegative integer. Let p_ν be the genus of $g_\nu(z)$ and c_ν the degree of $Q_\nu(z)$. Put $p = \max_\nu p_\nu$, $c = \max_\nu c_\nu$. Let t_ν be the genus of $G_\nu(z)$, then $p_\nu = \max(c_\nu, t_\nu)$. Then in view of (2.5) we have

$$q = \max_j q_j \geqq p_\nu .$$

Hence

$$q \geqq p . \tag{2.6}$$

On the other hand by solving the given equations (2.5) we have

$$A_j(z) = \sum_{\nu=1}^n b_{\nu, j} g_\nu(z) , \qquad j = 1, \cdots , n ,$$

which implies similarly

$$p = \max_\nu p_\nu \geqq q_j , \tag{2.7}$$

and hence

$$p \geqq q .$$

Combining (2.6) and (2.7) we deduce

$$p=q\,. \tag{2.8}$$

3. Proof of Theorem 1. Let

$$g_\nu(z)=z^{m_\nu}e^{Q_\nu(z)}G_\nu(z)\,. \tag{3.1}$$

Now the same arguments as in [2] does work. We assume then that

$$p_\nu<\rho<p_\nu+1\,.$$

By the main lemma of Hellerstein and Williamson we know that

$$T(r,\ G_\nu)=\frac{1}{\pi}\int_{C_\nu(r)}\log|G_\nu(re^{i\theta})|\,d\theta\,, \tag{3.2}$$

where $C_\nu(r)$ is defined as follows:

$$C_\nu(r)=\{\theta\in[0,\ \pi]:\log|G_\nu(re^{i\theta})|\geqq 0\}\,.$$

Then the well known lemma due to Edrei and Fuchs [1] we can write

$$\log|G_\nu(re^{i\theta})|\leqq\log|P_{\nu,R}(re^{i\theta})|+o(r^{p_\nu})+14\Big(\frac{r}{R}\Big)^{p_\nu+1}T(2R,\ G_\nu)\,, \tag{3.3}$$

where if $\{a_\mu\}_{\mu=1}^{\infty}$ denotes the zeros of $G_\nu(z)$,

$$P_{\nu,R}(z)=\prod_{|a_\mu|\leq R}\Big(1+\frac{z}{|a_\mu|}\Big)\exp\Big(-\frac{z}{|a_\mu|}+\cdots+\frac{(-1)^{p_\nu}}{p_\nu}\frac{z^{p_\nu}}{|a_\mu|}\Big) \tag{3.4}$$

and where

$$0<r=|z|\leqq\frac{R}{2}\,. \tag{3.5}$$

From (3.2) and (3.3) we have

$$T(r,\ G_\nu)\leqq\frac{1}{\pi}\int_{C_\nu(r)}\log|P_{\nu,R}(re^{i\theta})|\,d\theta+O(r^{p_\nu})+14\Big(\frac{r}{R}\Big)^{p_\nu+1}T(2R,\ G_\nu)\,.$$

Hence we get

$$T(r,\ G_\nu)+m(r,\ e^{Q_\nu})\leqq\frac{1}{\pi}\int_{C_\nu(r)}\log|P_{\nu,R}(re^{i\theta})|\,d\theta+O(r^{p_\nu})+O(r^{c_\nu})$$
$$+O(\log r)+14\Big(\frac{r}{R}\Big)^{p_\nu+1}T(2R,\ G_\nu)$$

in view of (3.1). Since $G_\nu(z)$ has only negative zeros, then

$$r^{p_\nu}=o(T(r,\ G_\nu(z)))\qquad(r\longrightarrow\infty)\,.$$

Thus, since we are assuming $c_\nu\leqq p_\nu$,

$$T(r,\ g_\nu)\leqq\frac{1}{\pi}\int_{C_\nu(r)}\log|P_{\nu,R}(re^{i\theta})|\,d\theta+O(r^{P_\nu})+14\Big(\frac{r}{R}\Big)^{p_\nu+1}T(2R,\ g_\nu)\,. \tag{3.6}$$

