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SUBMANIFOLDS OF AN ALMOST PRODUCT RIEMANNIAN MANIFOLD

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§0. Introduction. When a Riemannian manifold $\bar{M}$ admits a tensor field F of type (1, 1) such that $F^2=I$ (F is non-trivial), $\bar{M}$ is called an almost product Riemannian manifold. Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. We denote by $T_p(M)$ the tangent space of M at $P\in M$ and by $T_p(M)^\perp$ the normal space of M at P. If $FT_p(M)\subset T_p(M)$ for any point $P\in M$, then M is called an invariant submanifold. If $FT_p(M)\subset T_p(M)^\perp$ for any point P, then M is called an anti-invariant submanifold. In this paper, we shall study non-invariant, invariant and anti-invariant submanifolds of an almost product Riemannian manifold.

In §1 and §2, we obtain for later use fundamental formulas for submanifolds of an almost product Riemannian manifold $\bar{M}$. In §3, we study hypersurfaces of an almost product Riemannian manifold $\bar{M}$. In §4 and §5, we mainly investigate non-invariant submanifolds of $\bar{M}$. We devote §6 to the study of invariant submanifolds of $\bar{M}$. In the last §7, we consider anti-invariant submanifolds of $\bar{M}$.

§1. An almost product Riemannian manifold. Let $\bar{M}$ be an almost product Riemannian manifold of dimension m. Then, by definition, there exist a non-trivial tensor field F of type (1, 1) and a positive definite Riemannian metric G satisfying

$$F^2=I\,,\quad G(F\bar{X},\, F\bar{Y})=G(\bar{X},\, \bar{Y})\,,\quad \bar{X},\, \bar{Y}\in\mathfrak{X}(\bar{M})\,,$$

where I is the identity and $\mathfrak{X}(\bar{M})$ is the Lie algebra of vector fields on $\bar{M}$. It is well known that

$$G(F\bar{X},\, \bar{Y})=G(\bar{X},\, F\bar{Y})\,,$$

that is, Φ is symmetric, where $\Phi(\bar{X},\, \bar{Y})=G(F\bar{X},\, \bar{Y})$.

Let M be an n-dimensional manifold immersed in $\bar{M}$ ($m-n=s$) and i_* the differential of the immersion i of M into $\bar{M}$. The induced Riemannian metric g of M is given by

$$g(X,\, Y)=G(i_*X,\, i_*Y)\,,\qquad X,\, Y\in\mathfrak{X}(M), \tag{1.1}$$

where $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M. Let $\{N_1, N_2, \cdots, N_s\}$ be an orthonormal basis of the normal space $T_P(M)^\perp$ at a point $P\in M$.

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The transform Fi_*X of $X\in T_P(M)$ by F and FN_i of N_i by F can be respectively written in the next form:

$$Fi_*X=i_*fX+\sum_{i=1}^{s} u_i(X)N_i\,, \qquad X\in\mathfrak{X}(M), \tag{1.2}$$

$$FN_i=i_*U_i+\sum_{j=1}^{s} \lambda_{ij}N_j\,, \tag{1.3}$$

where f, u_i, U_i and λ_{ij} are respectively a linear transformation, 1-forms, vector fields and functions on M. Using (1.1) and (1.2),

$$g(fX,\ Y)=G(i_*fX,\ i_*Y)=G(Fi_*X,\ i_*Y)=G(i_*X,\ Fi_*Y)=G(i_*X,\ i_*fY)\,.$$

Therefore we have $g(fX, Y)=g(X, fY)$. Furthermore, from $G(Fi_*X, N_i)=G(i_*X, FN_i)$ and $G(FN_i, N_j)=G(N_i, FN_j)$, we can respectively get the equations

$$u_i(X)=g(X,\ U_i)\,, \qquad X\in\mathfrak{X}(M),$$

$$\lambda_{ij}=\lambda_{ji}\,.$$

LEMMA 1.1. *In submanifold M of an almost product Riemannian manifold $\bar{M}$,*

$$f^2X=X-\sum_{i=1}^{s} u_i(X)U_i \quad or \quad f^2=I-\sum_{i=1}^{s} u_i\otimes U_i\,, \tag{1.4}$$

$$u_i(fX)+\sum_{j=1}^{s} \lambda_{ij}u_j(X)=0\,, \qquad X\in\mathfrak{X}(M), \tag{1.5}$$

$$fU_i+\sum_{j=1}^{s} \lambda_{ij}U_j=0\,, \tag{1.6}$$

$$u_j(U_i)=\delta_{ji}-\sum_{k=1}^{s} \lambda_{jk}\lambda_{ki}\,. \tag{1.7}$$

Proof. From (1.2),

$$F^2i_*X=F(i_*fX+\sum_i u_i(X)N_i)=i_*(f^2X+\sum_i u_i(X)U_i)+\sum_j \{u_j(fX)+\sum_i \lambda_{ij}u_i(X)\}N_j\,.$$

Since $F^2i_*X=i_*X$, we get (1.4) and (1.5). Similarly,

$$F^2N_i=i_*(fU_i+\sum_j \lambda_{ij}U_j)+\sum_k (u_k(U_i)+\sum_j \lambda_{ij}\lambda_{jk})N_k\,.$$

Thus we get (1.6) and (1.7).

(1.5) and (1.6) are equivalent.

Using (1.2), for X, $Y\in\mathfrak{X}(M)$,

$$G(Fi_*X,\ Fi_*Y)=G(i_*fX,\ i_*fY)+G(\sum_i u_i(X)N_i,\ \sum_j u_j(Y)N_j)$$

$$=g(fX, fY)+\sum_{i} u_i(X)u_i(Y),$$

from which

$$g(fX, fY)=g(X, Y)-\sum_{i} u_i(X)u_i(Y). \tag{1.8}$$

§2. A locally product Riemannian manifold. We denote the covariant differentiation in $\bar{M}$ by $\bar{\nabla}$ and the covariant differentiation in M determined by the induced metric on M by ∇. Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_{\iota_* X}\iota_* Y=\iota_*\nabla_X Y+\sum_{i=1}^{s} h_i(X, Y)N_i, \qquad X, Y\in\mathfrak{X}(M),$$

$$\bar{\nabla}_{\iota_* X}N_i=-\iota_* H_i X+\sum_{j=1}^{s} \mu_{ij}(X)N_j,$$

where $h_i(i=1, 2, \cdots, s)$ are the second fundamental tensors corresponding to N_i respectively and $h_i(X, Y)=h_i(Y, X)$.

