M. DE LEON
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CONNECTIONS AND f-STRUCTURES ON T^2M

BY MANUEL DE LEON

Introduction

Grifone [11] defines a connection on M as a differentiable vector 1-form Γ on TM verifying: $J\Gamma=J$, $\Gamma J=-J$, where J defines the canonical almost-tangent structure of TM. If T^2M denotes the tangent bundle of order 2 over a C^∞ differentiable manifold M, the existence of the vertical fiber bundles V^{π_2} and $V^{\pi_{12}}$ lead us to define connections on T^2M by means of complementary distributions. Taking into account the canonical endomorphisms J_1 and J_2 (J_2 defining an almost-tangent structure of order 2 on T^2M), and following Catz [5], we introduce a non-homogeneous connection on M of type 1 as given by a vector 1-form Γ verifying $J_1\Gamma=J_1$, $\Gamma J_2=-J_2$.

The connection Γ is said of type 2 if $J_2\Gamma=J_2$, $\Gamma J_1=-J_1$.

In § 5, we express the non homogeneous character of a connection by means of its tension. Thus, a connection is said homogeneous if its tension vanishes. In § 6, a semispray or a differential equation of third order, is shown to be canonically associated with any connection of the same type. Moreover, the paths of a connection are just the solutions of its associated semi-spray. The curvature of a connection is defined in § 8 and Bianchi's identities are derived. In particular, if a connection is homogeneous, its curvature is homogeneous, too.

It is well known that, associated with a linear connection on M, there exists an almost-complex structure on TM, the integrability of which is given through the curvature and torsion of the connection [8], [12], [15]. In § 9, it is shown that if Γ is a connection on M of type 1, there exists an f-structure F associated with Γ and determined by relations

$$FJ'=h, \quad Fh=-J', \quad FJ_1=0$$

where $J'=J_2h$.

In the same way, an f-structure G is associated with a connection of type 2 and defined by

$$GJ_1=h', \quad Gh'=-J_1, \quad GhX=0, \quad \text{if } X\notin V^{\pi_2}(T^2M)$$

where $h'=hJ_2$.

Integrability conditions for both f-structures are given in Theorem 9.6, 9.12 and 9.13.

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Finally, in § 10, prolongations of metrics given on $V^{\pi_2}(\mathcal{T}^2M)$ and $V^{\pi_{12}}(\mathcal{T}^2M)$ to $\mathcal{T}^2M$ are defined with respect to connections of type 1 or 2, respectively. In fact, these prolongations are shown to be hor-ehresmannian with respect to the f-structures which are canonically associated with each connection.

§ 1. Preliminaries.

Let M be a paracompact n-dimensional differentiable manifold. The tangent bundle of order 2, T^2M, of M is the $3n$-dimensional manifold of 2-jets at $0 \in \boldsymbol{R}$ of differentiable mappings $f: \boldsymbol{R} \to M$; T^2M has the natural bundle structure over M, $\pi_2: T^2M \to M$ denoting the canonical projection. The tangent bundle TM is nothing but the manifold of 1-jets at $0 \in \boldsymbol{R}$ of differentiable mappings $f: \boldsymbol{R} \to M$.

If we denote $\pi_{12}: T^2M \to TM$ the canonical projection, then T^2M has a bundle structure over TM with projection π_{12}.

Let $\{U, x^i\}$ be a coordinate neighborhood of M, and denote by (x^i, y^i, z^i) the induced system of coordinates in $\pi_2^{-1}(U)$. The two fiber bundle structures of T^2M, over M and TM respectively, lead to two exact sequences of vector bundles over T^2M:

$$0 \longrightarrow V^{\pi_2}(T^2M) \stackrel{i_1}{\longrightarrow} TT^2M \stackrel{s_1}{\longrightarrow} T^2M \times_M TM \longrightarrow 0$$

$$0 \longrightarrow V^{\pi_{12}}(T^2M) \stackrel{i_2}{\longrightarrow} TT^2M \stackrel{s_2}{\longrightarrow} T^2M \times_{TM} TTM \longrightarrow 0$$

where $V^{\pi_2}(T^2M)$ (respect. $V^{\pi_{12}}(T^2M)$) denotes the vector bundle of those vectors of TT^2M which are projected to 0 by π_2^T (respect. π_{12}^T). These sequences are called the first and second fundamental exact sequences, respectively.

There exist two canonical isomorphisms of vector bundles

$$h_1: T^2M \times_M TM \longrightarrow V^{\pi_{12}}(T^2M)$$

$$h_2: T^2M \times_{TM} TTM \longrightarrow V^{\pi_2}(T^2M)$$

Thus, two vector 1-forms on T^2M are defined:

$$J_1 = i_2 \circ h_1 \circ s_1, \quad J_2 = i_1 \circ h_2 \circ s_2$$

and they verify

$$J_2^2 = 2J_1, \quad J_2^3 = 0$$

Moreover, J_2 has constant rank $2n$ and determines an almost-tangent structure of order 2 on T^2M.

With respect to the induced coordinates, J_1 and J_2 are locally expressed by

$$J_1: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta_i^j & 0 & 0 \end{pmatrix}, \qquad J_2: \begin{pmatrix} 0 & 0 & 0 \\ \delta_i^j & 0 & 0 \\ 0 & 2\delta_i^j & 0 \end{pmatrix}$$

With each fundamental exact sequence, a canonical vector field is associated; in fact, let $\alpha=id_{T^2M}\times_M\pi_{12}$ be the canonical section of the vector bundle $T^2M\times_M TM$; we denote C_1 the vector field defined on T^2M by

$$C_1=\imath_2\circ h_1\circ\alpha\,.$$

Analogously, if $\jmath$ is the injection $T^2M\to TTM$, the canonical section $\beta=id_{T^2M}\times_{TM}j$ of the vector bundle $T^2M\times_{TM}TTM$ permits to define the vector field C_2 on T^2M by

$$C_2=i_1\circ h_2\circ\beta$$

C_1 and C_2 are called the canonical vector fields on T^2M. Locally, in a point of coordinates (x^i, y^i, z^i), the components of C_1 and C_2 are, respectively,

$$(0, 0, y^i),\quad (0, y^i, 2z^i)\,.$$

The formalism of Frölicher-Nijenhuis [9] will be useful in this paper. The following identities are verified:

$$J_1C_1=0\,;\quad J_1C_2=0\,;\quad J_2C_1=0\,;\quad J_2C_2=2C_1$$

$$[C_1, J_1]=0\,;\quad [C_2, J_1]=-2J_1\,;\quad [C_1, J_2]=-J_1\,;\quad [C_2, J_2]=-J_2$$

$$[J_1, J_1]=0\,;\quad [J_1, J_2]=0\,;\quad [J_2, J_2]=0.$$

Finally, we denote $\mathcal{T}^2M$ the bundle of all non-zero elements of T^2M.

§ 2. Homogeneous and semibasic forms.

Let us introduce the following definitions.

a) Homogeneous forms.

DEFINITION 2.1. A real-valued differentiable function f on $\mathcal{T}^2M$ is said homogeneous of degree k if $\mathcal{L}_{C_2}f=k\cdot f$.

Always, $\mathcal{L}$ denotes the Lie derivative.

Let $h_t:\boldsymbol{R}\to\boldsymbol{R}$ be the homothetia of ratio e^t and let $H_t: T^2M\to T^2M$ denote the fibre-preserving transformation deduced from h_t. Since C_2 generates the 1-parameter group of transformation H_t, the condition in Definition 2.1, is equivalent to

$$f\circ H_t=e^{kt}f.$$

DEFINITION 2.2. A scalar p-form ω on $\mathcal{T}^2M$ is said homogeneous of degree k if

$$\mathcal{L}_{C_2}\omega=k\cdot\omega\,.$$

DEFINITION 2.3. A vector l-form L on $\mathcal{T}^2M$ is said homogeneous of degree k if

$$[C_2, L]=(k-1)L .$$

b) Semibasic forms.

DEFINITION 2.4. A vector l-form L on T^2M, with $l\geqq 1$, is said:

1) Semibasic of type 1 if
   a) $L(X_1, \cdots, X_l)\in V^{\pi_{12}}(T^2M)$, for every $X_1, \cdots, X_l$ vector fields on T^2M.
   b) $L(X_1, \cdots, X_l)=0$, if X_1 belongs to $V^{\pi_{12}}(T^2M)$.
2) Semibasic of type 2 if
   a) $L(X_1, \cdots, X_l)\in V^{\pi_2}(T^2M)$, for every $X_1, \cdots, X_l$ vector fields on T^2M.
   b) $L(X_1, \cdots, X_l)=0$, if X_1 belongs to $V^{\pi_2}(T^2M)$.

A vector field belonging to $V^{\pi_{12}}(T^2M)$ (respectively, $V^{\pi_2}(T^2M)$) is said semibasic of type 1 (respect. semibasic of type 2).