If we let $N_{\nu,R}(t, 0)=N(t, 1/P_{\nu,R})$, then by the definition of H_{p_ν} as given in [2],

$$\frac{1}{\pi}\int_{C_\nu(r)} \log|P_{\nu,R}(re^{i\theta})|\,d\theta$$

$$=\chi_\nu(r)N_{\nu,R}(r, 0)+(-1)^{p_\nu}\int_0^\infty N_{\nu,R}(t, 0)H_{p_\nu}(t, r, \alpha_1, \cdots, \alpha_{p_\nu+1})dt\,, \quad (3.7)$$

where

$$\chi_\nu(r)=\begin{cases} 1 & \text{for } \alpha_{p_\nu+1}=\pi\,, \\ 0 & \text{for } \alpha_{p_\nu+1}<\pi\,. \end{cases}$$

Now

$$N_{\nu,R}(t, 0)=\begin{cases} N_\nu(t, 0)=N(t, 1/g_\nu) & \text{if } t\leqq R\,, \\ N_\nu(R, 0)+n_\nu(R, 0)\log t/R & \text{if } t>R\,. \end{cases} \quad (3.8)$$

If follows easily from (3.6)—(3.8) that

$$T(r, g_\nu)\leqq\chi_\nu(r)N_\nu(r, 0)+(-1)^{p_\nu}\int_0^R N_\nu(t, 0)H_{p_\nu}(t, r, \alpha_1, \cdots, \alpha_{p_\nu+1})dt$$

$$+O(r^{p_\nu})+A\left(\frac{r}{R}\right)^{p_\nu+1}T(2R, g_\nu)\,,$$

where A is a positive absolute constant.

Taking the maximum over ν in the both side, we obtain

$$T(r, f)\leqq\max_\nu \chi_\nu(r)nN(r\,;a_\nu, f)$$

$$+\max_\nu\,(-1)^{p_\nu}\int_0^R nN(t\,;a_\nu, f)H_{p_\nu}(t, r, \alpha_1, \cdots, \alpha_{p_\nu+1})dt$$

$$+O(r^q)+A\left(\frac{r}{R}\right)^{q+1}T(2R, f)$$

in view of (2.4) and (2.8).

For the simplicity, we put

$$k(\rho)=\begin{cases} \dfrac{|\sin\pi\rho|}{q+|\sin\pi\rho|} & (q\leqq\rho\leqq q+1/2)\,, \\ \dfrac{|\sin\pi\rho|}{q+1} & (q+1/2<\rho\leqq q+1)\,. \end{cases}$$

Assume that for all ν

$$\varlimsup_{r\to\infty}\frac{nN(r\,;a_\nu, f)}{T(r, f)}<k(\rho)\,.$$

Then

$$\frac{nN(r\,;a_\nu, f)}{T(r, f)}<k(\rho)-\varepsilon\equiv U\,, \qquad \varepsilon>0$$

for $r \geqq r_0$. Put $\max_\nu \chi_\nu(r) = \chi(r)$. Thus

$$T(r, f) < \chi(r) U T(r, f) + \max_\nu (-1)^{p_\nu} U \int_{r_0}^{R} T(t, f) H_{p_\nu}(t, r, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt$$

$$+ O(r^q) + A\left(\frac{r}{R}\right)^{q+1} T(2R, f). \tag{3.9}$$

Now we make use of the notion of Pólya peaks of the first kind, order ρ, for $T(t, f)$.

It is possible to find three positive sequences $\{a_m\}$, $\{A_m\}$ and $\{r_m\}$ such that

$$\lim_{r\to\infty} a_m = \lim_{r\to\infty} \frac{A_m}{r_m} = \lim_{m\to\infty} \frac{r_m}{a_m} = \infty \tag{3.10}$$

and we can choose m_0 so large that for $m > m_0$

$$r_m > a_m \geqq r_0 \quad \text{and} \quad A_m \geqq 4 r_m .$$

Fix $m \geqq m_0$ and set

$$r = r_m, \qquad R = R_m = \frac{1}{2} A_m .$$

With this choice of r and R, $r_m \leqq \frac{1}{2} R_m$, we deduce that

$$\max_\nu (-1)^{p_\nu} \int_{r_0}^{R_m} T(t, f) H_{p_\nu}(t, r_m, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt$$