Covariantly differentiating $G(\iota_* Y, N_i)=0$ on M,

$$G(\bar{\nabla}_{\iota_* X}\iota_* Y, N_i)+G(\iota_* Y, \bar{\nabla}_{\iota_* X}N_i)=0,$$

from which $h_i(X, Y)=g(H_i X, Y)$. Similarly, covariantly differentiating $G(N_i, N_j)=\delta_{ij}$ on M, we have $\mu_{ij}(X)+\mu_{ji}(X)=0$.

Next, we consider $\bar{\nabla}_{\iota_* X}F$.

$$\begin{aligned}(\bar{\nabla}_{\iota_* X}F)i_* Y&=\bar{\nabla}_{\iota_* X}(Fi_* Y)-F\bar{\nabla}_{\iota_* X}\iota_* Y\\ &=\bar{\nabla}_{\iota_* X}(i_* fY+\sum_{i} u_i(Y)N_i)-F(\iota_*\nabla_X Y+\sum_{i} h_i(X, Y)N_i)\\ &=\{i_*(\nabla_X f)Y-\sum_{i} u_i(Y)H_i X-\sum_{i} h_i(X, Y)U_i\}\\ &\quad+\sum_{i}\{h_i(X, fY)+(\nabla_X u_i)(Y)-\sum_{j} \mu_{ij}(X)u_j(Y)\\ &\quad-\sum_{j} \lambda_{ij}h_j(X, Y)\}N_i.\end{aligned}$$

When $\bar{M}$ is a locally product Riemannian manifold, that is, $\bar{\nabla}F=0$ ([2], [5]), we have

LEMMA 2.1. *If $\bar{M}$ is a locally product Riemannian manifold, then the next equations hold good*:

$$(\nabla_X f)Y=\sum_{i}\{u_i(Y)H_i X+h_i(X, Y)U_i\}, \tag{2.1}$$

$$h_i(X, fY)+(\nabla_X u_i)(Y)-\sum_{j} \mu_{ij}(X)u_j(Y)-\sum_{j} \lambda_{ij}h_j(X, Y)=0. \tag{2.2}$$

Similarly,

$$
\begin{aligned}
(\bar{\nabla}_{i_*X}F)N_i &= \bar{\nabla}_{i_*X}(FN_i) - F\bar{\nabla}_{i_*X}N_i \\
&= \bar{\nabla}_{i_*X}(i_*U_i + \sum_j \lambda_{ij}N_j) - F(-i_*H_iX + \sum_j \mu_{ij}(X)N_j) \\
&= i_*\{\nabla_X U_i + fH_iX - \sum_j \mu_{ij}(X)U_j - \sum_j \lambda_{ij}H_jX\} \\
&\quad + \{\sum_j \{h_j(X, U_i) + h_i(X, U_j) + \nabla_X \lambda_{ij} + \sum_k \lambda_{ik}\mu_{kj}(X) \\
&\quad + \sum_k \lambda_{jk}\mu_{ki}(X)\} N_j = 0 .
\end{aligned}
$$

Thus we have

LEMMA 2.2. *If $\bar{M}$ is a locally product Riemannian manifold, then the next equations hold good:*

$$fH_iX + \nabla_X U_i - \sum_j \mu_{ij}(X)U_j - \sum_j \lambda_{ij}H_jX = 0 , \tag{2.3}$$

$$h_j(X, U_i) + h_i(X, U_j) + \nabla_X \lambda_{ij} + \sum_k \lambda_{ik}\mu_{kj}(X) + \sum_k \lambda_{jk}\mu_{ki}(X) = 0 . \tag{2.4}$$

Calculating $(\nabla_X u_i)(Y)$,

$$(\nabla_X u_i)(Y) = \nabla_X\{u_i(Y)\} - u_i(\nabla_X Y) = \nabla_X\{g(Y, U_i)\} - g(\nabla_X Y, U_i) = g(\nabla_X U_i, Y) .$$

Hence, (2.2) and (2.3) are equivalent.

§3. Hypersurfaces of an almost product Riemannian manifold. Suppose that M is a hypersurface immersed in an almost product Riemannian manifold $\bar{M}$ [1], [3]. In this case, (1.2) and (1.3) are respectively written in the following forms:

$$Fi_*X = i_*fX + u(X)N , \qquad FN = i_*U + \lambda N,$$

where $N = N_1$, $u = u_1$, $U = U_1$, $\lambda = \lambda_{11}$ and $u(X) = g(X, U)$. From Lemma 1.1, we have

$$f^2 = I - u \otimes U , \tag{3.1}$$

$$fU = -\lambda U , \tag{3.2}$$

$$u(U) = 1 - \lambda^2 , \qquad 0 \leqq \lambda^2 \leqq 1. \tag{3.3}$$

The Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + h(X, Y)N , \qquad \bar{\nabla}_{i_*X} N = -i_*HX ,$$

where $h = h_1$, $H = H_1$ and $h(X, Y) = g(HX, Y)$.

When $\bar{M}$ is a locally product Riemannian manifold, from Lemma 2.1 and

Lemma 2.2 we have

$$(\nabla_X f)Y = u(Y)HX + h(X, Y)U, \tag{3.4}$$

$$h(X, fY) + (\nabla_X u)(Y) - \lambda h(X, Y) = 0 \quad \text{or} \quad fHX + \nabla_X U - \lambda HX = 0, \tag{3.5}$$

$$2h(X, U) + \nabla_X \lambda = 0. \tag{3.6}$$

When $\lambda^2=1$, U is a zero vector. Consequently, M is an invariant hypersurface and $f^2=I$. Furthermore, we get $FN=\lambda N$. Thus we have

THEOREM 3.1. *In order that M is an invariant hypersurface of an almost product Riemannian manifold $\bar{M}$, it is necessary and sufficient that the normal of M is an eigenvector of the matrix F.*

THEOREM 3.2. *In order that M is an invariant hypersurface of an almost product Riemannian manifold $\bar{M}$, it is necessary and sufficient that the induced structure (f, g) of M is an almost product Riemannian structure excepting the case where f is trivial.*

Proof. If $f^2=I$, we have $u(X)U=0$. Therefore, we get $u(X)g(U, X)= u(X)^2=0$, that is, $u(X)=0$. Hence M is invariant.