Local expressions

1) If L is a semibasic vector l-form of type 1, in an induced local system of coordinates, it is expressed by

$$L=L^{\alpha}_{i_1\cdots i_r j_1\cdots j_s}dx^{i_1}\otimes\cdots\otimes dx^{i_r}\oplus dy^{j_1}\otimes\cdots\otimes dy^{j_s}\otimes\frac{\partial}{\partial z^{\alpha}}$$

where $r+s=l$, and i's, j's and α running over the set $\{1, 2, \cdots, n\}$.

2) If L is semibasic of type 2, it is locally expressed by

$$L=L^{\alpha}_{i_1\cdots i_l}dx^{i_1}\otimes\cdots\otimes dx^{i_l}\otimes\frac{\partial}{\partial y^{\alpha}}+M^{\alpha}_{j_1\cdots j_l}dx^{j_1}\otimes\cdots\otimes dx^{j_l}\otimes\frac{\partial}{\partial z^{\alpha}} .$$

PROPOSITION 2.5. *Let L be a vector l-form. Then:*

1) *L is semibasic of type 1 if and only if*

$$J_2L=0 \quad and \quad i_{J_1X}L=0 , \qquad \forall X\in\mathfrak{X}(T^2M)$$

2) *L is semibasic of type 2 if and only if*

$$J_1L=0 \quad and \quad i_{J_2X}L=0 , \qquad \forall X\in\mathfrak{X}(T^2M) .$$

§ 3. Semi-sprays and potentials.

DDFINITION 3.1. Let S be a vector field on T^2M, differentiable C^{∞} on $\mathcal{T}^2M$. Then:

1) S is said a semi-spray over M of type 1 if, for every integral curve α of S, one has

$$(\pi_2\circ\alpha)'=\pi_{12}\circ\alpha$$

where $(\pi_2\circ\alpha)'$ denotes the canonical lift of $(\pi_2\circ\alpha)$ to TM.

2) S is said a semi-spray over M of type 2 if, for every integral curve α of S, one has

$$(\pi_2 \circ \alpha)'' = \alpha$$

where $(\pi_2 \circ \alpha)''$ denotes the canonical lift of $(\pi_2 \circ \alpha)$ to T^2M.

The following proposition is easily shown.

PROPOSITION 3.2. *Let S be a vector field on T^2M, differentiable C^∞ on $\mathfrak{T}^2M$. Then*

1) *S is a semi-spray of type 1 if and only if*

$$\pi_2^T \circ S = \pi_{12}$$

2) *S is a semi-spray of type 2 if and only if*

$$\pi_{12}^T \circ S = j$$

j being the canonical injection $T^2M \to TTM$.

Local expressions

With respect to an induced local system of coordinates, we have:

1) if S is a semi-spray of type 1, it is expressed by

$$S : (y^i,\ S_1^i(x, y, z),\ S_2^i(x, y, z))$$

where the functions S_1^i, S_2^i, $i=1, 2, \cdots, n$, are differentiable C^∞ on $\mathfrak{T}^2M$.

2) if S is a semi-spray of type 2, then

$$S : (y^i,\ z^i,\ S^i(x, y, z))$$

where the functions S^i, $i=1, 2, \cdots, n$, are as above.

Using these local expressions, we can easily prove

PROPOSITION 3.3. *Let S be a vector field on T^2M, differentiable C^∞ on $\mathfrak{T}^2M$. Then S is a semispray of type 1 (respectively, of type 2) if and only if $J_1S = C_1$ (respect. $J_2S = C_2$).*

Remark. Evidently, any semi-spray of type 2 is also of type 1.

We shall now express the non-homogeneity of a semi-spray.

DEFINITION 3.4. Let S be a semi-spray over M (indistinctly of type 1 or 2). We shall call deviation of S the vector field

$$S^* = [C_2, S] - S.$$

Then, using local components, we have

PROPOSITION 3.5. 1) *If S is of type* 1, *S^* belongs to $V^{\pi_2}(T^2M)$.*
2) *If S is of type* 2, *S^* belongs to $V^{\pi_{12}}(T^2M)$.*

DEFINITION 3.6. A semi-spray which has zero deviation, and of class C^2 on the zero cross-section, is said a spray.

From this definition on, it is easily deduced that a semi-spray of type 1 is a spray if and only if the functions S_1^i, S_2^i are homogeneous of degree 2 and 3 respectively; analogously, a semi-spray of type 2 is a spray if and only if the function S^i are homogeneous of degree 3.

DEFINITION 3.7. Let L be a semibasic vector l-form on T^2M, of type 1 (respectively, of type 2). We call potential of L the semibasic vector $(l-1)$-form, of type 1 (respect. of type 2) given by $L^O = i_S L$, S being an arbitrary semi-spray of type 2.

The fact that L^O is independent of the election of S and that L^O is semibasic and of the same type as L is easily verified.
This terminology is justified by the following

PROPOSITION 3.8. *Let L be a semibasic vector l-form on T^2M, of type* 2 *and homogeneous of degree k, with $l+k\neq 1$. Then*

$$L=\frac{1}{l+k-1}([J_2, L]^O+[J_2, L^O]).$$

Proof. Let S be an arbitrary semi-spray of type 2. We have

$$[i_S, d_{J_2}]=d_{J_2\bar{\wedge} S}-i_{[S, J_2]}=\mathcal{L}_{C_2}-i_{[S, J_2]}.$$

But

$$(i_{[S, J_2]}L)(X_1, \cdots, X_l)=(L\bar{\wedge}[S, J_2])(X_1, \cdots, X_l)$$
$$=\sum_{i=1}^{l} L(X_1, \cdots, X_{i-1}, [S, J_2]X_i, X_{i+1}, \cdots, X_l)$$

for any $X_1, \cdots, X_l \in \mathfrak{X}(\mathcal{T}^2M)$. On the other hand,

$$J_1[S, J_2X_i]=-J_1X_i, \qquad i=1, 2, \cdots, n$$

or, equivalently,

$$J_1([S, J_2X_i]+X_i)=0, \qquad i=1, 2, \cdots, n.$$

Then, taking into account that

$$[S, J_2]X_i=[S, J_2X_i]-J_2[S, X_i]$$

and the fact that L is semibasic, we deduce

$$(i_{[S,J_2]}L)(X_1, \cdots, X_l)=\sum_{i=1}^{l} L(X_1, \cdots, -X_i, \cdots, X_1)$$

$$=-l\cdot L(X_1, \cdots, X_i, \cdots, X_1).$$

Hence,

$$i_{[S,J_2]}L=-l\cdot L$$

and then

$$[i_S, d_{J_2}]L=\mathcal{L}_{C_2}L-i_{[S,J_2]}L=(k-1)L+l\cdot L=(l+k-1)L.$$

Finally,

$$L=\frac{1}{l+k-1}[i_S, d_{J_2}]L=\frac{1}{l+k-1}(i_S d_{J_2}L+d_{J_2}i_S L)$$

$$=\frac{1}{l+k-1}([J_2, L]^O+[J_2, L^O]).$$

COROLLARY 3.9. *Let L be a semibasic vector l-form of type 2 and homogeneous of degree k, with $l+k\neq 1$. Then, if L is J_2-closed,*

$$L=\frac{1}{l+k-1}[J_2, L^O]$$

i.e., if L is J_2-closed, then L is expressed as a function of the derivatives of its potential.

§4. Connections on M.

Following Catz [5], we introduce

DEFINITION 4.1. We shall call non-homogeneous connection on T^2M of type 1, or simply, connection on M of type 1, a vector 1-form Γ on T^2M, differentiable C^∞ on $\mathcal{T}^2M$, such that

1) $J_1\Gamma=J_1$, 2) $\Gamma J_2=-J_2$.

DEFINITION 4.2. We shall call non-homogeneous connection on T^2M of type 2, or simply, connection on M of type 2, a vector 1-form Γ on T^2M, differentiable C^∞ on $\mathcal{T}^2M$, such that

1) $J_2\Gamma=J_2$, 2) $\Gamma J_1=-J_1$.

PROPOSITION 4.3. *A vector 1-form Γ on T^2M is a connection on M of type 1 if and only if Γ defines an almost-product structure over T^2M, differentiable C^∞ on $\mathcal{T}^2M$, such that, for every point $\omega\in T^2M$, the eigenspace corresponding to the eigenvalue -1 of Γ_ω is the subspace $V_\omega^{\pi_2}(T^2M)$.*

Proof. Let Γ be a connection on M of type 1, then

$$J_1\Gamma=J_1 \quad \text{if and only if} \quad J_1(\Gamma-I)=0$$

$$\Gamma J_2=-J_2 \quad \text{if and only if } (\Gamma+I)J_2=0.$$

But

$$J_1=\imath_2\circ h_1\circ s_1, \quad J_2=\imath_1\circ h_2\circ s_2$$

hence

$$\imath_2\circ h_1\circ s_1\circ(\Gamma-I)=0 \quad \text{if and only if } s_1\circ(\Gamma-I)=0$$

because $\imath_2$ is a monomorphism and h_1 is an isomorphism; analogously,

$$(\Gamma+I)\circ\imath_1\circ h_2\circ s_2=0 \quad \text{if and only if } (\Gamma+I)\circ\imath_1=0$$

because s_2 is an epimorphism and h_2 is an isomorphism.

