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ON THE MINIMUM MODULUS OF A SUBHARMONIC OR AN ALGEBROID FUNCTION OF $\mu_* < 1/2$

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0. Introduction. Let $y(z)$ be an N-valued entire algebroid function defined by an irreducible equation

$$F(z, y) = y^N + A_1(z) y^{N-1} + \cdots + A_N(z) = 0. \tag{1}$$

Denoting the j-th determination of y by y_j, we set

$$M(r, y) = \max_{|z|=r} \max_{1 \le j \le N} |y_j(z)|, \qquad m^*(r, y) = \min_{|z|=r} \max_{1 \le j \le N} |y_j(z)|.$$

Let A be the system $(1, A_1, \cdots, A_N)$ and put

$$B(z) = \max_{1 \le j \le N} |A_j(z)|, \quad M(r, B) = \max_{|z|=r} B(z), \quad m^*(r, B) = \min_{|z|=r} B(z).$$

Then Ozawa [12] showed that

$$\frac{N \log^+ m^*(r, y)}{\log M(r, y)} \geqq \frac{\log m^*(r, B) + O(1)}{\log M(r, B) + O(1)}. \tag{2}$$

And he obtained the following theorem by making use of Kjellberg's method [10].

THEOREM A. *Let $y(z)$ be an N-valued entire algebroid function of lower order μ, $0 \leqq \mu < 1/2$. Then*

$$\overline{\lim_{r \to \infty}} \frac{N^2 \log m^*(r, y)}{\log M(r, y)} \geqq \cos \pi\mu. \tag{3}$$

We can improve his result by two different methods. The first method is due to Baernstein [3]. He proved there

THEOREM B. *Let f be a nonconstant entire function. Let β and λ be numbers with $0 < \lambda < \infty$, $0 < \beta \leqq \pi$, $\beta\lambda < \pi$. Then either*

(a) *there exist arbitrarily large values of r for which the set of θ satisfying $\log|f(re^{i\theta})| > \cos \beta\lambda \log M(r, f)$ contains an interval of length at least 2β, or else*

(b) $\lim_{r \to \infty} r^{-\lambda} \log M(r, f)$ *exists, and is positive or ∞.*

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It turns out by a minute observation of his papers [2], [3] that Theorem B still holds when we replace $|f|$, $\log|f(re^{i\theta})|$ and $\log M(r, f)$ by $B(z)$, $\log B(re^{i\theta})$ and $\log M(r, B)$, respectively. Hence choosing $\beta=\pi$ and $\lambda=\mu+\varepsilon$ in Theorem B, it follows from (2), Theorem B and the above remark that

$$\overline{\lim_{r\to\infty}}\frac{N\log m^*(r, y)}{\log M(r, y)}\geqq\cos\pi\mu .$$

The second method is to make use of the notion of a local form of the Phragmén-Lindelöf indicator. This notion was introduced by Edrei [7] and is closely related to Pólya peaks. Drasin and Shea [6] proved that Pólya peaks of order ρ exist if and only if $\rho\in[\mu_*, \lambda_*]$, $\rho<\infty$, where

$$\mu_*=\mu_*(T)=\inf\left\{\rho:\ \underline{\lim_{r, C\to\infty}}\frac{T(Cr, A)}{C^\rho T(r, A)}=0\right\},$$

(4)

$$\lambda_*=\lambda_*(T)=\sup\left\{\rho:\underline{\lim_{r, C\to\infty}}\frac{T(Cr, A)}{C^\rho T(r, A)}=\infty\right\}.$$

It is easy to see that $\mu_*\leqq\mu\leqq\lambda\leqq\lambda_*$, where λ and μ are the order and the lower order of T, respectively. Edrei defined a local indicator for a sequence $\{f_m(z)\}_1^\infty$ of analytic functions such that $f_m(z)$ is regular and single-valued in the annulus: $r'_m\leqq|z|\leqq r''_m$ $(m=1, 2, \cdots)$. However, his definition is naturally extended for a sequence $\{B_m(z)\}_1^\infty$ of subharmonic functions. Exact definition of the local indicator for a sequence $\{B_m(z)\}_1^\infty$ will be stated in § 1. In § 2, we shall state some elementary facts on subharmonic functions defined in $\boldsymbol{C}$. In § 3 we shall prove the following Theorem 1. The case when $u(z)=\log|f(z)|$, and $f(z)$ is entire, is due to Edrei [7, Theorem 1]. In what follows, for a subharmonic function u in $\boldsymbol{C}$, we put

$$N(r, u)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}u(re^{i\theta})d\theta,\quad M(r, u)=\sup_{-\pi\leqq\theta\leqq\pi}u(re^{i\theta}),\quad m^*(r, u)=\inf_{-\pi\leqq\theta\leqq\pi}u(re^{i\theta}).$$

THEOREM 1. *Let $u(z)$ be a nonconstant subharmonic function in $\boldsymbol{C}$ and let $T(r, u)=N(r, u^+)$. Assume that $\mu_*=\mu_*(T)<1$. Let $\{r_m\}_1^\infty$ be a sequence of Pólya peaks of order ρ $(\mu_*\leqq\rho\leqq\lambda_*,\ 0<\rho<1)$ of $T(r, u)$. Then given $\varepsilon>0$, it is possible to find a bound $s=s(\varepsilon)>0$, independent of m, and such that, in $\bigcup_{m=1}^\infty[r_me^{-s}, r_me^s]$ there exist arbitrarily large values of r satisfying the inequality*:

(5) $$m^*(r, u)>(\cos\pi\rho-\varepsilon)M(r, u).$$

COROLLARY 1. *Let $y(z)$ be an N-valued entire algebroid function and have $\mu_*<1/2$. Let $\{r_m\}_1^\infty$ be a sequence of Pólya peaks of order ρ of $T(r, y)$ $(\mu_*\leqq\rho\leqq\lambda_*)$ and let $0<\rho<1/2$. Then given $\varepsilon>0$, it is possible to find a bound $s=s(\varepsilon)$, independent of m, and such that, in $\bigcup_{m=1}^\infty[r_me^{-s}, r_me^s]$ there exist arbitrarily large values* *of r satisfying the inequality*:

(6) $$N \log m^*(r, y) > (\cos \pi\rho - \varepsilon) \log M(r, y) .$$

This is also an improvement of Theorem A. However, since $\mu_* \leqq \mu$ (Equality does not always hold.), the second method is superior to the first one for this problem.