In the next place, we consider the case where M is not invariant, that is, $\lambda^2 \neq 1$. Since eigenvalues of f are ± 1 and $-\lambda$, we have

$$\mathrm{Tr}(f) = -\lambda + \text{const.}$$

When $\lambda=0$, the following equations hold good.

$$f^2 = I - u \otimes U, \qquad u(U) = 1,$$
$$u(X) = g(X, U), \quad g(fX, fY) = g(X, Y) - u(X)u(Y).$$

Thus, we get the following theorem [4].

THEOREM 3.3. *Let M be a hypersurface of an almost product Riemannian manifold $\bar{M}$. If FN is tangent to M, then M admits an almost paracontact Riemannian structure.*

THEOREM 3.4 [1]. *When M is non-invariant hypersurface of a locally product Riemannian manifold $\bar{M}$, the following conditions are equivalent.*
(i) $\nabla_X f=0$, (ii) *M is totally geodesic,* (iii) *U is parallel in M.*

Proof. (i) When $\nabla_X f=0$, we get from (3.4) $u(Y)HX+h(X, Y)U=0$, from which $u(Y)HX=-h(X, Y)U$. Therefore, for $X, Y, Z \in \mathfrak{X}(M)$

$$u(Y)h(X, Z) = -u(Z)h(X, Y).$$

Thus, since $u(Y)h(X, Z)$ is symmetric in X and Y,

$$u(Y)h(X, Z)=u(X)h(Y, Z)=-u(Y)h(X, Z).$$

By virtue of $u(U)=1-\lambda^2\neq 0$, we get $h(X, Z)=0$, that is, M is totally geodesic. Furthermore, from (3.5) we have $\nabla_x U=0$.

(ii) When $h(X, Y)=0$, from (3.4) and (3.5), we have $\nabla_X f=0$, $\nabla_X U=0$.

(iii) When $\nabla_X U=0$, from (3.5) we have $fHX=\lambda HX$. Therefore $f^2HX=\lambda fHX=\lambda^2 HX$, from which

$$HX-u(HX)U=\lambda^2 HX.$$

Since we have λ=const. from (3.3) and $\nabla_X U=0$, we find $h(X, U)=u(HX)=0$ from (3.6). Thus we get

$$HX=\lambda^2 HX,$$

from which $HX=0$. Consequently, $\nabla_X f=0$.

We denote by $\{e_1, e_2, \cdots, e_n\}$ an orthonormal basis of the tangent space $T_P(M)$ at a point $P\in M$. Then we have

THEOREM 3.5. *Let M be a non-invariant hypersurface of a locally product Riemannian manifold $\bar{M}$. If $\sum_{\lambda=1}^{n}(\nabla_{e_\lambda}f)e_\lambda=0$ and* Tr(f)=const., *then M is minimal.*

Proof. Since we have λ=const. from Tr(f)=const., we get $h(X, U)=0$ from (3.6). From (3.4)

$$\sum_\lambda(\nabla_{e_\lambda}f)e_\lambda=\sum_\lambda(u(e_\lambda)He_\lambda+h(e_\lambda, e_\lambda)U)=0.$$

Consequently,

$$\sum_\lambda\{u(e_\lambda)h(e_\lambda, U)+h(e_\lambda, e_\lambda)u(U)\}=\sum_{\mu=1}^{n}h(e_\mu, e_\mu)(1-\lambda^2)=0,$$

from which $\sum_\lambda h(e_\lambda, e_\lambda)=0$. Hence M is minimal.

§4. **Submanifolds of an almost product Riemannian manifold** (I). We consider a non-invariant submanifold M immersed in an almost product Riemannian manifold $\bar{M}$ and assume that U_i $(i=1, 2, \cdots, s)$ are linearly independent. Consequently we have

$$\sum_k(\lambda_{ik})^2<1 \ (i=1, 2, \cdots, s) \quad \text{and} \quad s\leqq n.$$

Let $\{\bar{N}_1, \bar{N}_2, \cdots, \bar{N}_s\}$ be the another orthonormal basis of $T_P(M)^\perp$ at $P\in M$. We put

$$\bar{N}_i=\sum_{l=1}^{s}k_{li}N_l. \tag{4.1}$$

By means of $G(\bar{N}_i, \bar{N}_j)=\sum_l k_{li}k_{lj}$, we have

$$\sum_{l=1}^{s} k_{li}k_{lj}=\delta_{ij},$$

from which

$$\sum_{h=1}^{s} k_{ih}k_{jh}=\delta_{ij}.$$

Consequently matrix (k_{ij}) is an orthogonal matrix. Thus from (4.1), we have

$$N_j=\sum_h k_{jl}\bar{N}_l.$$

Making use of (4.1), equations (1.2) and (1.3) are respectively written in the next forms:

$$Fi_*X=i_*fX+\sum_l \bar{u}_l(X)\bar{N}_l, \tag{4.2}$$

$$F\bar{N}_l=i_*\bar{U}_l+\sum_h \bar{\lambda}_{lh}\bar{N}_h, \tag{4.3}$$

where

$$\bar{u}_i=\sum_l k_{li}u_l, \qquad \bar{U}_i=\sum_l k_{li}U_l, \tag{4.4}$$

$$\bar{\lambda}_{lh}=\sum_{i,j} k_{il}\lambda_{ij}k_{jh}. \tag{4.5}$$

From (4.4), we obtain

LEMMA 4.1. *Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. When the orthonormal basis $\{N_i\}$ of $T_P(M)^\perp$ is transformed to the another orthonormal basis $\{\bar{N}_i\}$ of $T_P(M)^\perp$, if U_i $(i=1, 2, \cdots, s)$ are linearly independent, then $\bar{U}_i$ $(i=1, 2, \cdots, s)$ are also linearly independent, and vice versa.*

It is clear that if a submanifold M of a locally product Riemannian manifold $\bar{M}$ is totally geodesic, then $\nabla_X f=0$ is satisfied. Conversely, we have the following