Thus, we obtain

$$\mathrm{Im}(\Gamma-I)\subset\mathrm{Ker}\, s_1=\mathrm{Im}\,\imath_1, \quad \mathrm{Im}\,\imath_1\subset\mathrm{Ker}(\Gamma+I)$$

i. e.

$$\mathrm{Im}(\Gamma-I)\subset\mathrm{Ker}(\Gamma+I)$$

and, consequently

$$(\Gamma+I)(\Gamma-I)=\Gamma^2-I=0\,.$$

On the other hand, if $X\in T(T^2M)$ is such that

$$X=-\Gamma X$$

we have

$$J_1X=-J_1\Gamma X=-J_1X$$

and thus $X\in V^{\pi_2}_\omega(T^2M)$. Conversely, if $X\in V^{\pi_2}_\omega(T^2M)$, there exists $Y\in T_\omega(T^2M)$ such that $X=J_2Y$; hence

$$\Gamma X=\Gamma J_2Y=-J_2Y=-X$$

and X is associated with the eigenvalue -1.

The sufficiency of the condition is shown as follows; let $X\in\mathfrak{X}(T^2M)$, then $J_2X\in V^{\pi_2}(T^2M)$ and $\Gamma J_2X=-J_2X$, and thus $\Gamma J_2=-J_2$. Moreover, $X-\Gamma X\in V^{\pi_2}(T^2M)$ since $\Gamma(X-\Gamma X)=-(X-\Gamma X)$, and consequently

$$0=J_1(X-\Gamma X)=J_1X-J_1\Gamma X$$

and so $J_1\Gamma=J_1$.

By similar devices, we also have

PROPOSITION 4.4. *A vector 1-form Γ on T^2M is a connection on M of type 2 if and only if Γ defines an almost-product structure over T^2M, differentiable C^∞ on $\mathcal{T}^2M$, such that, for every point $\omega\in T^2M$ the eigenspace corresponding to the eigenvalue -1 of Γ_ω is the subspace $V^{\pi_{12}}_\omega(T^2M)$.*

To each connection Γ on M (of type 1 or 2) there are canonically associated two projection operators

$$h=\frac{1}{2}(I+\Gamma), \quad v=\frac{1}{2}(I-\Gamma)$$

which are called the horizontal and vertical projectors of Γ, respectively.

Therefore, we have a decomposition of the tangent bundle of T^2M,

$$T(T^2M)=\text{Im}\, v\oplus\text{Im}\, h$$

and, since

$$\text{Im}\, v=\text{Ker}\, h=\{X\in T(T^2M)/\Gamma X=-X\}$$

and accordingly with Propositions 4.3 and 4.4, we obtain:

$$\text{Im}\, v=V^{\pi_2}(T^2M), \text{ if } \Gamma \text{ is of type 1}; \quad \text{Im}\, v=V^{\pi_{12}}(T^2M), \text{ if } \Gamma \text{ is of type 2}.$$

Let us denote $\text{Im}\, h=H(T^2M)$; then, we have the following decompositions:

a) for Γ of type 1: $T(T^2M)=V^{\pi_2}(T^2M)\oplus H(T^2M)$ (I)

b) for Γ of type 2: $T(T^2M)=V^{\pi_{12}}(T^2M)\oplus H(T^2M)$ (II)

Conversely, decompositions of $T(T^2M)$ as in (I) or (II) determine connections on M of type 1 or 2, respectively.

If Γ is a connection of type 1, we have

$$J_1h=J_1, \qquad hJ_2=0$$
$$J_1v=0, \qquad vJ_2=J_2$$

and, if Γ is of type 2,

$$J_2h=J_2, \qquad hJ_1=0$$
$$J_2v=0, \qquad vJ_1=J_1.$$

PROPOSITION 4.5. *A connection Γ on M of type 1 defines a splitting, differentiable C^∞ on $\mathcal{T}^2M$, of the exact sequence of vector bundles*

$$0 \longrightarrow V^{\pi_2}(T^2M) \xrightarrow{i_1} T(T^2M) \xrightarrow{s_1} T^2M\times_M TM \longrightarrow 0.$$

Conversely, such a splitting determines a connection Γ on M of type 1.

Proof. Let Γ be a connection on M of type 1, with horizontal projector h, and let j be an arbitrary splitting of the exact sequence above, i.e.

$$j: T^2M\times_M TM \longrightarrow TT^2M$$

and $s_1\circ j=id_{T^2M\times_M TM}$.

Put $\gamma=h\circ j$; then γ is well-defined, since if j' is another splitting, $s_1(j-j')=0$ and then $j-j'\in\text{Ker}\, s_1=V^{\pi_2}(T^2M)$; hence, $h(j-j')=0$, i.e. $h\circ j=h\circ j'$. Moreover, γ is a splitting, since

$$J_1 \circ h = \imath_2 \circ h_1 \circ s_1 \circ h = \imath_2 \circ h_1 \circ s_1$$

and taking into account the fact that $\imath_2$ is a monomorphism and h_1 is an isomorphism, we deduce $s_1 \circ h = s_1$, and, thus,

$$s_1 \circ \gamma = s_1 \circ h \circ \jmath = s_1 \circ \jmath = \imath d_{T^2M \times_M TM} .$$

Conversely, let γ be a splitting of the exact sequence and put $\Gamma = 2\gamma \circ s_1 - I$, then

$$J_1 \Gamma = 2\imath_2 \circ h_1 \circ s_1 \circ \gamma \circ s_1 - J_1 = J_1$$

$$\Gamma J_2 = 2\gamma \circ s_1 \circ i_1 \circ h_2 \circ s_2 - J_2 = -J_2$$

and so Γ is a connection in M of type 1.

A similar Proposition is obtained for connections of type 2.

PROPOSITION 4.6. *A connection Γ on M of type 2 defines a splitting, differentiable C^∞ on $\mathcal{T}^2M$, of the exact sequence of vector bundles*

$$0 \longrightarrow V^{\pi_{12}}(T^2M) \xrightarrow{i_2} T(T^2M) \xrightarrow{s_2} T^2M \times_{TM} TTM \longrightarrow 0$$

Conversely, such a splitting determines a connection Γ on M of type 2.

Local expressions

Let (U, x^i) be a coordinate neighborhood of M, and (x^i, y^i, z^i) the induced coordinates in $\pi_2^{-1}(U)$. If $X \in \mathcal{X}(T^2M)$, in $\pi^{-1}(U)$ the local components of X are $(x^i, y^i, z^i; a^i, b^i, c^i)$. We shall separately discuss the case of a connection Γ of type 1 or of type 2.

a) Connections of type 1.

In this case, h being the horizontal projector of Γ, we have

$$hX = (x^i, y^i, z^i; \alpha^i, \beta^i, \gamma^i)$$

where $\alpha^j, \beta^j, \gamma^j$ are functions of $(x^i, y^i, z^i; a^i, b^i, c^i)$. The linearity of h implies that $\alpha^j, \beta^j, \gamma^j$ are also linear on a^i, b^i, c^i.

Since $J_1 h = J_1$, we deduce $\alpha^i = a^i$. Moreover, $hJ_2 = 0$, and, therefore, $\beta^j(0, a^i, 2b^i) = \gamma^j(0, a^i, 2b^i) = 0$; thus β^j and γ^j do not depend on b^i and c^i.

We denote

$$\beta(x, y, z, a) = -\Gamma_i^j(x, y, z)a^i, \quad \gamma(x, y, z, a) = -\bar{\Gamma}_i^j(x, y, z)a^i$$

where Γ_i^j, $\bar{\Gamma}_i^j$ are functions on T^2M, differentiable C^∞ on $\mathcal{T}^2M$; then, we have

$$h(x, y, z; a, b, c) = (x, y, z; a^j, -\Gamma_i^j a^i, -\bar{\Gamma}_i^j a^i)$$

and, consequently

$$\Gamma(x, y, z; a, b, c)=(2h-I)(x, y, z; a, b, c)$$
$$=(x, y, z; a^j, -2\Gamma_i^j a^i-b^j, -2\bar{\Gamma}_i^j a^i-c^j)$$

and, thus, Γ can be represented by the matrix

$$\Gamma: \begin{pmatrix} \delta_i^j & 0 & 0 \\ -2\Gamma_i^j & -\delta_i^j & 0 \\ -2\bar{\Gamma}_i^j & 0 & -\delta_i^j \end{pmatrix}.$$

b) Connections of type 2.