It is natural to consider an analogous problem to Theorem 1 for δ-subharmonic functions—differences of subharmonic functions. That is, for a δ-subharmonic function $v(z) = u^{(1)}(z) - u^{(2)}(z)$ of $\mu_* < 1/2$, what can we say about the relation between $m^*(r, v) = \inf_{-\pi \leqq \theta \leqq \pi} v(re^{i\theta})$ and $T(r, v) = N(r, v^+) + N(r, u^{(2)})$? In [1], Anderson and Baernstein considered a more general problem for δ-subharmonic functions. The following theorem is a part of their consideration. Here we put for a δ-subharmonic function $v = u^{(1)} - u^{(2)}$ in $\boldsymbol{C}$

$$\delta(\infty, v) = 1 - \overline{\lim_{r \to \infty}} \frac{N(r, u^{(2)})}{T(r, v)} .$$

THEOREM B. *Let $v(z) = u^{(1)}(z) - u^{(2)}(z)$ be a δ-subharmonic function in $\boldsymbol{C}$ of lower order μ, $0 \leqq \mu < 1/2$. And assume that* $\cos \pi\mu - 1 + \delta(\infty, v) > 0$. *Then it is possible to find a positive number R and Pólya peak sequence $\{r_m\}_1^\infty$ of order μ of $T(r, v)$ satisfying the inequality*:

$$m^*(Rr_m, v) > \left\{ \frac{\pi\mu[\cos \pi\mu - (1 - \delta(\infty, v))]}{\sin \pi\mu} - \varepsilon \right\} T(Rr_m, v) .$$

Using the concept of a local indicator, we can prove the following Theorem 2. The case when $v(z) = \log|f(z)| = \log|f_1(z)| - \log|f_2(z)|$, where $f = f_1/f_2$ is meromorphic, is due to Edrei [7, Theorem 2]. (The proof of Theorem 2 will be omitted.)

THEOREM 2. *Let $v(z) = u^{(1)}(z) - u^{(2)}(z)$ be a δ-subharmonic function in $\boldsymbol{C}$ and have $\mu_* < 1/2$. Assume that $v(z)$ satisfies the following conditions* (i) *and* (ii):

(i) $N(r, u^{(1)}) \sim T(r, v)$ $(r \to \infty)$,

(ii) $\delta(\infty, v) + \cos \pi\rho - 1 = k > 0$, *where* $\mu_* \leqq \rho \leqq \lambda_*$, $0 < \rho < 1/2$.

And let $\{r_m\}_1^\infty$ be a sequence of Pólya peaks of order ρ of $T(r, v)$. Then given $\varepsilon > 0$, it is possible to find a bound $s = s(\varepsilon) > 0$, independent of m, and such that, in $\bigcup_{m=1}^\infty [r_m e^{-s}, r_m e^s]$ there exist arbitrarily large values of r satisfying the inequality:

(7) $$m^*(r, v) > \frac{\pi\rho(k - \varepsilon)}{\sin \pi\rho} T(r, v) .$$

COROLLARY 2. *Let $y(z)$ be an N-valued algebroid function and have $\mu_* < 1/2$.*

Assume that ρ *satisfies the following three conditions*;

(i) $\mu_* \leqq \rho \leqq \lambda_*$, (ii) $0<\rho<1/2$, (iii) $\delta(\infty, y)+\cos \pi\rho-1=k>0$.

Let $\{r_m\}_1^\infty$ *be a sequence of Pólya peaks of order* ρ *of* $T(r, y)$. *Then given* $\varepsilon>0$, *it is possible to find a bound* $s=s(\varepsilon)>0$, *independent of* m, *and such that in* $\bigcup_{m=1}^{\infty}[r_m e^{-s}, r_m e^s]$ *there exist arbitrarily large values of* r *satisfying the inequality*:

$$\log m^*(r, y)>\frac{\pi\rho(k-\varepsilon)}{\sin \pi\rho}T(r, y). \tag{8}$$

The derivation of Corollary 2 will be done in § 4.

Finally, in § 5, as another application of a local indicator, we shall show the following theorem.

THEOREM 3. *Let* $v=u^{(1)}-u^{(2)}$ *be a* δ*-subharmonic function in* $\boldsymbol{C}$ *and have* $\mu_*<1/2$. *Assume that* $N(r, u^{(1)})\sim T(r, v)$ $(r\to\infty)$ *and let* ρ *satisfy the following three conditions*:

(i) $\mu_* \leqq \rho \leqq \lambda_*$, (ii) $0<\rho<1/2$, (iii) $\cos \pi\rho-1+\delta(\infty, v)/(2-\delta(\infty, v))=k_2>0$.

Further let $\{r_m\}_1^\infty$ *be a sequence of Pólya peaks of order* ρ *of* $T(r, v)$, *and let*

$$m_2(r, v)=\{N(r, v^2)\}^{1/2}.$$

Then given $\varepsilon>0$, *it is possible to find a bound* $s=s(\varepsilon)>0$, *independent of* m, *and such that in* $\bigcup_{m=1}^{\infty}[r_m e^{-s}, r_m e^s]$ *there exist arbitrarily large values of* r *satisfying the inequality*:

$$m^*(r, v)>\left\{\frac{k_2}{\sqrt{1/2+(\sin 2\pi\rho)/4\pi\rho}}-\varepsilon\right\}m_2(r, v). \tag{9}$$

In particular, if v *is subharmonic, then the assumption*: $N(r, u^{(1)})\sim T(r, v)$ *can be dropped.*

If $\delta(\infty, v)=1$, the estimate (9) is best possible. For example, consider a subharmonic function:

$$v(z)=\frac{\pi r^\rho}{\sin \pi\rho}\cos \rho\theta.$$

For an N-valued algebroid function $y(z)$, we introduce the following quantity:

$$C(r, y)=\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}\sum_{j=1}^{N}\{\log^+|y_j(re^{i\theta})|\}^2 d\theta\right]^{1/2} (y_j: j\text{-th determination of } y).$$

COROLLARY 3. *Let $y(z)$ be an N-valued algebroid function and have $\mu_*<1/2$. And let ρ satisfy the following three conditions:*

(i) $\mu_*\leqq\rho\leqq\lambda_*$, (ii) $0<\rho<1/2$, (iii) $k_2=\cos\pi\rho-1+\delta(\infty, y)/(2-\delta(\infty, y))>0$.

Further let $\{r_m\}_1^\infty$ be a sequence of Pólya peaks of order ρ of $T(r, y)$. Then given $\varepsilon>0$, it is possible to find a bound $s=s(\varepsilon)>0$, independent of m, and such that in $\bigcup_{m=1}^{\infty}[r_m e^{-s}, r_m e^s]$ there exist arbitrarily large values of r satisfying the inequality:

$$N\log m^*(r, y)>\frac{k_2-\varepsilon}{\sqrt{1/2+(\sin 2\pi\rho)/4\pi\rho}}C(r, y).$$

1. Definition of the local indicator of order ρ of a sequence $\{B_m(z)\}_1^\infty$ of subharmonic functions.

(i) three infinite sequences of positive numbers $\{r'_m\}_1^\infty$, $\{r_m\}_1^\infty$, $\{r''_m\}_1^\infty$ such that $r'_m<r_m<r''_m<r'_{m+1}$ $(m=1, 2, \cdots)$, and such that, as $m\to\infty$

$$r_m/r'_m \longrightarrow \infty, \qquad r''_m/r_m \longrightarrow \infty.$$

(ii) a sequence $\{B_m(z)\}_1^\infty$ such that $B_m(z)$ is subharmonic in the annulus: $r'_m<|z|<r''_m$.