THEOREM 4.2. *Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. If U_i $(i=1, 2, \cdots, s)$ are linearly independent and $\nabla_X f=0$, then M is totally geodesic.*

Proof. Since we have from (2.1) $\sum_i \{u_i(Y)H_iX+h_i(X, Y)U_i\}=0$, we get the equation

$$\sum_i \{u_i(Y)g(H_iX, Z)+h_i(X, Y)g(U_i, Z)\}=0, \qquad X, Y, Z\in\mathfrak{X}(M),$$

from which

$$\sum_i \{u_i(Y)h_i(X, Z)\} = -\sum_i \{u_i(Z)h_i(X, Y)\}$$

and consequently

$$\sum_i u_i(Y)h_i(X, Z) = -\sum_i u_i(X)h_i(Y, Z).$$

Therefore $\sum_i u_i(Y)h_i(X, Z)$ is symmetric and at the same time skew-symmetric in X, Y. Thus we have

$$\sum_i u_i(Y)h_i(X, Z)=0.$$

Since U_i $(i=1, 2, \cdots, s)$ are linearly independent, we get $h_i(X, Z)=0$ $(i=1, 2, \cdots, s)$, that is, M is totally geodesic.

In $T_p(M)$, we denote by $V(U_i)$ an s-dimensional vector space spanned by U_i $(i=1, 2, \cdots, s)$ and by V an eigenvector of f perpendicular to the vector space $V(U_i)$. Then the following equation holds good:

$$fV=\rho V,$$

where ρ is an eigenvalue of f. Therefore, $f^2V=\rho fV=\rho^2V$, from which $(I-\sum_i u_i\otimes U_i)V=\rho^2V$, that is, $V=\rho^2V$. Hence we have $\rho^2=1$.

When $s<n$, in $T_P(M)$, we denote eigenvectors of f, which are perpendicular to $V(U_i)$ and mutually orthogonal, by V_A $(A=s+1, \cdots, n)$. We put

$$fV_A=\varepsilon_A V_A \qquad (A=s+1, \cdots, n),$$

where ${\varepsilon_A}^2=1$.

Next, if we take an eigenvector U of f in the vector space $V(U_i)$, the following equation holds good:

$$fU=\sigma U,$$

where σ is an eigenvalue of f. Since we can put $U=\sum_i c_iU_i$, from (1.6)

$$fU=f\sum_i c_iU_i=-\sum_{i,j} c_i\lambda_{ij}U_j,$$

from which $\sum_i c_i\lambda_{ij}=-\sigma c_j$. Therefore, if we denote by σ an eigenvalue of f in $V(U_i)$, then -σ is an eigenvalue of the matrix (λ_{ij}). The converse is also true.

LEMMA 4.3. *Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. If U_i $(i=1, 2, \cdots, s)$ are linearly independent, then we have*

$$\begin{aligned} \operatorname{Tr}(f) &= -\operatorname{Tr}(\lambda_{ij})+\sum_A \varepsilon_A \qquad (s<n), \\ &= -\operatorname{Tr}(\lambda_{ij}) \qquad\qquad (s=n), \end{aligned} \tag{4.6}$$

where $\varepsilon_A^2=1$ $(A=s+1, \cdots n)$.

Proof. We shall prove the case of $s<n$. Since from (1.6) we have $fU_i=-\sum_j \lambda_{ij}U_j$, matrices (f), (λ_{ij}) and $(U_1U_2\cdots U_s)$ satisfy the relations

$$(f)(U_1U_2\cdots U_s)=(U_1U_2\cdots U_s)(-\lambda_{ij}).$$

We define matrices $\tilde{U}$, L by

$$\tilde{U}=(U_1U_2\cdots U_sV_{s+1}\cdots V_n),$$

$$L=\begin{pmatrix} -\lambda_{ij} & 0 \\ 0 & \varepsilon_A\delta_{AB} \end{pmatrix},$$

where $\delta_{AA}=1$, $\delta_{AB}=0$ $(A\neq B)$ $(A, B=s+1, \cdots, n)$. Then we have $(f)\tilde{U}=\tilde{U}L$. Since $|\tilde{U}|\neq 0$, we have $(f)=\tilde{U}L\tilde{U}^{-1}$. If we denote components of (f), L, $\tilde{U}$ and $\tilde{U}^{-1}$ by f^λ_μ, $l_{\mu\lambda}$, u^λ_μ and v^λ_μ respectively, then we get

$$f^\lambda_\mu=\sum_{\omega,\nu} u^\lambda_\omega l_{\omega\nu}v^\nu_\mu \qquad (\lambda, \mu, \nu, \omega=1, 2, \cdots, n).$$

Thus we have

$$\mathrm{Tr}(f)=\sum_\mu f^\mu_\mu=\sum_\nu l_{\nu\nu}=-\sum_i \lambda_{ii}+\sum_A \varepsilon_A.$$

LEMMA 4.4. *Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. If U_i $(i=1, 2, \cdots, s)$ are linearly independent and $\nabla_X f=0$, then* $\mathrm{Tr}(\lambda_{ij})=$const..

Proof. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of $T_P(M)$ and extend e_λ $(\lambda=1, 2, \cdots, n)$ to local vector fields E_λ which are covariantly constant at $P\in M$. Then at $P\in M$,

$$\begin{aligned}\nabla_X\mathrm{Tr}(f)&=\nabla_X\sum_\lambda g(fe_\lambda, e_\lambda)=\{\sum_\lambda \nabla_X g(fE_\lambda, E_\lambda)\}_P\\ &=\{\sum_\lambda g(\nabla_X f)E_\lambda, E_\lambda)+2\sum_\lambda g(\nabla_X E_\lambda, fE_\lambda)\}_P=\sum_\lambda g((\nabla_X f)e_\lambda, e_\lambda)=0.\end{aligned}$$

Thus we have $\mathrm{Tr}(f)=$const., from which, $\mathrm{Tr}(\lambda_{ij})=$const..