By similar devices, we obtain the following expression for a connection Γ of type 2

$$\Gamma: \begin{pmatrix} \delta_i^j & 0 & 0 \\ 0 & \delta_i^j & 0 \\ -2\Gamma_i^j & -2\bar{\Gamma}_i^j & -\delta_i^j \end{pmatrix}$$

§ 5. The tension of a connection.

We shall now express the non-homogeneity of a connection.

DEFINITION 5.1. Let Γ be a connection on M (indistinctly of type 1 or 2). We shall call tension of Γ the vector 1-form on T^2M, differentiable C^∞ on $\mathcal{T}^2M$, given by

$$H=\frac{1}{2}[C_2, \Gamma].$$

Note that, if h is the horizontal projector of Γ, then

$$H=[C_2, h].$$

Local expressions

1) Suppose Γ of type 1. Then

$$H=\left(\Gamma_i^j-y^k\frac{\partial\Gamma_i^j}{\partial y^k}-2z^k\frac{\partial\Gamma_i^j}{\partial z^k}\right)dx^i\otimes\frac{\partial}{\partial y^j}$$
$$+\left(2\bar{\Gamma}_i^j-y^k\frac{\partial\bar{\Gamma}_i^j}{\partial y^k}-2z^k\frac{\partial\bar{\Gamma}_i^j}{\partial z^k}\right)dx^i\otimes\frac{\partial}{\partial z^j}$$

or, in a matrix form

$$H=\begin{pmatrix} 0 & 0 & 0 \\ \Gamma_i^j-y^k\frac{\partial\Gamma_i^j}{\partial y^k}-2z^k\frac{\partial\Gamma_i^j}{\partial z^k} & 0 & 0 \\ 2\bar{\Gamma}_i^j-y^k\frac{\partial\bar{\Gamma}_i^j}{\partial y^k}-2z^k\frac{\partial\bar{\Gamma}_i^j}{\partial z^k} & 0 & 0 \end{pmatrix}.$$

2) Suppose Γ of type 2. Then

$$H=\left(2\Gamma_i^j-y^k\frac{\partial\Gamma_i^j}{\partial y^k}-2z^k\frac{\partial\Gamma_i^j}{\partial z^k}\right)dx^i\otimes\frac{\partial}{\partial z^j}$$

$$+\left(\bar{\Gamma}_i^j-y^k\frac{\partial\bar{\Gamma}_i^j}{\partial y^k}-2z^k\frac{\partial\bar{\Gamma}_i^j}{\partial z^k}\right)dy^i\otimes\frac{\partial}{\partial z^j}$$

or, in a matrix form

$$H=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\Gamma_i^j-y^k\frac{\partial\Gamma_i^j}{\partial y^k}-2z^k\frac{\partial\Gamma_i^j}{\partial z^k} & \bar{\Gamma}_i^j-y^k\frac{\partial\bar{\Gamma}_i^j}{\partial y^k}-2z^k\frac{\partial\bar{\Gamma}_i^j}{\partial z^k} & 0 \end{pmatrix}.$$

From these local expressions, we deduce the following

PROPOSITION 5.2. *Let Γ be a connection on M of type* 1 (*respect. of type* 2). *Then, the tension H of Γ is a semibasic vector* 1-*form of type* 2 (*respect. of type* 1).

DEFINITION 5.3. A connection Γ on M is said homogeneous if its tension vanishes.

Thus, a connection Γ on M is homogeneous if Γ is an homogeneous vector 1-form.

Once more, from the local expressions above for H, we deduce that a connection Γ on M of type 1 is homogeneous if and only if the functions Γ_i^j and $\bar{\Gamma}_i^j$ are also homogeneous of degree 1 and 2; respectively. In the same way, Γ of type 2 is homogeneous if and only if Γ_i^j and $\bar{\Gamma}_i^j$ are homogeneous of degree 2 and 1, respectively.

DEFINITION 5.4. An homogeneous connection on M (indistinctly of type 1 or 2) is said linear if it is of class C^2 on the zero cross-section.

§ 6. Semi-spray associated to a connection.

PROPOSITION 6.1. *To any connection Γ on M of type* 1 (*respect. of type* 2) *and tension H, there is canonically associated a semi-spray S of type* 1 (*respect. of type* 2) *such that the deviation S^* of S is equal to the potential H^0 of H, i.e. $S^*=H^0$.*

Proof. We shall discuss the case of a connection of type 1; the case of type 2 is shown by a similar device.

Let S' be an arbitrary semi-spray of type 1 and let h denote the horizontal projector of Γ. Let us consider the semi-spray of type 1 given by $S=hS'$. Note that S is independent of S', since if S'' is another semi-spray of type 1, $S''-S' \in V^{\pi_2}(T^2M)$, and, therefore, $hS'=hS''$.

Thus, the semi-spray S of type 1 is canonically associated with Γ. Now, we shall prove $S^*=H^0$.

In fact

$$H^0=i_S H=H(S)=\frac{1}{2}([C_2, \Gamma S]-\Gamma[C_2, S]).$$

But

$$\Gamma S=\Gamma hS'=hS'=S,$$

$$\Gamma[C_2, S]=\Gamma(h[C_2, S]+v[C_2, S])=h[C_2, S]-v[C_2, S]$$

and

$$0=hS^*=h([C_2, S]-S)=h[C_2, S]-S.$$

Consequently

$$\begin{aligned} H^0&=\frac{1}{2}([C_2, S]-S+v[C_2, S]) \\ &=\frac{1}{2}([C_2, S]-S-h[C_2, S]+h[C_2, S]+v[C_2, S]) \\ &=[C_2, S]-S=S^*. \end{aligned}$$

Remark. The semi-spray associated with an homogeneous connection is a spray of the same type.

Local expressions

If Γ is a connection of type 1, its associated semi-spray S is locally expressed by

$$S=(y^j, -y^i\Gamma_i^j, -y^i\bar{\Gamma}_i^j).$$

If Γ is of type 2,

$$S=(y^j, z^j, -y^i\Gamma_i^j-z^i\bar{\Gamma}_i^j).$$

THEOREM 6.2. *Let S be a semi-spray of type 2 and let us define*

$$\Gamma_1=\frac{1}{3}\{2[J_2, S]+2[[J_1, S], S]-I\}, \qquad \Gamma_2=\frac{1}{3}\{2[J_2, S]+I\}.$$

Then, we have:

1) *Γ_1 is a connection on M of type 1, its associated semi-spray being*

$$\frac{1}{3}\{2S+S^*+[[C_1, S], S]\}.$$

2) *Γ_2 is a connection on M of type 2, its associated semi-spray being $S+\frac{1}{3}S^*$.*

3) *If S is a spray, then*

a) Γ_1 *is homogeneous and its associated spray is reduce to*

$$\frac{1}{3}\{2S+[[C_1, S], S]\}.$$

b) Γ_2 *is homogeneous and its associated spray is exactly S.*

Proof. 1) For every $X\in\mathfrak{X}(T^2M)$, we have

$$\Gamma_1X=\frac{1}{3}\{2J_2[S, X]-2[S, J_2X]+2J_1[S, [S, X]]-4[S, J_1[S, X]]$$
$$+2[S, [S, J_1X]]-X\}.$$

But $J_1[S, J_2X]=-J_1X$, hence

$$\Gamma_1J_2X=\frac{1}{3}\{2J_2[S, J_2X]+2J_1[S, [S, J_2X]]-J_2X\}.$$

Moreover

$$J_2X=2J_1[S, X]-J_2[S, J_2X]-2J_1[S, [S, J_2X]],$$
$$J_2[S, J_2X]+2J_1[S, X]=-J_2X$$

and consequently

$$\Gamma_1J_2X=\frac{1}{3}\{J_2[S, J_2X]+2J_1[S, X]-2J_2X\}$$
$$=\frac{1}{3}(-J_2X-2J_2X)=-J_2X.$$

On the other hand

$$J_1\Gamma_1X=\frac{1}{3}\{-2J_1[S, J_2X]-4J_1[S, J_1[S, X]]+2J_1[S, [S, J_1X]]-J_1X\}$$

and, since

$$J_1X=J_1[S, [S, J_1X]]-2J_1[S, J_1[S, X]]$$

we deduce $J_1\Gamma_1X=J_1X$ and, thus, Γ_1 is a connection of type 1.