(iii) a strictly positive sequence $\{V(r_m)\}_1^\infty$ and a quantity ρ $(0<\rho<\infty)$. We then define a sequence $\{V_m(z)\}_1^\infty$ of analytic "comparison functions":

$$V_m(z)=V_m(r)e^{i\rho\theta}\equiv V(r_m)\left(\frac{r}{r_m}\right)e^{i\rho\theta} \qquad (z=re^{i\theta}).$$

The symbol $V_m(r)$ always refers to the choice of $\theta=0$.

(iv) Consider the intervals $I_m=[r'_m, r''_m]$ $(m=1, 2, \cdots)$ as well as the intervals $I_m(s)=[r_m e^{-s}, r_m e^s]$ $(m=1, 2, \cdots, s=1, 2, \cdots)$, and let

$$\Lambda=\bigcup_{m=1}^{\infty}I_m, \qquad \Lambda(s)=\bigcup_{m=1}^{\infty}I_m(s) \qquad (s=1, 2, \cdots).$$

(v) Let the sequence $\{B_m(z)\}_1^\infty$ be chosen so that

$$\varlimsup_{\substack{r\to\infty \\ r\in\Lambda}}\frac{M(r, B)}{V(r)}<\infty,$$

where $B(z)$ stands for $B_m(z)$ in the annulus: $r'_m<|z|<r''_m$. $(m=1, 2, \cdots)$. We set for every real value of θ,

$$h_s(\theta)=\varlimsup_{\substack{r\to\infty \\ r\in\Lambda(s)}}\frac{B(re^{i\theta})}{V(r)} \qquad (s=1, 2, \cdots),$$

and consider

$$h(\theta)=\lim_{s\to\infty}h_s(\theta).$$

The real function $h(\theta)$ is, by definition, the local indicator of order ρ of $\{B_m(z)\}_1^\infty$ at the peaks $\{r_m\}_1^\infty$. With this definition, Edrei's Fundamental Lemma can be extended straightforwardly for the sequence $\{B_m(z)\}_1^\infty$ of subharmonic functions (For the proof, cf. [7, pp. 159-162]).

Fundamental Lemma. *Let $h(\theta)$ be the local indicator of order ρ $(0<\rho<\infty)$ of $\{B_m(z)\}_1^\infty$ at the peaks $\{r_m\}_1^\infty$. Let θ_1, θ_2 be given such that $0<\theta_2-\theta_1<\pi/\rho$, and let the constants a, b be such that the sinusoid $H(\theta)=a\cos\rho\theta+b\sin\rho\theta$ satisfies the conditions $h(\theta_1)\leqq H(\theta_1)$, $h(\theta_2)\leqq H(\theta_2)$. Then given $\varepsilon>0$ and any integer $s>0$, there exists a bound $r_0=r_0(\varepsilon, s, a, b, \theta_1, \theta_2)$, independent of θ, such that for $r\in\Lambda(s)$, $\theta_1\leqq\theta\leqq\theta_2$, $r\geqq r_0$*

$$B(re^{i\theta})\leqq(H(\theta)+\varepsilon)V(r).$$

From Fundamental Lemma, we immediately have $h(\theta)\leqq H(\theta)$ $(\theta_1\leqq\theta\leqq\theta_2)$, that is, the subtrigonometric character of $h(\theta)$. It is known that many important properties of an indicator depend only on its subtrigonometric character (cf. [5]). For example, we have the following fact (cf. [5, pp. 42-45]).

Let $h(\theta)$ be the local indicator of order ρ of $\{B_m(z)\}_1^\infty$. Assume that $h(\theta)\not\equiv-\infty$, and let θ_1, θ_2, θ_3 be such that $0<\theta_2-\theta_1<\pi/\rho$, $0<\theta_3-\theta_2<\pi/\rho$. Then

$$\begin{vmatrix} h(\theta_1) & \cos\rho\theta_1 & \sin\rho\theta_1 \\ h(\theta_2) & \cos\rho\theta_2 & \sin\rho\theta_2 \\ h(\theta_3) & \cos\rho\theta_3 & \sin\rho\theta_3 \end{vmatrix}\geqq 0.$$

In particular, if $0\leqq\theta<\pi/\rho$, then

$$\frac{h(-\theta)+h(\theta)}{2}\geqq h(0)\cos\rho\theta. \tag{10}$$

2. Some elementary facts on subharmonic functons defined in C. Since we are interested in results for large values of r in Theorem 1, we may assume that $u(z)$ is harmonic in a neighborhood of the orign. Further we may prove Theorem 1 for $u(0)=0$. In fact, assume that Theorem 1 is valid for an arbitrary subharmonic function $v(z)$ of $\mu_*<1/2$ which is harmonic in a neighborhood of $z=0$ and satisfies $v(0)=0$. Take an arbitrary subharmonic function $u(z)$ of $\mu_*<1/2$ which is harmonic in a neighborhood of $z=0$. Put $v(z)=u(z)-u(0)$. By the Riesz representation theorem there exists a positive Borel measure ν and C such that for $|z|<R$ $(0<R<\infty)$

$$\begin{aligned} v(z)&=h(z)+\int_{|\zeta|<R}\log|z-\zeta|\,d\nu(\zeta) \\ &=h(z)+\int_{|\zeta|<R}\log|\zeta|\,d\nu(\zeta)+\int_{|\zeta|<R}\log\left|1-\frac{z}{\zeta}\right|d\nu(\zeta), \end{aligned} \tag{11}$$

where $h(z)$ is harmonic in $|z|<R$. Let $n(r)=\nu(|\zeta|<r)$. Then Jensen's formula for subharmonic functions (cf. [9]) gives

$$(12)\qquad N(r, v)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}v(re^{i\theta})d\theta=\int_0^r\frac{n(t)}{t}dt\leqq T(r, v)\leqq M(r, v),$$

which implies that "$N(r, v)$ is bounded. $\Leftrightarrow v(z)$ is harmonic in $\boldsymbol{C}$." Assume now that $T(r, v)$ is bounded. Then v^+ is harmonic in $\boldsymbol{C}$. Since $v^+\geqq 0$, this shows that v^+ is a constant. Therefore $N(r, v)$ is bounded, so that v is harmonic in $\boldsymbol{C}$. However, since v is bounded above, v must be a constant. Hence the nonconstancy of v implies that $T(r, v)\nearrow\infty$ and that $M(r, v)\nearrow\infty$ $(r\to\infty)$. Thus there exists a $r_0=r_0(\varepsilon)>0$ such that $r\geqq r_0$ implies

$$(13)\qquad |u(0)|\left(1-\cos\pi\rho+\frac{\varepsilon}{2}\right)<\frac{\varepsilon}{2}M(r, v).$$