THEOREM 4.5. *Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. If U_i $(i=1, 2, \cdots, s)$ are linearly independent,* $\mathrm{Tr}(f)=$const. *and M is totally umbilical, then M is totally geodesic.*

Proof. If we put $i=j$ in (2.4), we can get

$$2\sum_i h_i(X, U_i)+\nabla_X\sum_i \lambda_{ii}+2\sum_{i,k}\lambda_{ik}\mu_{ki}(X)=0.$$

Since λ_{ij} is symmetric and μ_{ij} skew-symmetric in i, j, $\sum_{i,k}\lambda_{ik}\mu_{ki}(X)=0$. And by means of $\mathrm{Tr}(f)=$const. and (4.6), we have $\sum_i \lambda_{ii}=$const.. Hence we find

$$\sum_i h_i(X, U_i)=0 .$$

Putting $h_i(X, Y)=\sigma_i g(X, Y)$ and substituting in the above equation, we have $\sum_i \sigma_i g(X, U_i)=0$, from which $\sum_i \sigma_i U_i=0$. Thus we have $\sigma_i=0$, that is, M is totally geodesic.

THEOREM 4.6. *Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. If U_i ($i=1, 2, \cdots, s$) are linearly independent, $\sum_\lambda(\nabla_{e_\lambda}f)e_\lambda=0$ and* $\mathrm{Tr}(f)=\mathrm{const.}$, *then M is minimal.*

Proof. If we put $X=Y=e_\lambda$ in (2.1), we have

$$(\nabla_{e_\lambda}f)e_\lambda=\sum_i (u_i(e_\lambda)H_i e_\lambda+h_i(e_\lambda, e_\lambda)U_i) ,$$

from which

$$\sum_\lambda (\nabla_{e_\lambda}f)e_\lambda=\sum_i (H_i \sum_\lambda u_i(e_\lambda)e_\lambda+\sum_\lambda h_i(e_\lambda, e_\lambda)U_i)=\sum_i (H_i U_i+\sum_\lambda h_i(e_\lambda, e_\lambda)U_i)=0 .$$

By means of $\mathrm{Tr}(f)=\mathrm{const.}$ and (4.6), we have $\sum_i h_i(X, U_i)=0$, which was shown in the proof of Theorem 4.5. Therefore $\sum_i g(H_i X, U_i)=\sum_i g(H_i U_i, X)=0$, from which $\sum_i H_i U_i=0$. Thus we find

$$\sum_i \sum_\lambda h_i(e_\lambda, e_\lambda)U_i=0$$

and consequently $\sum_\lambda h_i(e_\lambda, e_\lambda)=0$. Hence M is minimal.

Next, we consider the case of $\lambda_{ij}=\lambda_i\delta_{ij}$ ($\lambda_i^2<1$). In this case, since from (1.7) we have

$$u_j(U_i)=\delta_{ji}-\lambda_j\lambda_i\delta_{ji} ,$$

U_i ($i=1, 2, \cdots, s$) are mutually orthogonal. Consequently these vectors are linearly independent. Furthermore, since from (1.6) we have $fU_i=-\lambda_i U_i$, U_i is an eigenvector of f and $-\lambda_i$ is the corresponding eigenvalue of f.

Thus we have

THEOREM 4.7. *Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. If $\lambda_{ij}=\lambda_i\delta_{ij}$ ($\lambda_i^2<1$) and $\nabla_X f=0$, then M is totally geodesic.*

Similarly, in Theorem 4.5 and Theorem 4.6, we can replace the condition that U_i ($i=1, 2, \cdots, s$) are linearly independent by $\lambda_{ij}=\lambda_i\delta_{ij}$ ($\lambda_i^2<1$).

Especially, we put $\lambda_{ij}=0$, that is, $u_j(U_i)=\delta_{ji}$. In this case, FN_i is tangent to M. Since we have $f^3X=fX$ by means of $fU_i=0$, we obtain the following

THEOREM 4.8. *In a submanifold M of an almost product Riemannian manifold*

$\bar{M}$, if FN_i ($i=1, 2, \cdots, s$) are tangent to M, then the induced structure tensor f satisfies $f^3-f=0$.

It is obvious that the following theorems hold good.

THEOREM 4.9. *In a submanifold M of a locally product Riemannian manifold $\bar{M}$, if FN_i ($i=1, 2, \cdots, s$) are tangent to M and $\nabla_X f=0$, then M is totally geodesic.*

THEOREM 4.10. *In a submanifold M of a locally product Riemannian manifold $\bar{M}$, if FN_i is tangent to M and M is totally umbilical, then M is totally geodesic.*

THEOREM 4.11. *In a submanifold M of a locally product Riemannian manifold $\bar{M}$, if FN_i is tangent to M and $\sum_\lambda (\nabla_{e_\lambda} f)e_\lambda=0$, then M is minimal.*

Furthermore, we assume $s=n$. Since $fU_i=0$ and U_i ($i=1, 2, \cdots, n$) are linearly independent, we get $f=0$. Thus we have

THEOREM 4.12. *In an n-dimensional submanifold M of a locally product Riemannain manifold $\bar{M}$ of dimension $2n$, if FN_i is tangent to M, then M is anti-invariant and totally geodesic.*

§ 5. Submanifolds of an almost product Riemannian manifold (II).

In this section, we assume that U_i ($i=1, 2, \cdots, s$) are not always linearly independent. Let $\{N_1, N_2, \cdots, N_s\}$, $\{\bar{N}_1, \bar{N}_2, \cdots, \bar{N}_s\}$ be orthonormal bases of the normal space $T_P(M)^\perp$. If we put

$$\bar{N}_i=\sum_l k_{li}N_l \,, \tag{5.1}$$

then the matrix (k_{ij}) is an orthogonal matrix and we have (4.2), (4.3), where

$$\bar{u}_i=\sum_l k_{li}u_l \,, \qquad \bar{U}_i=\sum_l k_{li}U_l \,, \tag{5.2}$$

$$\bar{\lambda}_{lh}=\sum_{i,j} k_{il}\lambda_{ij}k_{jh} \,. \tag{5.3}$$

By means of (5.2), if U_i ($i=1, 2, \cdots, s$) are linearly dependent, then $\bar{U}_i$ ($i=1, 2, \cdots, s$) are also linearly dependent. And the greatest number of the linearly independent vector fields in U_i ($i=1, 2, \cdots, s$) is invariant under the transformation (5.1).