The semi-spray associated with Γ can be calculated as follows; let h_1 be the horizontal projector of Γ_1; then

$$h_1S=\frac{1}{2}(I+\Gamma_1)S=\frac{1}{3}(S-[S, J_2S]+[S, [S, J_1S]])$$
$$=\frac{1}{3}(S-[S, C_2]+[S, [S, C_1]])=\frac{1}{3}(S^*+2S+[[C_1, S], S]).$$

2) For Γ_2 we have

$$\Gamma_2 X=\frac{1}{3}(2J_2[S, X]-2[S, J_2X)+X)$$

and therefore

$$J_2\Gamma_2 X=\frac{1}{3}\{4J_1[S, X]-2J_2[S, J_2X]+J_2X\}.$$

But

$$4J_1[S, X]-2J_2[S, J_2X]=2J_2X$$

and, consequently,

$$J_2\Gamma_2 X=J_2X.$$

On the other hand

$$\Gamma_2 J_1X=\frac{1}{3}\{2J_2[S, J_1X]+J_1X\}=\frac{1}{3}\{-4J_1X+J_1X\}=-J_1X$$

and, thus, Γ_2 is a connection of type 2.

If h_2 denotes the horizontal projector of Γ_2, we obtain its associated semi-spray as given by

$$h_2S=\frac{1}{2}(I+\Gamma_2)S=\frac{1}{2}\Big(S-\frac{2}{3}[S, J_2S]+\frac{1}{3}S\Big)=\frac{2}{3}S+\frac{1}{3}[C_2, S]$$

$$=\frac{2}{3}S+\frac{1}{3}S+\frac{1}{3}[C_2, S]-\frac{1}{3}S=S+\frac{1}{3}S^*.$$

3) Suppose now that S is a spray of type 2. From Jacobi's identity

$$[C_2, [J_1, S]]+[J_1, [S, C_2]]+[S, [C_2, J_1]]=0$$

from which we find

$$[C_2, [J_1, S]]+[J_1, S]=0$$

and, consequently, if $H_1=1/2[C_2, \Gamma_1]$ is the tension of Γ_1, we obtain

$$6H_1=2[C_2, [J_2, S]]+2[C_2, [[J_1, S], S]]-[C_2, I]$$

$$=2[C_2, [[J_1, S], S]].$$

Applying once more Jacobi's identity we have

$$[C_2, [[J_1, S], S]]+[[J_1, S], [S, C_2]]+[S, [C_2, [J_1, S]]]=0$$

and thus

$$[C_2, [[J_1, S], S]=0.$$

Analogously, if $H_2=1/2[C_2, \Gamma_2]$ is the tension of Γ_2, we deduce

$$6H_2=[C_2, 2[J_2, S]]+[C_2, I]=2[C_2, [J_2, S]]$$

and, from Jacobi's identity

$$[C_2, [J_2, S]]+[J_2, [S, C_2]]+[S, [C_2, J_2]]=0$$

or, equivalently

$$[C_2, [J_2, S]]-[J_2, S]-[S, J_2]=0 \text{ i.e. } [C_2, [J_2, S]]=0.$$

§ 7. Paths of semi-sprays and connections.

DEFINITION 7.1. A path of a semi-spray S is a parametric curve $f: I \to M$ such that $(f'')'=S \circ f''$ i.e., such that the canonical lift f'' of f to T^2M is an integral curve of S.

If S is a spray, its paths are called geodesics.

If S is a semi-spray of type 1, its paths are the solutions of the system of differential equations

$$\left.\begin{aligned} &\frac{d^2x^i}{dt^2}-S_1^i\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)=0 \\ &\frac{d^3x^i}{dt^3}-S_2^i\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)=0 \end{aligned}\right. \qquad i=1, 2, \cdots, n.$$

The paths of a semi-spray of type 2 are the solutions of the system of differential equations

$$\frac{d^3x^i}{dt^3}-S^i\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)=0, \qquad i=1, 2, \cdots, n.$$

DEFINITION 7.2. A parametric curve f in M is said path of a connection Γ on M if

$$v \circ (f'')'=0$$

v being the vertical projector of Γ.

If Γ is homogeneous, its paths are called geodesics.

The paths of a connection Γ of type 1 satisfy the system of differential equations

$$\frac{d^2x^j}{dt^2}=-\Gamma_i^j\frac{dx^i}{dt}, \qquad \frac{d^3x^j}{dt^3}=-\bar{\Gamma}_i^j\frac{dx^i}{dt}$$

and if Γ is of type 2, they satisfy

$$\frac{d^3x^j}{dt^3}=-\Gamma_i^j\frac{dx^i}{dt}-\bar{\Gamma}_i^j\frac{d^2x^i}{dt^2}.$$

PROPOSITION 7.3. *The paths of a connection Γ are the paths of its associated semi-spray.*

The proof is an immediate consequence of the local expressions previously obtained.

§ 8. Curvature.

DEFINITION 8.1. Let Γ be a connection on M (indistinctly of type 1 or 2). The curvature of Γ is the vector 2-form R, differentiable C^∞ on $\mathcal{T}^2M$, defined by $R=-1/2[h, h]$, h being the horizontal projector of Γ.

Local expressions

Let $X, Y \in x(T^2M)$ be locally expressed by

$$X=(x^i, y^i, z^i; a^i, b^i, c^i), \qquad Y=(x^i, y^i, z^i; \alpha^i, \beta^i, \gamma^i).$$

Then, if Γ is of type 1, we find

$$R(X, Y)=a^i\alpha^j\Big(\frac{\partial\Gamma^k_i}{\partial x^i}-\frac{\partial\Gamma^k_j}{\partial x^j}+\Gamma^l_j\frac{\partial\Gamma^k_i}{\partial y^l}-\Gamma^l_j\frac{\partial\Gamma^k_j}{\partial y^l}+\bar\Gamma^l_j\frac{\partial\Gamma^k_i}{\partial z^l}-\bar\Gamma^l_i\frac{\partial\Gamma^k_j}{\partial z^l}\Big)\frac{\partial}{\partial y^k}$$

$$+a^i\alpha^j\Big(\frac{\partial\bar\Gamma^k_j}{\partial x^i}-\frac{\partial\bar\Gamma^k_i}{\partial x^j}+\Gamma^l_j\frac{\partial\bar\Gamma^k_i}{\partial y^l}-\Gamma^l_i\frac{\partial\bar\Gamma^k_j}{\partial y^l}+\bar\Gamma^l_j\frac{\partial\bar\Gamma^k_i}{\partial z^l}-\bar\Gamma^l_i\frac{\partial\bar\Gamma^k_j}{\partial z^l}\Big)\frac{\partial}{\partial z^k}.$$

If Γ is of type 2, we find

$$R(X, Y)=\Big[a^i\alpha^j\Big(\frac{\partial\Gamma^k_j}{\partial x^i}-\frac{\partial\Gamma^k_i}{\partial x^j}+\Gamma^l_j\frac{\partial\Gamma^k_i}{\partial z^l}-\Gamma^l_i\frac{\partial\Gamma^k_j}{\partial z^l}\Big)$$

$$+b^i\beta^j\Big(\frac{\partial\bar\Gamma^k_j}{\partial y^i}-\frac{\partial\bar\Gamma^k_j}{\partial y^j}+\bar\Gamma^l_j\frac{\partial\bar\Gamma^k_i}{\partial z^l}-\bar\Gamma^l_i\frac{\partial\bar\Gamma^k_j}{\partial z^l}\Big)$$

$$+(a^i\alpha^j-b^j\beta^i)\Big(\frac{\partial\Gamma^k_i}{\partial y^j}+\frac{\partial\Gamma^k_j}{\partial x^i}+\Gamma^l_j\frac{\partial\Gamma^k_i}{\partial z^l}+\bar\Gamma^l_j\frac{\partial\Gamma^k_i}{\partial z^l}\Big)\Big]\cdot\frac{\partial}{\partial z^k}.$$

The following proposition is easily deduced from the local expressions of the curvature.

PROPOSITION 8.2. *If Γ is a connection on M of type* 1 (*respectively, of type* 2), *the curvature of Γ is a semibasic form of type* 2 (*respect. of type* 1).

PROPOSITION 8.3. (Bianchi's identities) *Let Γ be a connection on M* (*indistinctly of type* 1 *or* 2). *Then, the following identities are verified*

$$\text{I}.\ [J_1, R]=[h, [J_1, h]] \qquad \text{II}.\ [h, R]=0.$$

$$[J_2, R]=[h, [J_2, h]]$$

Proof. Let us recall Jacobi's identity for vector 1-forms L, M, N:

$$[L, [M, N]]+[M, [N, L]]+[N, [L, M]]=0.$$

If we put $L=J_1$, $M=N=h$, we obtain

$$[J_1, [h, h]]+[h, [h, J_1]]+[h, [J_1, h]]=0$$

i.e.

$$[J_1, [h, h]]=-2[h, [J_1, h]]$$

or, equivalently $[J_1, R]=[h, [J_1, h]]$.

In the same way, if we put $L=J_2$, $M=N=h$, we obtain $[J_2, R]=[h, [J_2, h]]$.