Further by the above assumption, there exists a sequence $\{\chi_n\}_1^\infty\nearrow\infty$ contained in $\bigcup_{m=1}^{\infty}[r_m e^{-s}, r_m e^s]$ ($\{r_m\}_1^\infty$: a Pólya peak sequence of order ρ of $T(r, u)$; $s=s(\varepsilon)$: a positive integer) satisfying the inequality:

$$(14)\qquad m^*(\chi_n, v)>\left(\cos\pi\rho-\frac{\varepsilon}{2}\right)M(\chi_n, v).$$

It follows from (13) and (14) that

$$m^*(\chi_n, u)>(\cos\pi\rho-\varepsilon)M(\chi_n, u)\qquad(\chi_n\geqq r_0).$$

In what follows we may assume that $u(z)$ is harmonic in a neighborhood of 0 and satisfies $u(0)=0$.

Now, we put

$$(15)\qquad u_1(z, R)=\int_{|\zeta|<R}\log\left|1-\frac{z}{\zeta}\right|d\nu(\zeta),$$

$$u_2(z, R)=\int_{|\zeta|<R}\log\left|1+\frac{z}{|\zeta|}\right|d\nu(\zeta)=\int_0^R\log\left|1+\frac{z}{t}\right|dn(t).$$

Then $u_1(z, R)$ and $u_2(z, R)$ are subharmonic in $\boldsymbol{C}$ and they satisfy

$$(16)\qquad m^*(r, u_2)\leqq m^*(r, u_1)\leqq M(r, u_1)\leqq M(r, u_2).$$

Next, let

$$(17)\qquad u_3(z, R)=u(z)-u_1(z, R).$$

Then, using (17), (11), and (15), we have

$$u_3(z, R)=h(z)+\int_{|\zeta|<R}\log|\zeta|d\nu(\zeta)\qquad(|z|<R).$$

which shows that $u_3(z, R)$ is harmonic in $|z|<R$. Let $f(z)$ be regular in $|z|<R$ such that $\mathrm{Re}\, f(z)=u_3(z, R)$ and $f(0)=0$. Hence by a theorem of Carathéodory

$$|f(z)|\leqq\frac{2|z|}{R-|z|}M(R, u_3) \qquad (|z|<R). \tag{18}$$

Further, an estimate due to Kjellberg [10, p. 92] or Barry [4, p. 182] gives

$$M(2R, u_3)\leqq M(2R, u). \tag{19}$$

Combining (18) and (19) we obtain

$$|u_3(z, R)|\leqq|f(z)|<\frac{4M(2R, u)}{R}r \qquad \left(|z|=r<\frac{R}{2}\right). \tag{20}$$

3. Proof of Theorem 1. Since we are mainly interested in Corollary 1, we shall prove only for the case of $\mu_*<1/2$, $\mu_*\leqq\rho\leqq\lambda_*$ and $0<\rho<1/2$. Let $\{r_m\}_1^\infty$ be a sequence of Pólya peaks of order ρ of $T(r, u)$. And let $\{r'_m\}_1^\infty$, $\{r''_m\}_1^\infty$, $\{\varepsilon_m\}_1^\infty$ be the associated sequences with Pólya peaks $\{r_m\}_1^\infty$ of order ρ. Choose $\{V(r_m)\}_1^\infty$ as follows.

$$V(r_m)=(1+\varepsilon_m)T(r_m, u) \qquad (m=1, 2, \cdots). \tag{21}$$

This implies

$$T(r, u)<V(r) \qquad (r\in\Lambda). \tag{22}$$

Put

$$B_m(z)=u_2(z, r''_m/4)=\int_0^{r''_m/4}\log\left|1+\frac{z}{t}\right|dn(t) \qquad (r'_m\leqq|z|\leqq r''_m), \tag{23}$$

and we consider the local indicator $h(\theta)$ of order ρ of $\{B_m(z)\}_1^\infty$ at the peaks $\{r_m\}_1^\infty$.

(i) Existence of $h(\theta)$: By definition we may show that

$$\varlimsup_{\substack{r\to\infty\\ r\in\Lambda}}\frac{B(r)}{V(r)}<\infty. \tag{24}$$

Put

$$n_m^{(0)}(t)=\begin{cases} n(t) & \left(t\leqq\dfrac{r''_m}{4}\right)\\ n\left(\dfrac{r''_m}{4}\right) & \left(t>\dfrac{r''_m}{4}\right)\end{cases}, \qquad N_m^{(0)}(t)=\int_0^t\frac{n_m^{(0)}(r)}{r}dr.$$

Then

$$B_m(r)=\int_0^\infty \log\Big(1+\frac{r}{t}\Big)dn_m^{(0)}(t)=r\int_0^\infty \frac{n_m^{(0)}(t)}{(t+r)t}dt=r\int_0^\infty \frac{dN_m^{(0)}(t)}{t+r}$$

$$=r\int_0^\infty \frac{N_m^{(0)}(t)}{(t+r)^2}dt=r\Big(\int_0^{r'_m}+\int_{r'_m}^{r''_m/4}+\int_{r''_m/4}^\infty\Big)\frac{N_m^{(0)}(t)}{(t+r)^2}dt\,.$$

Since

$$N_m^{(0)}(t)=\begin{cases} N(t) & \Big(t\leqq \dfrac{r''_m}{4}\Big) \\ N\Big(\dfrac{r''_m}{4}\Big)+n\Big(\dfrac{r''_m}{4}\Big)\log\Big(\dfrac{4t}{r''_m}\Big) & \Big(t>\dfrac{r''_m}{4}\Big), \end{cases}$$

we easily obtain

$$B_m(r)\leqq \frac{r'_m}{r}N(r'_m)+A\frac{r}{r''_m}N(r''_m)+r\int_{r'_m}^{r''_m/4}\frac{N(t)}{(t+r)^2}dt$$

$$(A:\text{ an absolute constant}).$$

By (22) we have

$$N(r,\ u)\leqq T(r,\ u)< V(r)\qquad (r\in I_m)\,.$$

Thus for $r\in I_m$

$$B_m(r)\leqq \frac{r'_m}{r}V(r'_m)+A\frac{r}{r''_m}V(r''_m)+r\int_{r'_m}^{r''_m/4}\frac{V(t)}{(t+r)^2}dt$$

$$\leqq\Big(\frac{r'_m}{r}\Big)V(r)\Big(\frac{r'_m}{r}\Big)^\rho+A\Big(\frac{r}{r''_m}\Big)V(r)\Big(\frac{r''_m}{r}\Big)^\rho+V(r)\int_0^\infty\frac{x^\rho}{(1+x)^2}dx$$

$$<V(r)\Big[1+A+\int_0^\infty\frac{x^\rho}{(1+x)^2}dx\Big]\,.$$

This shows (24).