Furthermore, because λ_{ij} is symmetric in i and j, from (5.3), we can find that under a suitable transformation (5.1) λ_{ij} reduces to $\bar{\lambda}_{ij}=\lambda_i\delta_{ij}$, where λ_i ($i=1, 2, \cdots, s$) are eigenvalues of (λ_{ij}). In this case, we have $\bar{u}_j(\bar{U}_i)=\delta_{ji}-\lambda_j\lambda_i\delta_{ji}$, that is,

$$\bar{u}_i(\bar{U}_i)=1-\lambda_i^2 \,, \qquad \bar{u}_j(\bar{U}_i)=0 \quad (i\neq j) \,, \tag{5.4}$$

and

(5.5) $$f\bar{U}_i=-\lambda_i\bar{U}_i\,.$$

For h such as $\lambda_h{}^2=1$, $\bar{U}_h$ is a zero vector and $F\bar{N}_h=\lambda_h\bar{N}_h$. Consequently $\bar{N}_h$ is an eigenvector of F. When $\lambda_l{}^2\neq 1$ $(l=1, 2, \cdots, p\leqq s)$, $\bar{U}_l$ $(l=1, 2, \cdots, p)$ are linearly independent, because these vectors are mutually orthogonal.

Therefore Fi_*X, $F\bar{N}_i$ can be written as follows:

(5.6) $$Fi_*X=i_*fX+\sum_{l=1}^{p}\bar{u}_l(X)\bar{N}_l\,, \qquad (p\leqq \min(s, n)),$$

(5.7) $$\begin{cases} F\bar{N}_l=i_*\bar{U}_l+\lambda_l\bar{N}_l & (l=1, 2, \cdots, p), (p<s, p\leqq n), \\ F\bar{N}_h=\qquad\lambda_h\bar{N}_h & (h=p+1, \cdots, s), \end{cases}$$

respectively, where $\lambda_l{}^2\neq 1$ $(l=1, 2, \cdots, p)$ and $\lambda_h{}^2=1$ $(h=p+1, \cdots, s)$. Especially, when $p=s$, in place of (5.7) the following equation holds good:

(5.7)′ $$F\bar{N}_l=i_*\bar{U}_l+\lambda_l\bar{N}_l\,, \qquad \lambda_l{}^2\neq 1 \quad (l=1, 2, \cdots, s).$$

LEMMA 5.1. *Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. A necessary and sufficient condition for U_i $(i=1, 2, \cdots, s)$ to be linearly independent is that at every point of M normals are not the eigenvector of F.*

Proof. We consider the condition for U_i $(i=1, 2, \cdots, s)$ to be linearly dependent. Let N be a unit normal of M which is an eigenvector of F. If we put $\bar{N}_s=N$ and transform the orthonormal basis $\{N_1, N_2, \cdots, N_s\}$ of $T_P(M)^\perp$ to another orthonormal basis $\{\bar{N}_1, \bar{N}_2, \cdots, \bar{N}_s\}$, then $\bar{U}_i$ $(i=1, 2, \cdots, s)$ are linearly dependent, because $\bar{U}_s$ is a zero vector. Consequently, U_i $(i=1, 2, \cdots, s)$ are linearly dependent.

Conversely, if U_i $(i=1, 2, \cdots, s)$ are linearly dependent, then by a suitable transformation of $\{N_1, N_2, \cdots, N_s\}$, we get a zero vector $\bar{U}_s$ and the normal $\bar{N}_s$ corresponding to $\bar{U}_s$ is an eigenvector of F. Thus, the lemma was proved.

LEMMA 5.2. *In a submanifold M of an almost product Riemannian manifold $\bar{M}$, we have*

(5.8) $$\mathrm{Tr}(f)=-\sum_{l=1}^{p}\lambda_l+\sum_{A=p+1}^{n}\varepsilon_A \qquad (p<s, p\leqq n),$$

where λ_l $(l=1, 2, \cdots, p)$ are eigenvalues of (λ_{ij}) satisfying $\lambda_l{}^2\neq 1$ and $\varepsilon_A{}^2=1$ $(A=p+1, \cdots, n)$. Especially, when $p=s$,

$$\mathrm{Tr}(f)=-\mathrm{Tr}(\lambda_{ij})+\sum_{A=s+1}^{n}\varepsilon_A\,, \qquad \varepsilon_A{}^2=1 \quad (s<n),$$

or $$=-\mathrm{Tr}(\lambda_{ij}) \qquad (s=n).$$

Proof. We prove the case of $p<s$. From (5.5), we have $f\bar{U}_l=-\lambda_l\bar{U}_l$ $(l=1, 2, \cdots, p)$, where $\bar{U}_l$ $(l=1, 2, \cdots, p)$ are linearly independent. Thus we get

(5.8).

From Theorem 4.2 and Lemma 5.1, we have

THEOREM 5.3. *Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. If at every point of M normals are not the eigenvector of F and $\nabla_X f=0$, then M is totally geodesic.*

Similarly, in Theorem 4.5 and Theorem 4.6, the condition that U_i $(i=1, 2, \cdots, s)$ are linearly independent can be replaced by the condition that at every point of M normals are not the eigenvector of F.

THEOREM 5.4. *Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. If $\lambda_{ij}=\lambda_i\delta_{ij}$, where $\lambda_l^2\neq 1$ $(l=1, 2, \cdots, p)$, $\lambda_h^2=1$ $(h=p+1, \cdots, s)$ $(p<s,\ p\leqq n)$, and $\nabla_X f=0$, then $h_l(X, Y)=0$ $(l=1, 2, \cdots, p)$.*

Proof. From (2.1) we have

$$\sum_{l=1}^{p}\{u_l(Y)H_lX+h_l(X, Y)U_l\}=0\,.$$

Consequently,

$$\begin{aligned}\sum_l u_l(Y)h_l(X, Z)&=-\sum_l u_l(Z)h_l(X, Y)=\sum_l u_l(X)h_l(Y, Z)\\&=-\sum_l u_l(Y)h_l(X, Z)\,. \qquad (X, Y, Z\in\mathfrak{X}(M))\end{aligned}$$

Thus, we get $\sum_l u_l(Y)h_l(X, Z)=0$, from which $h_l(X, Z)=0$ $(l=1, 2, \cdots, p)$.