Finally, if $M=N=L=h$, we have $[h, [h, h]]=0$, and thus $[h, R]=0$.

PROPOSITION 8.4. *Let Γ be a connection on M. Then $[C_2, R]=-[h, H]$.*

Proof. From Jacobi's identity we obtain

$$[C_2, [h, h]]+[h, [h, C_2]]-[h, [C_2, h]]=0$$

and, thus $[C_2, [h, h]]=2[h, [C_2, h]]$. But $[C_2, h]=H$, hence $[C_2, R]=-[h, H]$.

COROLLARY 8.5. *If Γ is an homogeneous connection, its curvature R is also an homogeneous vector form.*

§ 9. f-structure associated with a connection.

PROPOSITION 9.1. *Let Γ be a connection on M of type 1, with horizontal projector h. Then, there exists one and only one vector 1-form F on T^2M, differentiable C^∞ on $\mathcal{T}^2M$, such that*

$$FJ'=h, \quad Fh=-J', \quad FJ_1=0,$$

where $J'=J_2h$.

In fact, F is well defined from these identities, and it is uniquely determined by its action on vertical and horizontal vector fields.

Local expression of F.

Let U be a coordinate neighborhood of M and (x^i, y^j, z^i) the induced coordinate functions on $\pi_2^{-1}(U)$. Then, we have

$$F\left(\frac{\partial}{\partial z^i}\right)=FJ_1\left(\frac{\partial}{\partial x^i}\right)=0,$$

$$F\left(\frac{\partial}{\partial y^i}\right)=F\left(J'\left(\frac{\partial}{\partial x^i}\right)+2\Gamma_i^j\frac{\partial}{\partial z^j}\right)=FJ'\left(\frac{\partial}{\partial x^i}\right)+2\Gamma_i^jF\left(\frac{\partial}{\partial z^j}\right)$$

$$=FJ'\left(\frac{\partial}{\partial x^i}\right)=h\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial x^i}-\Gamma_i^j\frac{\partial}{\partial y^j}-\bar{\Gamma}_i^j\frac{\partial}{\partial z^j}.$$

On the other hand

$$-J'\left(\frac{\partial}{\partial x^i}\right)=Fh\left(\frac{\partial}{\partial x^i}\right)=F\left(\frac{\partial}{\partial x^i}-\Gamma_i^j\frac{\partial}{\partial y^j}-\bar{\Gamma}_i^j\frac{\partial}{\partial z^j}\right)$$

$$=F\left(\frac{\partial}{\partial x^i}\right)-\Gamma_i^jF\left(\frac{\partial}{\partial y^j}\right)=F\left(\frac{\partial}{\partial x^i}\right)-\Gamma_i^j\left(\frac{\partial}{\partial x^j}-\Gamma_j^k\frac{\partial}{\partial y^k}-\bar{\Gamma}_j^k\frac{\partial}{\partial z^k}\right)$$

$$=F\left(\frac{\partial}{\partial x^i}\right)-\Gamma_i^j\frac{\partial}{\partial x^j}+\Gamma_i^j\Gamma_j^k\frac{\partial}{\partial y^k}+\Gamma_i^j\bar{\Gamma}_j^k\frac{\partial}{\partial z^k}$$

and then

$$F\left(\frac{\partial}{\partial x^i}\right)=\Gamma_i^j\frac{\partial}{\partial x^j}-(\delta_i^j+\Gamma_i^k\Gamma_k^j)\frac{\partial}{\partial y^j}+(2\Gamma_i^j-\Gamma_i^k\bar{\Gamma}_k^j)\frac{\partial}{\partial z^j}\,.$$

In a matrix form, F is given by

$$F:\begin{pmatrix} \Gamma_i^j & \delta_i^j & 0 \\ -\delta_i^j-\Gamma_k^j\Gamma_i^k & -\Gamma_i^j & 0 \\ 2\Gamma_i^j-\bar{\Gamma}_k^j\Gamma_i^k & -\bar{\Gamma}_i^j & 0 \end{pmatrix}.$$

PROPOSITION 9.2. *The vector* 1-*form* F *defines on* T^2M *an* f-*structure of constant rank* $2n$, *which we call the* f-*structure associated with connection* Γ *of type* 1.

Proof. From the local expression above for F, it is easily derived that rank $F=2n$ and $F^3+F=0$.

We shall now study the integrability of this f-structure F, following Yano-Ishihara [20].

Let $l=-F^2$, $m=F^2+I$ be the projection operators of F and $\boldsymbol{L}=\operatorname{Im} l$, $\boldsymbol{M}=\operatorname{Im} m$ denote the complementary distributions associated with l and m; they have dimension $2n$ and n respectively.

Since $\boldsymbol{M}=V^{\pi_{12}}(T^2M)$, the distribution $\boldsymbol{M}$ is always completely integrable.

Before proceeding further, we shall prove the following three lemmas.

LEMMA 9.3. *The vector* 2-*form* $[J', h]$ *is semibasic of type* 2.

Proof. If $X, Y\in\mathfrak{X}(T^2M)$, we have

$$[J', h](J_2X, Y)=J'[J_2X, Y]-J'[J_2X, hY]$$

$$=J'[J_2X, hY+vY]-J'[J_2X, hY]=J'[J_2X, vY]=0$$

On the other hand

$$J_1[J', h](X, Y)=J_1[J'X, hY]+J_1[hX, J'Y]-J_1[J'X, Y]-J_1[X, J'Y]$$

$$=J_1([J'X, hY]-[J'X, Y])+J_1([hX, J'Y]-[X, J'Y])=0\,.$$

LEMMA 9.4. *The vector* 2-*form* $N_{J'}=1/2[J', J']$ *is semibasic of type* 2.

Proof. If $X, Y \in \mathfrak{X}(T^2M)$, we have

$$N_{J'}(J_2X, Y) = -J'[J_2X, J'Y] = 0$$

since $[J_2X, J'Y]$ is vertical. Moreover,

$$J_1N_{J'}(X, Y) = J_1[J'X, J'Y] = 0 .$$

LEMMA 9.5. $J_2 \circ [J', h] = N_{J'}$.

Proof. We have, for every $X, Y \in \mathfrak{X}(T^2M)$,

$$\begin{aligned}[J', h](X, Y) &= [J', h](hX, hY) \\ &= [J'X, hY] + [hX, J'Y] - J'[hX, hY] - h[J'X, hY] - h[hX, J'Y]\end{aligned}$$

and then

$$\begin{aligned}(J_2 \circ [J', h])(X, Y) &= J_2[J'X, hY] + J_2[hX, J'Y] - 2J_1[hX, hY] \\ &\quad - J'[J'X, hY] - J'[hX, J'Y] .\end{aligned}$$

On the other hand

$$\begin{aligned}N_{J'}(X, Y) = N_{J'}(hX, hY) &= [J'X, J'Y] - J_2[J'X, hY] - J_2[hX, J'Y] \\ &\quad + 2J_1[hX, hY] .\end{aligned}$$

Moreover, since J_2 is integrable

$$0 = N_{J_2}(hX, hY) = [J'X, J'Y] - J_2[J'X, hY] - J_2[hX, J'Y] + 2J_1[hX, hY]$$

and we obtain

$$J_2 \circ [J', h] = N_{J'} .$$

THEOREM 9.6. *Let Γ be a connection on M of type 1, with curvature form R. If the distribution $\boldsymbol{L}$ is completely integrable, $R=0$ and $[J', h]=0$, then the f-structure F associated with Γ is partially integrable.*

Proof. For every $X, Y \in \mathfrak{X}(T^2M)$, taking into account Lemma 9.3, we have

$$\begin{aligned}[J', h](X, Y) &= [J', h](hX, hY) \\ &= [J'X, hY] + [hX, J'Y] - J'[hX, hY] - h[J'X, hY] - h[hX, J'Y]\end{aligned}$$

Therefore

$$\begin{aligned}(F \circ [J', h])(hX, hY) &= F[J'X, hY] + F[hX, J'Y] + J'[J'X, hY] \\ &\quad + J'[hX, J'Y] - h[hX, hY] .\end{aligned}$$

On the other hand

$$(h^*N_F)(X, Y)=N_F(hX, hY)$$
$$=[J'X, J'Y]+F[J'X, hY]+F[hX, J'Y]+F^2[hX, hY]$$

and

$$N_{J'}(X, Y)=N_{J'}(hX, hY)=[J'X, J'Y]-J'[J'X, hY]-J'[hX, J'Y].$$

Since $R=0$, it follows

$$h[hX, hY]=[hX, hY]$$

and then

$$F^2[hX, hY]=-[hX, hY].$$

Thus

$$(h^*N_F)(X, Y)=[J'X, J'Y]+F[J'X, hY]+F[hX, J'Y]-[hX, hY]$$
$$=(F\circ[J', h])(X, Y)+N_{J'}(X, Y)=(F\circ[J', h]+N_{J'})(X, Y)$$

i. e.

$$h^*N_F=F\circ[J', h]+N_{J'}$$

and, by using Lemma 9.5, we deduce $h^*N_F=0$.