(ii) $h(0)\geqq 1$: By definition we may prove

$$\varliminf_{m\to\infty}\frac{B(r_m)}{V(r_m)}\geqq 1\,. \tag{25}$$

From (21), (17), (16), (23) and (20) it follows that

$$\frac{V(r_m)}{1+\varepsilon_m}=T(r_m,\ u)\leqq M(r_m,\ u)\leqq M(r_m,\ u_1)+M(r_m,\ u_3)$$

$$\leqq M(r_m,\ u_2)+M(r_m,\ u_3)\leqq B(r_m)+\frac{4M(r''_m/2,\ u)}{r''_m/4}r_m$$

$$=B(r_m)+16M(r''_m/2,\ u)\frac{r_m}{r''_m}$$

$$\leqq B(r_m)+48T(r''_m, u)\frac{r_m}{r''_m}\leqq B(r_m)+48V(r_m)\left(\frac{r_m}{r''_m}\right)^{1-\rho},$$

where we used the fact that $M(r, u)\leqq 3T(2r, u)$ (cf. [9, Chapter 3]). Since $r_m/r''_m \to 0$ as $m\to\infty$, (25) follows.

(iii) By (23) we have $B_m(re^{i\theta})=B_m(re^{-i\theta})$ for $0\leqq\theta\leqq\pi$, which implies $h(\theta)=h(-\theta)$ $(0\leqq\theta\leqq\pi)$. It follows from this, (10) and (ii) that

$$h(\theta)\geqq h(0)\cos\rho\theta\geqq\cos\rho\theta>0. \tag{26}$$

(iv) By (17), (16), (20) and (22) we have for $r\leqq r''_m/8$

$$\begin{aligned} m^*(r, u)&\geqq m^*(r, u)+m^*(r, u_3)\\ &\geqq m^*(r, u_2)-\frac{4M(r''_m/2, u)}{r''_m/4}r\geqq m^*(r, u_2)-48V(r''_m)\frac{r}{r''_m}\\ &=m^*(r, u_2)-48\left(\frac{r}{r''_m}\right)^{1-\rho}V(r). \end{aligned} \tag{27}$$

In the same way we obtain

$$M(r, u)\leqq M(r, u_2)+48\left(\frac{r}{r''_m}\right)^{1-\rho}V(r) \qquad (r\leqq r''_m/8). \tag{28}$$

(v) For given $\eta>0$ (small enough), choose s (a positive integer) such that $h_s(\pi)>h(\pi)-\eta$. By the definition of $h_s(\pi)$, there exists a sequence $\{\chi_n\}_1^\infty(\nearrow\infty) \subset\bigcup_{m=1}^{\infty}[r_me^{-s}, r_me^{s}]$ satisfying $B(-\chi_n)>(h_s(\pi)-\eta)V(\chi_n)>(h(\pi)-2\eta)V(\chi_n)$. Hence by (27) and (26)

$$\begin{aligned} m^*(\chi_n, u)&>(h(\pi)-3\eta)V(\chi_n) \qquad (n\geqq n_0(\eta, s))\\ &>(h(0)\cos\pi\rho-3\eta)V(\chi_n)\geqq(\cos\pi\rho-3\eta)V(\chi_n). \end{aligned} \tag{29}$$

We may assume that $\cos\pi\rho-3\eta>0$. On the other hand, by the definition of $h(0)$ and (28)

$$M(\chi_n, u)<(h(0)+\eta)V(\chi_n)+\eta V(\chi_n) \qquad (n\geqq n_1(\eta, s)). \tag{30}$$

It follows from (29) and (30) that

$$\frac{m^*(\chi_n, u)}{M(\chi_n, u)}>\frac{h(\pi)-3\eta}{h(0)+2\eta}>\cos\pi\rho-\varepsilon.$$

Proof of Corollary 1. Let $y(z)$ be an N-valued entire algebroid function defined by (1). And let A be the system $(1, A_1, \cdots, A_N)$. Then Valiron [13] proved that

(31) $$T(r, A)=NT(r, y)+O(1).$$

Next, put $u(z)=\log B(z)=\max_{1\leq j\leq N}\log|A_j(z)|$. Evidently

(32) $$T(r, A)=N(r, u^+)=T(r, u).$$

From (31) and (32) we deduce that $\{r_m\}_1^\infty$ is a Pólya peak sequence of order ρ of $T(r, u)(\rho<1/2)$. Hence Theorem 1 implies the existence of a positive integer $s=s(\varepsilon)$ and a sequence $\{\chi_n\}_1^\infty\nearrow\infty$ contained in $\Lambda(s)$ such that

$$\log m^*(\chi_n, B)>(\cos\pi\rho-\varepsilon)\log M(\chi_n, B) \qquad (n=1, 2, \cdots).$$

Combining this and (2), we have the desired result.

4. Proof of Carollary 2. Let $y(z)$ be an N-valued algebroid function defined by the irreducible equation

$$F(z, y)=A_0(z)y^N+\cdots+A_N(z)=0.$$

And let $A=(A_0, \cdots, A_N)$. Then

$$\min_{|z|=r}\max_{1\leq j\leq N}\log|A_j(z)/A_0(z)|\leqq N\log^+m^*(r, y)+O(1).$$

For the proof, cf. [12, p. 167]. Since (31) holds also in this case, we have

(33) $$\frac{\log^+ m^*(r, y)}{T(r, y)}\geqq\frac{\min_{|z|=r}\max_{1\leq j\leq N}\log^+|A_j(z)/A_0(z)|+O(1)}{T(r, A)+O(1)}.$$

Now, let $v=u^{(1)}-u^{(2)}$, where $u^{(1)}(z)=\max_{0\leq j\leq N}\log|A_j(z)|$, $u^{(2)}(z)=\log|A_0(z)|$. Then it is clear that

(34) $$\begin{aligned}T(r, v)&=N(r, v^+)+N(r, u^{(2)})=N(r, v)+N(r, u^{(2)})\\&=N(r, u^{(1)})=T(r, A)=NT(r, y)+O(1),\end{aligned}$$

and

(35) $$1-\delta(\infty, v)=\overline{\lim_{r\to\infty}}\frac{N(r, u^{(2)})}{T(r, v)}=\overline{\lim_{r\to\infty}}\frac{N(r, 0, A_0)}{T(r, A)}=\overline{\lim_{r\to\infty}}\frac{N(r, \infty, y)}{T(r, y)}=1-\delta(\infty, y).$$

We deduce from (34) that $\{r_m\}_1^\infty$ is a sequence of Pólya peaks of order ρ of $T(r, v)$. Further note that the condition (ii) in Theorem 2 follows from (35) and the condition (iii). Hence Theorem 2 guarantees the existence of a positive integer $s=s(\varepsilon)$ and a sequence $\{\chi_n\}_1^\infty\subset\Lambda(s)$ tending to ∞ such that

$$\min_{|z|=r}\max_{1\leq j\leq N}\log^+|A_j(z)/A_0(z)|>\frac{\pi\rho(k-\varepsilon)}{\sin\pi\rho}T(\chi_n, A).$$

Combinig this and (33) we have the desired result.