$$\leqq \max_\nu (-1)^{p_\nu} \int_{a_m}^{A_m} T(t, f) H_{p_\nu}(t, r_m, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt$$

$$+ \max_\nu (-1)^{p_\nu} \int_{r_0}^{a_m} T(t, f) H_{p_\nu}(t, r_m, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt$$

$$\leqq \max_\nu (-1)^{p_\nu} (1 + o(1)) T(r_m, f) \int_{a_m}^{A_m} \left(\frac{t}{r_m}\right)^\rho H_{p_\nu}(t, r_m, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt$$

$$+ \max_\nu (-1)^{p_\nu} \int_{r_0}^{a_m} T(r, f) H_{p_\nu}(t, r_m, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt . \tag{3.11}$$

Thus

$$\chi(r_m) + \max_\nu (-1)^{p_\nu} \int_{a_m}^{A_m} \left(\frac{t}{r_m}\right)^\rho H_{p_\nu}(t, r_m, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt$$

$$\leqq \max_\nu \left\{ \chi_\nu(r_m) + (-1)^{p_\nu} \int_0^\infty s^\rho H_{p_\nu}(s, 1, \alpha_1, \cdots, \alpha_{p_\nu+1}) \right\} dt$$

$$= \max_\nu \frac{(-1)^{p_\nu}}{\sin \pi\rho} \sum_{j=0}^{[(p_\nu+1)/2]} (\sin \alpha_{2j+1}\rho - \sin \alpha_{2j}\rho)$$

$$= \max_\nu \left\{ \frac{p_\nu}{|\sin \pi\rho|} + (-1)^{p_\nu} \frac{\sin \alpha_{p_\nu+1}\rho}{|\sin \pi\rho|} \right\} . \tag{3.12}$$

From (3.9), (3.11) and (3.12) we have

$$T(r_m, f) \leqq U(1+o(1))T(r_m, f) \max_{\nu} \left\{ \frac{p_\nu}{|\sin \pi\rho|} + (-1)^{p_\nu} \frac{\sin \alpha_{p_\nu+1}\rho}{|\sin \pi\rho|} \right\} + \eta(a_m, r_m, A_m), \tag{3.13}$$

where

$$\eta(a_m, r_m, A_m) = O(r_m{}^q) + A\left(\frac{2r_m}{A_m}\right)^{q+1} T(A_m, f) + \max_{\nu} (-1)^{p_\nu} \int_{r_0}^{a_m} T(t, f) H_{p_\nu}(t, r_m, \alpha_1, \cdots, \alpha_{p_\nu+1}) dt .$$

By (3.11) and the definition of Pólya peaks of the first kind, order ρ, we can see

$$\eta(a_m, r_m, A_m) = o(T(r_m, f)) \qquad (m \longrightarrow \infty)$$

by means of the same process as in [2].

Hence in view of (3.13) we get

$$1 \leqq U(1+o(1)) \max_{\nu} \left\{ \frac{p_\nu}{|\sin \pi\rho|} + (-1)^{p_\nu} \frac{\sin \alpha_{p_\nu+1}\rho}{|\sin \pi\rho|} \right\} + o(1) \quad (m \longrightarrow \infty). \tag{3.14}$$

If $p_\nu < \rho \leqq p_\nu + 1/2$, then

$$(-1)^{p_\nu} \sin \alpha_{p_\nu+1}\rho \leqq (-1)^{p_\nu} \sin \pi\rho = |\sin \pi\rho| .$$

Thus for $p_\nu < \rho \leqq p_\nu + 1/2$, consequently for $q < \rho \leqq q + 1/2$ (3.14) implies

$$1 \leqq U\left\{ \frac{q}{|\sin \pi\rho|} + 1 \right\} .$$

By definition of U we have

$$1 \leqq (k(\rho) - \varepsilon) \frac{q + |\sin \pi\rho|}{|\sin \pi\rho|} = 1 - \varepsilon \cdot k(\rho) < 1$$

which is a contradiction. If $p_\nu + 1/2 < \rho \leqq p_\nu + 1$, then $(-1)^{p_\nu} \sin \alpha_{p_\nu+1}\rho \leqq 1$. Consequently for $q + 1/2 < \rho \leqq q + 1$ (3.14) implies

$$1 \leqq U\left\{ \frac{q+1}{|\sin \pi\rho|} \right\},$$

which is a contradiction. Hence Theorem 1 follows.