We also have

$$(J')^*N_F(X, Y)=[hX, hY]-F[hX, J'Y]-F[J'X, hY]+F^2[J'X, J'Y]$$

and, since $N_{J'}=0$, $[J'X, J'Y]\in \operatorname{Im} J'$; thus

$$(J'^*)N_F(X, Y)=[hX, hY]-F[hX, J'Y]-F[J'X, hY]-[J'X, J'Y]$$
$$=-(h^*N_F)(X, Y)$$

i. e.

$$(J')^*N_F=-h^*N_F=0.$$

Finally, taking into account the integrability of $\boldsymbol{L}$, and by a similar device, we obtain

$$N_F(J'X, hY)=(F\circ h^*N_F)(X, Y)=0.$$

We shall now consider the case of connections of type 2.

PROPOSITION 9.7. *Let Γ be a connection on M of type 2, with horizontal projector h. Then, there exists one and only one vector 1-form G on T^2M, differentiable C^∞ on $\mathcal{T}^2M$, such that*

$$GJ_1=h',\quad Gh'=-J_1,\quad Gh(X)=0,\quad \textit{if}\ X\notin V^{\pi_2}(T^2M)$$

where $h'=hJ_2$.

In fact, G is well defined from these identities, and it is uniquely determined by its action on vertical and horizontal vector fields.

Local expression of G.

As in the case of a connection of type 1, and by similar devices, the following expression of G in a matrix form is obtained

$$G:\begin{pmatrix} 0 & 0 & 0 \\ \Gamma_i^j & \bar{\Gamma}_i^j & \delta_i^j \\ -\Gamma_i^k\bar{\Gamma}_k^j & -\delta_i^j-\bar{\Gamma}_i^k\bar{\Gamma}_k^j & -\bar{\Gamma}_i^j \end{pmatrix}.$$

PROPOSITION 9.8. *The vector 1-form G defines on T^2M an f-structure of constant rank $2n$, which we call the f-structure associated with connection Γ of type 2.*

Proof. It is easily derived from the local expression of G above.

As before, let $l=-G^2$, $m=G^2+I$ be the projection operators of G and $\boldsymbol{L}=\operatorname{Im} l$, $\boldsymbol{M}=\operatorname{Im} m$ denote the complementary distributions associated with l and m; they have dimension $2n$ and n, respectively.

LEMMA 9.9. $$J_2G=2v,$$ *v being the vertical projector of Γ.*

Proof. It is easily checked since

$$J_2GJ_1=J_2h'=2J_1,\quad J_2Gh'=0.$$

LEMMA 9.10. *The vector 2-form $R'=1/2[h', h']$ is semibasic of type 1.*

Proof. Obviously, $(h')^2=0$; then, for every $X, Y\in\mathfrak{X}(T^2M)$,

$$R'(X, Y)=[h'X, h'Y]-h'[h'Y, Y]-h'[X, h'Y]$$

and, hence, $R'(J_1X, Y)=0$.

Moreover

$$J_2R'(X, Y)=J_2[h'X, h'Y]-2J_1[h'X, Y]-2J_1[X, h'Y]$$

and, a simple calculation involving local coordinates leads us to

$$J_2R'(X, Y)=0.$$

LEMMA 9.11. *The vector 2-form $[J_1, h']$ is semibasic of type 1.*

Proof. For every $X, Y\in\mathfrak{X}(T^2M)$, we have

$$[J_1, h'](J_1X, Y)=-J_1[J_1X, h'Y]-h'[J_1X, J_1Y]=0$$

and, moreover,

$$(J_2[J_1, h'])(X, Y)=J_2[J_1X, h'Y]+J_2[h'X, J_1Y]-2J_1[X, J_1Y]-2J_1[J_1X, Y]$$
$$=J_2[J_1X, h'Y]+J_2[h'X, J_1Y]-2[J_1X, J_1Y]=0\,.$$

THEOREM 9.12. *If the f-structure G is integrable, then $R'=0$ and $[J_1, h']=0$.*

Proof. Putting $N_G=1/2[G, G]$, we have, for every X, $Y\in\mathfrak{X}(T^2M)$

$$(h')^*N_G(X, Y)=[J_1X, J_1Y]+G[J_1X, h'Y]+G[h'X, J_1Y]+G^2[h'X, h'Y]$$
$$=[J_1X, J_1Y]+G[J_1X, h'Y]+G[h'X, J_1Y]-[h'X, h'Y]\,.$$

On the other hand,

$$[J_1, h'](X, Y)=[J_1X, h'Y]+[h'X, J_1Y]-J_1[X, h'Y]$$
$$-J_1[h'X, Y]-h'[X, J_1Y]-h'[J_1X, Y]$$

and, therefore,

$$G[J_1, h'](X, Y)=G[J_1X, h'Y]+G[h'X, J_1Y]-h'[X, h'Y]$$
$$-h'[h'X, Y]+[J_1X, J_1Y]\,.$$

Thus

$$((h')^*N_G-G\circ[J_1, h'])(X, Y)=-[h'X, h'Y]+h'[X, h'Y]$$
$$+h'[h'X, Y]=-R'(X, Y)$$

i. e.

$$(h')^*N_G=G\circ[J_1, h']-R'\,.$$

Operating J_2 on both sides of this identity, we obtain

$$J_2(h')^*N_G=2v[J_1, h']=2[J_1, h']$$

since $[J_1, h']$ and R' are semibasic forms.

Now, the result follows from the fact that G is integrable if and only if $N_G=0$.

A partial converse of this theorem can be stablished as follows:

THEOREM 9.13. *If $R'=0$ and $[J_1, h']=0$, then the f-structure G is partially integrable.*

Proof. Firstly, from the proof of Theorem 9.12, we have

$$(h')^*N_G=G\circ[J_1, h']=R'$$

and, thus $$(h')^* N_G = 0 .$$

Secondly, for every $X, Y \in \mathfrak{X}(T^2M)$,

$$\begin{aligned} N_G(h'X, J_1Y) &= -[J_1X, h'Y] - G[h'X, h'Y] + G[J_1X, J_1Y] - [h'X, J_1Y] \\ &= (G \circ (h')^* N_G)(X, Y) \end{aligned}$$

and, then $$N_G(h'X, J_1Y) = 0 .$$

Thirdly,

$$\begin{aligned} N_G(J_1X, J_1Y) &= [h'X, h'Y] - G[h'X, J_1Y] - G[J_1X, h'Y] - [J_1X, J_1Y] \\ &= -(h')^* N_G(X, Y) = 0 . \end{aligned}$$

These three identities together imply the partial integrability of G.

Remark. Note that the vanishing of curvature R of Γ implies that of R'; in fact

$$\begin{aligned} R'(X, Y) = R'(hX, hY) &= [h'X, h'Y] - h'[h'X, hY] - h'[hX, h'Y] \\ &= [hJ_2X, hJ_2Y] - hJ_2[hJ_2X, hY] - hJ_2[hX, hJ_2Y] \\ &= h[J_2X, J_2Y] - hJ_2[J_2X, Y] - hJ_2[X, J_2Y] = h(-2J_1[X, Y]) = 0 . \end{aligned}$$

§ 10. Prolongation of metrics on the vertical bundles to $\mathcal{T}^2M$.

Let $\bar{g}$ be a Riemannian metric on the vertical bundle $V^{\pi_2}(\mathcal{T}^2M)$. Then, fixed a point $\omega \in \mathcal{T}^2M$, we can define a metric $\bar{g}_\omega$ on TM as follows:

$$\bar{g}_\omega(u, v) = \bar{g}(h_2(\omega, u), h_2(\omega, v)), \quad \forall u, v \in T_{\pi_{12}(\omega)}(TM)$$

where h_2 is the canonical isomorphism introduced in § 1.

Therefore, a Riemannian metric on the vertical bundle $V^{\pi_2}(\mathcal{T}^2M)$ can be considered as a Riemannian metric on TM, the latter depending not only on the point but also on a previously fixed point $\omega \in T^2M$, with ω non belonging to the zero cross-section.

Given on M a connection Γ of type 1, it is possible to extend $\bar{g}$ to the whole fibre bundle $T(\mathcal{E}^2M)$, that is, to a Riemannian metric g_Γ on $\mathcal{T}^2M$, by putting

$$g_\Gamma(X, Y) = \bar{g}(J'X, J'Y) + \bar{g}(vX, vY), \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}^2M)$$

being h, v and J' as defined in the previous sections.