5. Proof of Theorem 3. We may assume that $u^{(1)}$ and $u^{(2)}$ are harmonic in a neighborhood of 0 and that $u^{(1)}(0)=u^{(2)}(0)=0$. In fact, assume that Theorem 3 holds for the set $\mathcal{F}$ of such δ-subharmonic functions. Take an arbitrary nonconstant δ-subharmonic function $v=u^{(1)}-u^{(2)}$ satisfying the assumption of Theorem 3. Since we are interested in results for large values of r, we may assume that $u^{(1)}$ and $u^{(2)}$ are harmonic in a neighborhood of 0. Next put $\tilde{u}^{(1)}(z)=u^{(1)}(z)-u^{(1)}(0)$, $\tilde{u}^{(2)}(z)=u^{(2)}(z)-u^{(2)}(0)$, and $\tilde{v}=\tilde{u}^{(1)}-\tilde{u}^{(2)}$. Since nonconstancy of v implies that $T(r, v)\nearrow\infty$ as $r\to\infty$ (For the proof, cf. § 2), we easily have

$$T(r, v)=T(r, \tilde{v})+O(1)=(1+o(1))T(r, \tilde{v}) \qquad (r\to\infty). \tag{36}$$

From (36), if $\{r_m\}_1^\infty$ is a sequence of Pólya peaks of order ρ of $T(r, v)$, it is also a Pólya peak sequence of order ρ of $T(r, \tilde{v})$. Further evidently (36) implies that "All the assumptions of Theorem 3 are satisfied for $v(z)$. $\Leftrightarrow$ All the assumptions of Theorem 3 are satisfied for $\tilde{v}(z)$." Hence by assumption Theorem 3 guarantees the existence of a positive integer $s=s(\varepsilon)>0$ and a sequence $\{\chi_n\}_1^\infty\subset\Lambda(s)$ tending to ∞ such that

$$m^*(\chi_n, \tilde{v})>\frac{k_2-\varepsilon/2}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}m_2(\chi_n, \tilde{v}) \qquad n=1, 2, \cdots). \tag{37}$$

Next, it is clear that

$$m^*(\chi_n, v)=m^*(\chi_n, \tilde{v})+u^{(1)}(0)-u^{(2)}(0), \tag{38}$$

and

$$m_2(r, v)\geqq N(r, |v|)=N(r, v^+)+N(r, v^-)=2T(r, v)-N(r, u^{(1)})-N(r, u^{(2)}). \tag{39}$$

It follows from (39) and $\delta(\infty, v)>0$ that $m_2(r, v)\to\infty$ as $r\to\infty$. Since $\tilde{v}=v-(u^{(1)}(0)-u^{(2)}(0))\equiv v-c$, we have

$$m_2^2(r, \tilde{v})=m_2^2(r, v-c)\geqq m_2^2(r, v)+c^2-2|c|m_2(r, v)=\{m_2(r, v)-|c|\}^2,$$

so that

$$m_2(r, \tilde{v})\geqq m_2(r, v)-|c| \qquad (r\geqq r_0(|c|)). \tag{40}$$

Now, noting that $m_2(r, v)\to\infty$ as $r\to\infty$, there exists a $r_1>0$ such that $r\geqq r_1$ implies

$$|c|\left(1+\frac{k_2-\varepsilon/2}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}\right)<\frac{\varepsilon}{2\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}m_2(r, v). \tag{41}$$

Combining (37), (38), (40) and (41) we deduce

$$m^*(\chi_n, v)>\frac{k_2-\varepsilon}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}m_2(\chi_n, v) \qquad (n\geqq n_0).$$

From now on, we assume that $v\in\mathcal{F}$. Let $\nu^{(j)}$ be the Riesz mass associated with

$u^{(j)}$ $(j=1, 2)$, and let $n^{(j)}(t)=\nu^{(j)}(|\zeta|<t)$. Further let $\{r'_m\}_1^\infty$, $\{r''_m\}_1^\infty$, $\{\varepsilon_m\}_1^\infty$ be the associated sequences with $\{r_m\}_1^\infty$. Choose $V(r_m)=(1+\varepsilon_m)T(r_m, v)$, which implies

(42) $$T(r, v)<V(r) \qquad (r\in\Lambda).$$

Put

$$B_m(z)=u_2^{(1)}(z, r''_m/4)+u_2^{(2)}(z, r''_m/4)=\int_0^{r''_m/4}\log\left|1+\frac{z}{t}\right|d\{n^{(1)}(t)+n^{(2)}(t)\}.$$

Now, we consider the local indicator $h(\theta)$ of order ρ of $\{B_m(z)\}_1^\infty$ at the peaks $\{r_m\}_1^\infty$. As in the proof of Theorem 1, we can easily see the existence of $h(\theta)$. Here we shall show $h(0)\geqq 1$. By our assumptions, as $m\to\infty$

$$\frac{V(r_m)}{1+\varepsilon_m}=T(r_m, v)\sim N(r_m, u^{(1)})=N(r_m, u_2^{(1)})\leqq N(r_m, u_2^{(1)}+u_2^{(2)})\leqq B_m(r_m).$$

Hence by the definition of $h(0)$

$$h(0)\geqq\varliminf_{m\to\infty}\frac{B_m(r_m)}{V(r_m)}\geqq 1.$$

Next, using (17) we have

(43) $$v(z)=u^{(1)}(z)-u^{(2)}(z)=u_{1,m}^{(1)}(z)-u_{1,m}^{(2)}(z)+u_{3,m}^{(1)}(z)-u_{3,m}^{(2)}(z)$$
$$\equiv u_{1,m}^{(1)}(z)-u_{1,m}^{(2)}(z)+W_m(z).$$

Since $W_m(z)$ is harmonic in $|z|<r''_m/4\equiv R_m$ and $W_m(0)=0$, it is the real part of a regular function $f_m(z)$ which may be taken to satisfy $f_m(0)=0$. Let