4. Proof of Theorem 2. When $\mu = \lambda$, we are able to prove with slightly modification of proof in the case $\mu < \lambda$, with remark for making of sequence of Pólya peaks of the second kind.

Then it is enough to prove when $\mu < \lambda$. We assume, therefore, that

$$p_\nu < \mu_\nu \leqq \rho \leqq \lambda_\nu < p_\nu + 1 \tag{4.1}$$

for the canonical products $G_\nu(z)$ of genus p_ν.

In view of the definition of $T(r, G_\nu)$ we know that if $\alpha'_1, \alpha'_2, \cdots, \alpha'_{p_\nu+1}$ are any $p_\nu+1$ numbers satisfying

$$\frac{2j-1}{2(p_\nu+1)}\pi<\alpha'_j<\frac{2j-1}{2p_\nu}\pi, \qquad j=1, \cdots, p_\nu;$$

$$\frac{2p_\nu+1}{2(p_\nu+1)}\pi<\alpha'_{p_\nu+1}\leqq\pi$$

then,

$$T(r, G_\nu)\geqq\begin{cases}\displaystyle\sum_{i=1}^{(p_\nu+1)/2}\frac{1}{\pi}\int_{\alpha'_{2i-1}}^{\alpha'_{2i}}\log|G_\nu(re^{i\theta})|\,d\theta & \text{if } p_\nu \text{ is odd},\\ \displaystyle\sum_{i=0}^{p_\nu/2}\frac{1}{\pi}\int_{\alpha'_{2i}}^{\alpha'_{2i+1}}\log|G_\nu(re^{i\theta})|\,d\theta & \text{if } p_\nu \text{ is even}.\end{cases} \tag{4.2}$$

From Shea's Lemma [5] we see that (4.2) implies,

$$T(r, G_\nu)\geqq\chi_\nu(\alpha'_{p+1})N_\nu(r, 0)+(-1)^{p_\nu}\int_0^\infty N_\nu(t, 0)H_{p_\nu}(t, r, \alpha'_1, \cdots, \alpha'_{p_\nu+1})dt,$$

where

$$\chi_\nu(\alpha'_{p_\nu+1})=\begin{cases}1 & \text{if } \alpha'_{p_\nu+1}=\pi,\\ 0 & \text{if } \alpha'_{p_\nu+1}<\pi.\end{cases}$$

Hence

$$T(r, G_\nu)+m(r, e^{Q_\nu(z)})\geqq\chi_\nu(\alpha'_{p+1})N_\nu(r, 0)+O(r^{c_\nu})+O(\log r)$$
$$+(-1)^{p_\nu}\int_0^\infty N_\nu(t, 0)H_{p_\nu}(t, r, \alpha'_1, \cdots, \alpha'_{p_\nu+1})dt.$$

This implies

$$T(r, g_\nu)\geqq\chi_\nu(\alpha'_{p_\nu+1})N_\nu(r, 0)+O(r^{c_\nu})+(-1)^{p_\nu}\int_0^\infty N_\nu(t, 0)H_{p_\nu}(t, r, \alpha'_1, \cdots, \alpha'_{p_\nu+1})dt.$$

Taking maximum over ν in the both sides

$$nT(r, f)\geqq\max_\nu \chi_\nu(\alpha'_{p_\nu+1})nN(r\,;a_\nu, f)+O(r^c)$$
$$+\max_\nu(-1)^{p_\nu}\int_0^\infty nN(t\,;a_\nu, f)H_{p_\nu}(t, r, \alpha'_1, \cdots, \alpha'_{p_\nu+1})dt$$

in view of (2.4).