Similarly, when $\lambda_{ij}=\lambda_i\delta_{ij}$, where $\lambda_l^2\neq 1$ $(l=1, 2, \cdots, p)$ and $\lambda_h^2=1$ $(h=p+1, \cdots, s)$ $(p<s,\ p\leqq n)$, the following theorems hold good.
If $\mathrm{Tr}(f)=\text{const.}$ and M is totally umbilical, then $h_l(X, Y)=0$ $(l=1, 2, \cdots, p)$.
If $\sum_\lambda(\nabla_{e_\lambda}f)e_\lambda=0$ and $\mathrm{Tr}(f)=\text{const.}$, then $\sum_\lambda h_l(e_\lambda, e_\lambda)=0$ $(l=1, 2, \cdots, p)$.

§6. Invariant submanifolds of an almost product Riemannian manifold. Suppose that M is an invariant submanifold immersed in an almost product Riemannian manifold $\bar{M}$. Then U_i $(i=1, 2, \cdots, s)$ are zero vector fields and consequently (1.2), (1.3) are respectively written as follows.

$$Fi_*X=i_*fX\,, \qquad FN_i=\sum_j\lambda_{ij}N_j\,,$$

where

$$\sum_k\lambda_{jk}\lambda_{ki}=\delta_{ji}\,, \tag{6.1}$$

that is $\sum_k\lambda_{jk}^2=1$, $\sum_k\lambda_{jk}\lambda_{ki}=0$ $(i\neq j)$.

Moreover, from (1.4) and (1.8), we get

$$f^2=I\,,\qquad g(fX, fY)=g(X, Y)\,,\qquad X, Y\in\mathfrak{X}(M)\,.$$

Hence, M is an almost product Riemannian manifold excepting the case where f is trivial.

LEMMA 6.1. *If M is an invariant submanifold of an almost product Riemannian manifold $\bar{M}$, the next equations hold good.*

$$\Phi(i_*X, i_*Y)=\phi(X, Y)\,,\quad (\bar{\nabla}_{i_*Z}\Phi)(i_*X, i_*Y)=(\nabla_Z\phi)(X, Y)\,,\quad X, Y, Z\in\mathfrak{X}(M),$$

where $\Phi(\bar{X}, \bar{Y})=G(F\bar{X}, \bar{Y})$, $\phi(X, Y)=g(fX, Y)$, $\bar{X}, \bar{Y}\in\mathfrak{X}(M)$.

Proof. $\Phi(i_*X, i_*Y)=G(Fi_*X, i_*Y)=G(i_*fX, i_*Y)=g(fX, Y)=\phi(X, Y)$.

Next,

$$(\bar{\nabla}_{i_*Z}\Phi)(i_*X, i_*Y)=\bar{\nabla}_{i_*Z}(\Phi(i_*X, i_*Y))-\Phi(\bar{\nabla}_{i_*Z}i_*X, i_*Y)-\Phi(i_*X, \bar{\nabla}_{i_*Z}i_*Y)\,.$$

On the other hand,

$$\begin{aligned}&\Phi(\bar{\nabla}_{i_*Z}i_*X, i_*Y)\\&=\Phi(i_*\nabla_ZX+\sum h_i(Z, X)N_i, i_*Y)=G(Fi_*\nabla_ZX+\sum h_i(Z, X)FN_i, i_*Y)\\&=G(i_*f\nabla_ZX, i_*Y)=g(f\nabla_ZX, Y)=\phi(\nabla_ZX, Y)\,.\end{aligned}$$

Therefore

$$(\bar{\nabla}_{i_*Z}\Phi)(i_*X, i_*Y)=\nabla_Z(\phi(X, Y))-\phi(\nabla_ZX, Y)-\phi(X, \nabla_ZY)=(\nabla_Z\phi)(X, Y)\,.$$

THEOREM 6.2. *Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. A necessary and sufficient condition for M to be invariant is that the induced structure (f, g) of M is an almost product Riemannian structure whenever f is non-trivial.*

Proof. It is clear that, if M is invariant, then M is an almost product Riemannian manifold whenever f is non-trivial. Conversely, suppose that M with the induced structure (f, g) is an almost product Riemannian manifold. Then, since $\sum_i u_i(X)U_i=0$, we get

$$\sum_i u_i(X)g(U_i, X)=\sum_i u_i(X)^2=0\,,$$

from which $u_i(X)=0$. Hence M is invariant.

THEOREM 6.3. *Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. A necessary and sufficient condition for M to be invariant is that* *the normal space $T_P(M)^{\perp}$ at every point $P \in M$ admits an orthonormal basis consisting of eigenvectors of the matrix F.*

Proof. Suppose that, by the transformation of the basis $\{N_1, N_2, \cdots, N_s\}$, N_i, U_i and λ_{ij} was respectively transformed into $\bar{N}_i$, $\bar{U}_i$ and $\lambda_i\delta_{ij}$, λ_i being eigenvalues of (λ_{ij}). If M is invariant, then we have $F\bar{N}_i=\lambda_i\bar{N}_i$, $\lambda_i^2=1$. Hence $\bar{N}_i$ $(i=1, 2, \cdots, s)$ are eigenvectors of F.

Conversely, suppose that $\bar{N}_i$ $(i=1, 2, \cdots, s)$ are eigenvectors of F. Then, by virtue of $F\bar{N}_i=\lambda_i\bar{N}_i$ $(\lambda_i^2=1)$, we obtain $\bar{U}_i=0$. Consequently M is invariant.

THEOREM 6.4. *If M is an invariant submanifold of a locally product Riemannian manifold $\bar{M}$, then M is a locally product Riemannian manifold whenever f is non-trivial.*

Proof. Making use of Lemma 6.1 (or (2.1)), from $\bar{\nabla}F=0$ we can easily obtain $\nabla_X f=0$.

THEOREM 6.5 [5]. *In a submanifold M of a locally product Riemannian manifold $\bar{M}$, if the equations*

$$\text{(i)}\ \begin{cases} Fi_*X=i_*X\,, \\ FN_i=-N_i \end{cases} \qquad \text{or} \qquad \text{(ii)}\ \begin{cases} Fi_*X=-i_*X\,, \\ FN_i=N_i \end{cases}$$

are satisfied, then M is totally geodesic.