(44) $$f_m(z)=\sum_{n=1}^{\infty}C_n(R_m)z^n \qquad (|z|<R_m).$$

Then

$$C_n(R_m)r^n=\frac{1}{\pi}\int_{-\pi}^{+\pi}W_m(re^{i\theta})e^{-in\theta}d\theta(r<R_m,\ n\geqq 1).$$
$$=\frac{1}{\pi}\int_{-\pi}^{+\pi}v(re^{i\theta})e^{-in\theta}d\theta-\frac{1}{\pi}\int_{-\pi}^{+\pi}u_{1,m}^{(1)}(re^{i\theta})e^{-in\theta}d\theta+\frac{1}{\pi}\int_{-\pi}^{+\pi}u_{1,m}^{(2)}(re^{i\theta})e^{-in\theta}d\theta$$
$$=\frac{1}{\pi}\int_{-\pi}^{+\pi}v(re^{i\theta})e^{-in\theta}d\theta+\frac{1}{n}\int_{|\zeta|\leq r}\left(\frac{\bar\zeta}{r}\right)^n d\nu^{(1)}(\zeta)+\frac{1}{n}\int_{r<|\zeta|\leq R_m}\left(\frac{r}{\zeta}\right)^n d\nu^{(1)}(\zeta)$$
$$-\frac{1}{n}\int_{|\zeta|\leq r}\left(\frac{\bar\zeta}{r}\right)^n d\nu^{(2)}(\zeta)-\frac{1}{n}\int_{r<|\zeta|\leq R_m}\left(\frac{r}{\zeta}\right)^n d\nu^{(2)}(\zeta). \qquad \text{(cf. [8])}$$

Evidently

$$\left|\frac{1}{\pi}\int_{-\pi}^{+\pi}v(re^{i\theta})e^{-in\theta}d\theta\right|\leqq 2N(r, |v|)\leqq 4T(r, v)-2N(r, u^{(1)})-2N(r, u^{(2)}),$$

$$\left|\int_{|\zeta|\le r}\left(\frac{\bar{\zeta}}{r}\right)^n d\nu^{(1)}(\zeta)\right| \leqq \int_0^r dn^{(1)}(t) = n^{(1)}(r), \qquad \text{etc.}$$

Hence by (45) we have

$$(46)\qquad \begin{aligned} |C_n(R_m)| &\leqq \frac{4T(R_m, v)}{R_m^n} + \frac{n^{(1)}(R_m)+n^{(2)}(R_m)}{n\cdot R_m^n} \\ &\leqq \frac{4T(2R_m, v)}{R_m^n} + \frac{2T(2R_m, v)}{n\cdot R_m^n\cdot \log 2} \end{aligned}$$

Substituting (46) into (44), we obtain for $r=|z|<R_m/2$

$$(47)\qquad \begin{aligned} |W_m(z)| &\leqq |f_m(z)| \leqq \sum_{n=1}^{\infty} |C_n(R_m)| r^n \\ &< 4T(2R_m, v)\sum_{n=1}^{\infty}\left(\frac{r}{R_m}\right)^n + 2T(2R_m, v)\frac{1}{\log 2}\sum_{n=1}^{\infty}\left(\frac{r}{R_m}\right)^n \\ &= 4T(2R_m, v)\frac{r/R_m}{1-r/R_m} + 2T(2R_m, v)\frac{1}{\log 2}\frac{r/R_m}{1-r/R_m} \\ &< \frac{T(2R_m, v)}{R_m} r\left(8+\frac{4}{\log 2}\right). \end{aligned}$$

From (43), (16), (47) and (42), we deduce that for any $\eta>0$ and any integer $s>0$, there exists a $m_0=m_0(\eta, s)>0$ such that $r\in I_m(s)$, $m\geqq m_0$ imply

$$(48)\qquad m^*(r, v) \geqq u_{2,m}^{(1)}(-r) - u_{2,m}^{(2)}(r) - \eta V(r).$$

Now, for given $\eta>0$, choose an integer $s>0$ such that $h_s(\pi)>h(\pi)-\eta$. By the definition of $h_s(\pi)$ there exists a sequence $\{\chi_n\}_1^{\infty}\subset \Lambda(s)$ tending to ∞ such that

$$(49)\qquad B(-\chi_n) > (h_s(\pi)-\eta)V(\chi_n) > (h(\pi)-2\eta)V(\chi_n) > (h(0)\cos\pi\rho - 2\eta)V(\chi_n).$$

We may suppose that $\cos\pi\rho-2\eta>0$. By (48) and (49) we have

$$(50)\qquad m^*(\chi_n, v) > (h(\pi)-3\eta)V(\chi_n) - \{u_{2,m}^{(2)}(-\chi_n) + u_{2,m}^{(2)}(\chi_n)\}.$$

Since $N(r, u^{(1)}) \sim T(r, v)$ $(r\to\infty)$, we obtain for any $\varepsilon>0$,

$$(51)\qquad N(r, u_{2,m}^{(2)}) \sim N(r, u^{(2)}) < (1-\delta(\infty, v)+\varepsilon)N(r, u^{(1)}) \qquad (r\in I_m(s),\ r\geqq r_0(\varepsilon)).$$

As we have shown in the proof of Theorem 1,

$$u_{2,m}^{(2)}(r) = r\int_0^{\infty}\frac{N(t, u_{2,m}^{(2)})}{(t+r)^2}dt.$$

Using this and (51), we easily have for $r\in I_m(s)$, $m>m_0(\eta, s)$

$$u_{2,m}^{(2)}(r)<(1-\delta(\infty, v))u_{1,m}^{(2)}(r)+\eta V(r).$$

Thus for $r\in I_m(s)$, $m>m_0(\eta, s)$

$$u_{2,m}^{(2)}(-r)+u_{2,m}^{(2)}(r)\leqq 2u_{2,m}^{(2)}(r)\leqq\left(\frac{2-2\delta(\infty, v)}{2-\delta(\infty, v)}+\eta'\right)(u_{2,m}^{(1)}(r)+u_{2,m}^{(2)}(r)) \tag{52}$$

$$<\left(\frac{2-2\delta(\infty, v)}{2-\delta(\infty, v)}+\eta''\right)h(0)V(r),$$

where η', $\eta''(>0)$ satisfy η', $\eta''\to 0$ as $\eta\to 0$. Substituting (52) into (50),

$$m^*(\chi_n, v)>\left\{h(\pi)-3\eta h(0)-\frac{2-2\delta(\infty, v)}{2-\delta(\infty, v)}h(0)-\eta'' h(0)\right\}V(\chi_n). \tag{53}$$