Assume that for all ν

$$\varliminf_{r\to\infty}\frac{N(r\,;a_\nu, f)}{T(r, f)}>k(\rho),$$

then

$$\frac{N(r: a_\nu, f)}{T(r, f)}>k(\rho)+\varepsilon\equiv V, \qquad (\varepsilon>0)$$

for $r\geqq r_0$. Thus

$$T(r, f) \geqq \max_{\nu} \chi_{\nu}(\alpha'_{p_\nu+1}) V T(r, f) + O(r^c)$$
$$+ \max_{\nu} (-1)^{p_\nu} \int_0^\infty V T(t, f) H_{p_\nu}(t, r, \alpha'_1, \cdots \alpha'_{p_\nu+1}) dt .$$

Letting $\{r_m\}$ be a sequence of Pólya peaks of the second kind, order ρ, for $T(t, f)$ with $\{a_m\}$, $\{A_m\}$ the associated sequences, we have

$$T(r_m, f) \geqq \max_{\nu} \chi_{\nu}(\alpha'_{p_\nu+1}) V T(r_m, f) + O(r_m{}^c)$$
$$+ \max_{\nu} (-1)^{p_\nu} V(1+o(1)) T(r_m, f) \int_{a_m}^{A_m} (t/r_m)^\rho H_{p_\nu}(t, r_m, \alpha'_1, \cdots, \alpha'_{p_\nu+1}) dt . \tag{4.3}$$

Setting $t = s r_m$, recalling that

$$\lim_{m\to\infty} A_m/r_m = \lim_{m\to\infty} r_m/a_m = \infty ,$$

and upon dividing in the both side of (4.3) by $T(r_m, f)$ and letting $m \to \infty$, it follows

$$1 \geqq V(1+o(1)) \max_{\nu} \frac{1}{|\sin \pi\rho|} \sum_{j=0}^{[(p_\nu+1)/2]} (\sin \alpha'_{2j+1}\rho - \sin \alpha'_{2j}\rho) , \qquad (\alpha'_{p_\nu+2} = 0) .$$

Selecting $\alpha'_k = \dfrac{(2k-1)}{2} \pi/\rho$ if $k = 1, 2, \cdots, p_\nu$; $\alpha'_{p_\nu+1} = \pi$ if $p_\nu < \rho \leqq p_\nu + 1/2$ and $\alpha'_{p_\nu+1} = \dfrac{2p_\nu+1}{2} \pi/\rho$ if $p_\nu + 1/2 < \rho < p_\nu + 1$, we obtain the following inequalities in view of (2.8)

$$1 \geqq V\left\{\frac{q}{|\sin \pi\rho|} + 1\right\} \quad \text{if} \quad q < \rho \leqq q + 1/2 ,$$
$$1 \geqq V\left\{\frac{q+1}{|\sin \pi\rho|}\right\} \qquad \text{if} \quad q + 1/2 < \rho < q + 1 ,$$

which are contradictions together. Hence we have the desired result.

5. Now we consider equality parts in the above Theorem 2. Let $f(z ; \rho)$ be the Lindelöf function

$$f(z ; \rho) = \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{b_\nu}\right), \qquad b_\nu = \nu^{1/\rho},\ \nu = 1, 2, 3, \cdots .$$

The asymptotic behaviour of $f(z ; \rho)$ is well known [3]. Now we consider

$$f^n + f(z ; \rho) - 1 = 0 .$$

Evidently we have

$$\lim_{r\to\infty} \frac{N(r ; a_\nu, f)}{T(r, f)} = \lim_{r\to\infty} \frac{(1/n) N(r ; 0, f(z ; \rho))}{(1/n) T(r, f(z ; \rho))}$$

$$
=\lim_{r\to\infty}\frac{N(r\,;0,\,f(z\,;\rho))}{T(r\,;f(z\,;\rho))}
$$

$$
=\begin{cases}\dfrac{|\sin\pi\rho|}{q+|\sin\pi\rho|}, & q\leqq\rho\leqq q+1/2,\ q=[\rho]\\[2ex] \dfrac{|\sin\pi\rho|}{q+1} & q+1/2<\rho<q+1,\ q=[\rho]\end{cases}
$$

for $a_\nu=\exp 2\pi\nu i/n,\ \nu=1,\,2,\,\cdots n.$

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