PROPOSITION 10.1. *g_Γ is a Riemannian metric on $\mathcal{T}^2M$, which will be called the prolongation of $\bar{g}$ along the connection Γ.*

Proof. Bilinearity and symmetry of g_Γ are immediate. Moreover, g_Γ is positive definite, since

$$g_\Gamma(X, X)=\bar{g}(J'X, J'Y)+\bar{g}(vX, vY)$$

and because $J'X$ and vX are simultaneously zero if and only if X is zero.

Finally, g_Γ extends $\bar{g}$, since $g_\Gamma(J_2X, J_2Y)=\bar{g}(J_2X, J_2Y)$ as consequence of the fact that $J'J_2=J_2hJ_2=0$.

PROPOSITION 10.2. *A Riemannian metric g on $\mathcal{T}^2M$ is the prolongation of a Riemannian metric $\bar{g}$ on $V^{\pi_2}(\mathcal{T}^2M)$ along a connection Γ on M of type* 1 *if and only if*

1) $g(hX, vY)=0$
2) $g(hX, hY)=g(J'X, J'Y)=\bar{g}(J'X, J'Y)$, $g(J_2X, J_2Y)=\bar{g}(J_2X, J_2Y)$

for every X, $Y\in\mathcal{X}(\mathcal{T}^2M)$.

Proof. Let g_Γ be the prolongation of $\bar{g}$ along a connection Γ of type 1. Then,

$$g_\Gamma(hX, vY)=\bar{g}(J'hX, J'vY)=0$$

since $J'v=0$. Moreover

$$g_\Gamma(hX, hY)=\bar{g}(J'hX, J'hY)=\bar{g}(J'X, J'Y)$$

and

$$g_\Gamma(J_2X, J_2Y)=\bar{g}(J_2X, J_2Y).$$

The converse is immediate.

PROPOSITION 10.3. *Let Γ be a connection on M of type* 1 *and $\bar{g}$ a Riemannian metric on the vertical bundle $V^{\pi_2}(\mathcal{T}^2M)$, such that $\bar{g}(J_1X, J_2Y)=0$, $\forall X, Y\in\mathcal{X}(\mathcal{T}^2M)$. Then, the prolongation g_Γ of g along Γ is a hor-ehresmannian metric with respect to the f-structure F associated to Γ.*

Proof. Let $l=-F^2$, $m=F^2+I$ be the projection operators of F. It is easily verified that

$$g_\Gamma(lX, mY)=0, \quad \forall X, Y\in\mathcal{X}(\mathcal{T}^2M)$$

that is, the distribution $\boldsymbol{L}$ and $\boldsymbol{M}$ are mutually orthogonal with respect to g_Γ.

Moreover,

$$g_\Gamma(X, FX)=0, \quad \forall X\in\mathcal{X}(\mathcal{T}^2M)$$

and, thus, g_Γ is hor-ehresmannian with respect to F. Note that there exist Riemannian metrics on $\mathcal{T}^2M$ verifying

$$g(J_1X, J_2Y)=0, \quad \forall X, Y\in\mathcal{X}(\mathcal{T}^2M).$$

In fact, given a Riemannian metric g on M, the second canonical lift g^{II} of g to $\mathcal{T}^2M$, [20], makes mutually orthogonal $V^{\pi_2}(\mathcal{T}^2M)$ and $V^{\pi_{12}}(\mathcal{T}^2M)$.

Under the hypothesis of Proposition 10.3., g_Γ permits to define the fundamental form K_Γ by putting

$$K_\Gamma(X, Y)=g_\Gamma(FX, Y), \quad \forall X, Y\in\mathcal{X}(\mathcal{T}^2M).$$

We then have

PROPOSITION 10.4. *Under the hypothesis of Proposition* 10.3., *the fundamental form* K_Γ *verifies*

$$K_\Gamma(X, Y)=g_\Gamma(X, J'Y)-g_\Gamma(J'X, Y), \quad \forall X, Y\in\mathcal{X}(\mathcal{T}^2M).$$

Proof. From previous definitions, we have

$$\begin{aligned} K_\Gamma(X, Y)&=g_\Gamma(FX, Y)=g_\Gamma(FhX+FvX, hY+vY)\\ &=g_\Gamma(FhX, hY)+g_\Gamma(FvX, hY)+g_\Gamma(FhX, vY)+g_\Gamma(FvX, vY)\\ &=-g_\Gamma(J'X, hY)+g_\Gamma(FvX, hY)-g_\Gamma(J'X, vY)+g_\Gamma(FvX, vY)\end{aligned}$$

for every $X, Y\in\mathcal{X}(\mathcal{T}^2M)$.

On the other hand

$$g_\Gamma(J'X, hY)=0$$

since v and h are mutually orthogonal with respect to g_Γ. But $Fv=hF$, hence

$$g_\Gamma(FvX, vY)=g_\Gamma(hFX, vY)=0$$

and, therefore

$$K_\Gamma(X, Y)=g_\Gamma(hFX, hY)-g_\Gamma(J'X, vY).$$

But

$$g_\Gamma(hFX, hY)=\bar{g}(J'FX, J'Y)=\bar{g}(vX, J'Y)=g_\Gamma(vX, J'Y)$$

and, consequently,

$$\begin{aligned} K_\Gamma(X, Y)&=g_\Gamma(vX, J'Y)-g_\Gamma(J'X, vY)=g_\Gamma(hX+vX, J'Y)-g_\Gamma(J'X, vY+hY)\\ &=g_\Gamma(X, J'Y)-g_\Gamma(J'X, Y).\end{aligned}$$

We shall now consider the case of the vertical bundle $V^{\pi_{12}}(\mathcal{T}^2M)$. Let $\bar{g}$ be a Riemannian metric on $V^{\pi_{12}}(\mathcal{T}^2M)$; as before, for a fixed point $\omega\in\mathcal{T}^2M$, we can define a metric $\bar{g}_\omega$ on M by putting

$$\bar{g}_\omega(u, v)=\bar{g}(h_1(\omega, u), h_1(\omega, v)), \quad \forall u, v\in T_{\pi_2(\omega)}(M)$$

where h_1 is the canonical isomorphism introduced in § 1. Thus, a Riemannian metric on the vertical bundle $V^{\pi_{12}}(\mathcal{T}^2M)$ can be considered as a Riemannian metric on M, the latter depending not only on the point but also on a previously fixed point $\omega \in T^2M$, with ω non belonging to the zero cross-section.

If Γ is a connection on M of type 2, we can extend $\bar{g}$ to the whole fibre bundle $T(\mathcal{T}^2M)$, that is, to a Riemannian metric g_Γ on $\mathcal{T}^2M$ by putting

$$g_\Gamma(X, Y)=\bar{g}(J_1X, J_1Y)+\bar{g}(vX, vY), \quad \forall X, Y\in\mathcal{X}(\mathcal{T}^2M).$$

PROPOSITION 10.5. *g_Γ is a Riemannian metric on $\mathcal{T}^2M$, which will be called the prolongation of $\bar{g}$ along the connection Γ.*

We omit the proof, which is analogous to that of Proposition 10.1.

The following Propositions are all similar to those in the case of metrics on $V^{\pi_2}(\mathcal{T}^2M)$.

PROPOSITION 10.6. *A Riemannian metric g in $\mathcal{T}^2M$ is the prolongation of a Riemannian metric $\bar{g}$ on the vertical bundle $V^{\pi_{12}}(\mathcal{T}^2M)$ along a connection Γ on M of type 2 if and only if*

$$1)\quad g(hX, vY)=0, \qquad 2)\quad g(hX, hY)=\bar{g}(J_1X, J_1Y)=g(J_1X, J_1Y)$$

for every $X, Y\in\mathcal{X}(\mathcal{T}^2M)$.

PROPOSITION 10.7. *Let Γ be a connection on M of type 2 and $\bar{g}$ a Riemannian metric on the vertical bundle $V^{\pi_{12}}(\mathcal{T}^2M)$. Then, the prolongation g_Γ of $\bar{g}$ along Γ is a hor-ehresmannian metric with respect to the f-structure G associated to Γ.*

Once more, under the hypothesis of Proposition 10.7., g_Γ permits to define the fundamental form K_Γ by putting

$$K_\Gamma(X, Y)=g_\Gamma(GX, Y), \quad \forall X, Y\in\mathcal{X}(\mathcal{T}^2M).$$

We then have

PROPOSITION 10.8. *Under the hypothesis of Proposition 10.7, the fundamental form K_Γ verifies*

$$K_\Gamma(X, Y)=g_\Gamma(GhX, Y), \quad \forall X, Y\in\mathcal{X}(\mathcal{T}^2M).$$

In particular,

$$K_\Gamma(h'X, Y)=-g_\Gamma(J_1X, Y), \quad K_\Gamma(J_1X, Y)=0.$$

Proof. It is proved by a similar calculation to that in the proof of Proposition 10.4.

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Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad de Santiago de Compostela
España