We may suppose that the right hand side of (53) is positive. On the other hand, by (43) and (47),

$$m_2^2(\chi_n, v)=m_2^2(\chi_n, u_{1,m}^{(1)}-u_{1,m}^{(2)})+\frac{1}{2\pi}\int_0^{2\pi}(W_m(\chi_n e^{i\theta}))^2 d\theta$$

$$+2\frac{1}{2\pi}\int_0^{2\pi}W_m(\chi_n e^{i\theta})(u_{1,m}^{(1)}(\chi_n e^{i\theta})-u_{1,m}^{(2)}(\chi_n e^{i\theta}))d\theta \tag{54}$$

$$\leqq m_2^2(\chi_n, u_{1,m}^{(1)}-u_{1,m}^{(2)})+\eta^2(V(\chi_n))^2+2m_2(\chi_n, u_{1,m}^{(1)}-u_{1,m}^{(2)})\eta V(\chi_n).$$

As Miles and Shea [11] proved,

$$m_2(\chi_n, u_{1,m}^{(1)}-u_{1,m}^{(2)})\leqq m_2(\chi_n, u_{2,m}^{(1)}+u_{2,m}^{(2)}). \tag{55}$$

Here we note that for $|z|=\chi_n$

$$0<u_{2,m}^{(1)}(z)+u_{2,m}^{(2)}(z)<(H(\theta)+\varepsilon)V(\chi_n),$$

where

$$H(\theta)=\frac{h(0)\sin(\pi-\theta)\rho+h(\pi)\sin\theta\rho}{\sin\pi\rho}\qquad(0\leqq\theta\leqq\pi).$$

Hence

$$m_2(\chi_n, u_{2,m}^{(1)}+u_{2,m}^{(2)})<V(\chi_n)\left\{\frac{1}{\pi}\int_0^{\pi}(H(\theta)+\varepsilon)^2 d\theta\right\}^{1/2}$$

$$<V(\chi_n)\left\{\frac{\left[\left\{(h(0))^2+(h(\pi))^2\right\}\left(\frac{1}{2}-\frac{\sin 2\pi\rho}{4\pi\rho}\right)+h(0)h(\pi)\left(\frac{\sin\pi\rho}{\pi\rho}-\cos\pi\rho\right)\right]^{1/2}}{\sin\pi\rho}+\eta'''\right\},$$

where $\eta'''\to 0$ as $\varepsilon\to 0$. Combining (53)—(56), we obtain

(57)
$$\frac{m^*(\chi_n, v)}{m_2(\chi_n, v)}$$
$$>\frac{\left(h(\pi)-\dfrac{2-2\delta(\infty, v)}{2-\delta(\infty, v)}h(0)-3\eta h(0)-\eta'' h(0)\right)\sin\pi\rho}{\left[\left\{(h(0))^2+(h(\pi))^2\right\}\left(\dfrac{1}{2}-\dfrac{\sin 2\pi\rho}{4\pi\rho}\right)+h(0)h(\pi)\left(\dfrac{\sin\pi\rho}{\pi\rho}-\cos\pi\rho\right)\right]^{1/2}+\eta'''}.$$

The function

$$\frac{t-\dfrac{2-2\delta(\infty, v)}{2-\delta(\infty, v)}}{\left\{(1+t^2)\left(\dfrac{1}{2}-\dfrac{\sin 2\pi\rho}{4\pi\rho}\right)+t\left(\dfrac{\sin\pi\rho}{\pi\rho}-\cos\pi\rho\right)\right\}^{1/2}}$$

increases as t increases, and therefore, in view of $h(\pi)>h(0)\cos\pi\rho$, the right hand side of (57) is not smaller than

$$\frac{\cos\pi\rho-\dfrac{2-2\delta(\infty, v)}{2-\delta(\infty, v)}}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}-\varepsilon.$$

Proof of Corollary 3. We make use of some estimates stated in § 4. As Valiron [13] showed

$$\sum_{j=1}^{N}\log^+|y_j|\leqq\max_{0\leqq j\leqq N}\log|A_j/A_0|+O(1).$$

Hence

(58)
$$C^2(r, y)\leqq\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left\{\max_{0\leqq j\leqq N}\log\left|\frac{A_j}{A_0}(re^{i\theta})\right|\right\}^2 d\theta\right](1+o(1)).$$

We apply Theorem 3 to $u^{(1)}=\max_{0\leqq j\leqq N}\log|A_j|$, $u^{(2)}=\log|A_0|$. Then there exist an integer $s=s(\varepsilon)>0$ and a sequence $\{\chi_n\}_1^\infty\subset\Lambda(s)$ tending to ∞ such that

(59)
$$\left\{\min_{|z|=\chi_n}\max_{1\leqq j\leqq N}\log|A_j/A_0|\right\}^2$$
$$>\left\{\frac{k_2}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}-\varepsilon/2\right\}^2\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left\{\max_{0\leqq j\leqq N}\log\left|\frac{A_j}{A_0}(\chi_n e^{i\theta})\right|\right\}^2 d\theta\right].$$

Combining (58), (59) and an estimate stated in § 4, we have

$$N\log m^*(\chi_n, y)>\left(\frac{k_2}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}}-\varepsilon\right)C(\chi_n, y).$$

REFERENCES

[1] ANDERSON, J. M. AND BAERNSTEIN II, A., The size of the set on which a meromorphic function is large, Proc. London Math. Soc. (3) **36** (1978), 518-539.

[2] BAERNSTEIN II, A., Proof of Edrei's spread conjecture, ibid, (3) **26** (1973), 418-434.

[3] BAERNSTEIN II, A., A generalization of the $\cos \pi\rho$ theorem, Trans. Amer. Math. Soc. **193** (1974), 181-197.

[4] BARRY, P. D., On a theorem of Kjellberg, Quart. J. Math. Oxford (2) **15** (1964), 179-191.

[5] CARTWRIGHT, M. L., Integral functions (Cambridge, 1956).

[6] DRASIN, D. AND SHEA, D. F., Pólya peaks and the oscillation of positive functions Proc. Amer. Math. Soc. **34** no. 2 (1972), 403-411.

[7] EDREI, A., A local form of the Phragmén-Lindelöf indicator; Mathematika **17** (1970), 149-172.

[8] EDREI, A. AND FUCHS, W. H. J., On the growth of meromorphic functions with several deficient values, Trans. Amer. Math Soc. **93** (1959), 292-328.

[9] HAYMAN, W. K. AND KENNEDY, P. B., Subharmonic functions, Academic Press (1976).

[10] KJELLBERG, B., On the minimum modulus of functions of lower order less than one, Math. Scand. **8** (1960), 189-197.

[11] MILES, J. AND SHEA, D. F., An extremal problem in value distribution theory, Quart. J. Math. Oxford (2) **24** (1973), 377-383.

[12] OZAWA, M., On the minimum modulus of an entire algebroid function of lower order less than one; Kōdai Math. Sem. Rep. **22** (1970), 166-171.

[13] VALIRON, G., Sur la dérivée des functions algébroides, Bull. Soc. Math. **59** (1931), 17-39.

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