Proof. In the case (i), we have $f=I$. Therefore from (2.3) we get $H_iX=0$. Hence M is totally geodesic.

Similarly, we obtain

THEOREM 6.6. *In a submanifold M of a locally product Riemannian manifold $\bar{M}$, if the equations*

$$\text{(i)}\ \begin{cases} Fi_*X=i_*X\,, \\ FN_l=-N_l\ (l=1, 2, \cdots, q) \\ FN_h=N_h\ (h=q+1, \cdots, s) \end{cases} \quad \text{or} \quad \text{(ii)}\ \begin{cases} Fi_*X=-i_*X\,, \\ FN_l=N_l\ (l=1, 2, \cdots, q)\,, \\ FN_h=-N_h\ (h=q+1, \cdots, s) \end{cases}$$

are satisfied, then $h_l(X, Y)=0$ $(l=1, 2, \cdots, q<s)$.

THEOREM 6.7. *Let M be an invariant submanifold of a locally product Riemannian manifold $\bar{M}$. If M is totally umbilical and $\{\mathrm{Tr}(f)\}^2 \neq n^2$ (or equivalently, f is non-trivial), then M is totally geodesic.*

Proof. From (2.2), we have $h_i(X, fY)=\sum_j \lambda_{ij}h_j(X, Y)$. If we put $h_i(X, Y)=\sigma_i g(X, Y)$, we get $\sigma_i g(X, fY)=\sum_j \lambda_{ij}\sigma_j g(X, Y)$. Substituting $X=Y=e_\lambda$, we find $\sigma_i \sum_\lambda g(e_\lambda, fe_\lambda)=n\sum \lambda_{ij}\sigma_j$, that is,

$$\mathrm{Tr}(f)\sigma_i=n\sum_j \lambda_{ij}\sigma_j\,. \tag{6.2}$$

We multiply the above equation by λ_{ji} and sum for i. Then we have

$$\mathrm{Tr}(f)\sum_i \lambda_{ji}\sigma_i=n\sum_i\sum_k \lambda_{ji}\lambda_{ik}\sigma_k=n\sigma_j$$

by virtue of (6.1). Consequently

$$\sigma_j=\frac{1}{n}\mathrm{Tr}(f)\sum_i \lambda_{ji}\sigma_i\,.$$

Substituting the above equation into (6.2), we have

$$\{\mathrm{Tr}(f)\}^2\sum_j \lambda_{ij}\sigma_j=n^2\sum_j \lambda_{ij}\sigma_j\,,$$

from which $\sum_j \lambda_{ij}\sigma_j=0$. Since $\sum_i\sum_j \lambda_{hi}\lambda_{ij}\sigma_j=\sigma_h=0$ $(h=1, 2, \cdots, s)$, M is totally geodesic.

§7. Anti-invariant submanifolds of an almost product Riemannian manifold. Last we consider an anti-invariant submanifold M immersed in an almost product Riemannian manifold $\bar{M}$. In this case, Fi_*X, FN_i are written as follows:

$$Fi_*X=\sum_i u_i(X)N_i\,, \tag{7.1}$$

$$FN_i=i_*U_i+\sum_j \lambda_{ij}N_j\,. \tag{7.2}$$

And (1.4), (1.6) become

$$\sum_i u_i\otimes U_i=I\,, \tag{7.3}$$

$$\sum_j \lambda_{ij}U_j=0\,. \tag{7.4}$$

In order that the solution of $u_i(X)=0$ $(i=1, 2, \cdots, s)$ does not exist except zero vector, it is necessary and sufficient that the rank of the matrix $(U_1U_2\cdots U_s)$ is n and consequently $s\geqq n$.

When $s=n$, U_i $(i=1, 2, \cdots, n)$ are linearly independent and we have $\lambda_{ij}=0$ from (7.4). Thus we obtain the following theorem by virtue of Theorem 4.2.

THEOREM 7.1. *In a locally product Riemannian manifold $\bar{M}$ of dimension $2n$, an antiinvariant submanifold M of dimension n is totally geodesic.*

When $s>n$, U_i $(i=1, 2, \cdots, s)$ are linearly dependent. Suppose that, by a suitable transformation (5.1) of the orthonormal basis $\{N_1, N_2, \cdots, N_s\}$, N_i, U_i and

λ_{ij} are transformed to $\bar{N}_i$, $\bar{U}_i$ and $\lambda_i\delta_{ij}$ respectively, which λ_i are eigenvalues of (λ_{ij}). Then, since (7.4) becomes $\lambda_i\bar{U}_i=0$ $(i=1, 2, \cdots, s)$, we can assume that $\bar{U}_l$ $(l=1, 2, \cdots, n)$ are linearly independent, $\bar{U}_h$ $(h=n+1, \cdots, s)$ zero vectors, $\lambda_l=0$ $(l=1, 2, \cdots, n)$ and $\lambda_h^2=1$ $(h=n+1, \cdots, s)$. Consequently $\bar{U}_l$ $(l=1, 2, \cdots, n)$ are unit vectors which are mutually orthogonal and $\bar{N}_h$ $(h=n+1, \cdots, s)$ are eigenvectors of F.

Now, denote by $\{e_1, e_2, \cdots, e_n\}$ the orthonormal basis of $T_P(M)$. If we put $Fi_*e_k=N'_k$, $Fi_*e_l=N'_l$ $(k, l=1, 2, \cdots, n\}$, then $G(N'_k, N'_l)=\delta_{kl}$. Therefore, we can take the normals $\bar{N}_l$ $(l=1, 2, \cdots, n)$ such as $Fi_*e_l=\bar{N}_l$.

Thus we obtain

THEOREM 7.2. *If M is an anti-invariant submanifold of a locally product Riemannian manifold $\bar{M}$, for the normals N_l $(l=1, 2, \cdots, n)$ corresponding to the orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of $T_P(M)$, $h_l(X, Y)=0$ $(l=1, 2, \cdots, n)$.*

On a Riemannian product manifold $\bar{M}=\bar{M}_1\times\bar{M}_2$, K. Yano and M. Kon proved Theorem 7.1 and Theorem 7.2 [6].

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