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THE SPECTRUM OF THE LAPLACIAN FOR SOME 6-DIMENSIONAL K-SPACES

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1. Introduction.

Let (M, g) be a compact orientable Riemannian manifold with metric tensor g. By Δ we denote the Laplacian acting on differentiable functions on M. Then we have the spectrum

$$\mathrm{Spec}(M, g)=\{0\geqq\lambda_1\geqq\lambda_2\geqq\cdots>-\infty\}$$

where each eigenvalue is repeated as many time as its multiplicity indicates. The spectrum Spec(M, g) exerts an influence on the geometry of (M, g). It is interesting to see the relation of Spec(M, g) on the geometry of (M, g). For the study of this, M. Berger and T. Sakai used the coefficients of the asymptotic expansion of Minakshisundaram-Pleijel. In [6], after a long calculation, Sakai obtained the following

THEOREM A. *Let (M, g) and (M', g') be compact connected orientable Einstein manifolds with dimension $M=6$. We assume that $\chi(M)=\chi(M')$ and* Spec$(M, g)=$ Spec(M', g') *hold where $\chi(M)$ denotes the Euler-Poincaré characteristic of M. Then (M, g) is locally symmetric if and only if (M', g') is locally symmetric.*

In the present paper, we shall prove the following

THEOREM B. *Let (M, g, J) and (M', g', J') be* 6*-dimensional complete, connected K-spaces which are non-Kählerian. We assume that $\chi(M)=\chi(M')$ and* Spec$(M, g)=$Spec(M', g'). *Then (M, g) is Riemannian locally* 3*-symmetric if and only if (M', g') is Riemannian locally* 3*-symmetric.*

It is well-known that the 6-dimensional non-Kähler K-space (M, g, J) is an Einstein manifold with positive scalar curvature [5]. Therefore M is compact by Myers' theorem. The study of Riemannian 3-symmetric space has been done by A. Gray [4]. We shall give some definitions and preliminary facts on Riemannian 3-symmetric spaces in § 2. Particularly we shall show the relationship between Riemannian 3-symmetric spaces and homogeneous K-spaces. In § 3, we shall prove Theorem B by slight modification of the proof of Theorem A.

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2. Riemannian 3-symmetric spaces.

Throughout this paper, manifolds and tensor fields are assumed to be of class C^∞ unless otherwise specified.

Let (M, g) be a Riemannian manifold with Riemannian connection ∇. By $R=(R_{abc}{}^d)$, $R_1=(R_{ab})$ and S we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively.

Suppose that (M, g) admits a local isometry $\theta_p: U_p \to U_p$ for each point p of M such that

i) $\theta_p{}^3=1$,

ii) p is an isolated fixed point of θ_p,

iii) the tensor field Θ defined by $\Theta_p=(d\theta_p)_p$ is C^∞.

Then we can define an almost complex structure J by

$$\frac{\sqrt{3}}{2}J_p=\Theta_p+\frac{1}{2}I_p, \tag{2.1}$$

where I_p denotes the identity of $T_p(M)$. Since each θ_p is an isometry, the Riemannian metric g is almost Hermitian with respect to J. Furthermore, we assume

iv) each θ_p is holomorphic with respect to J, i.e.,

$$d\theta_p \circ J = J \circ d\theta_p \quad \text{on } U_p.$$

DEFINITION 1. A Riemannian manifold (M, g) is called a *Riemannian locally 3-symmetric space* if (M, g) admits a family of local isometries $\{\theta_p\}$ satisfying the above conditions i), ii), iii) and iv). An almost complex structure J defined by (2.1) is said to be a *canonical* one.

DEFINITION 2. A Riemannian locally 3-symmetric space (M, g) is called a *Riemannian 3-symmetric space* if each θ_p can be extended to a global holomorphic isometry of M.

As an example, we shall consider the 6-dimensional unit sphere S^6. Let C be the Cayley algebra and E be a set of all pure imaginary Cayley numbers. Then E can be identified with the 7-dimensional Euclidean space. For any two points x, y of E, the inner product (x, y) and the vector product $x\times y$ are defined by

$$-(x, y)=\text{the real part of } xy,$$

$$x\times y=\text{the imaginary part of } xy,$$

where xy is a product of x and y in C. The 6-dimensional unit sphere S^6 is the set of all $x \in E$ such that $(x, x)=1$. For any point a of S^6, we define a map $\theta_a : S^6 \to S^6$ by

$$\theta_a(x)=\frac{3}{2}(a, x)a-\frac{1}{2}x+\frac{\sqrt{3}}{2}a \times x .$$

Then we can check by straightforward computation that $\theta_a{}^3=1$ and a is an isolated fixed point of θ_a. With this family $\{\theta_a\}$ and a canonical metric g_0, (S^6, g_0) becomes a Riemannian 3-symmetric space. It may be verified that the canonical almost complex structure J_0 of this family coincides with the one constructed by A. Frölicher [2], and hence (S^6, g_0, J_0) becomes a K-space [3].

DEFINITION 3. Let (M, g, J) be an almost Hermitian manifold. A tensor field T of type (1, 2) on M is called a *homogeneous structure* if it satisfies

(a) $(\nabla_X R)(Y, Z)=[T(X), R(Y, Z)]-R(T(X)Y, Z)-R(Y, T(X)Z)$,
(b) $(\nabla_X T)Y=[T(X), T(Y)]-T(T(X)Y)$,
(c) $\nabla_X J=[T(X), J]$,
(d) $g(T(X)Y, Z)+g(Y, T(X)Z)=0$.

In [8], Sekigawa proved the following

THEOREM 2.1. *Let (M, g, J) be a homogeneous almost Hermitian manifold. Then, there exists a homogeneous structure T on M. Conversely, if a connected, simply connected, complete almost Hermitian manifold (M, g, J) admits a homogeneous structure T, then (M, g, J) is a homogeneous almost Hermitian manifold.*

Now let (M, g, J) be a K-space. We put

$$\hat{T}(X)Y=\frac{1}{2}J(\nabla_X J)Y . \tag{2.2}$$

The tensor field $\hat{T}$ plays an important role in a K-space. It has been shown that $\hat{T}$ always satisfies the conditions (b), (c), (d) for the homogeneous structure [9]. Hence we shall consider only the condition (a). We define a tensor field L of type (1, 4) by

$$L(X, Y, Z)=(\nabla_X R)(Y, Z)-[\hat{T}(X), R(Y, Z)]+R(\hat{T}(X)Y, Z)+R(Y, \hat{T}(X)Z) .$$

Obviously, $L=0$ means that the tensor field $\hat{T}$ satisfies (a). In the case of $\dim M=6$, Sekigawa has calculated in his paper [9] the square of the length of L. By $|P|^2$ we denote the square of the length of a tensor P.

THEOREM 2.2. *Let (M, g, J) be a complete 6-dimensional complete non-Kähler K-space. Then we have*

$$\int_M |L|^2 dM=\int_M\left[|\nabla R|^2-\frac{1}{15}S\left(|R|^2-\frac{1}{15}S^2\right)\right]dM ,$$

where dM denotes the volume element of (M, g).

As for the relation between Riemannian 3-symmetric spaces and homogeneous K-spaces, the present author [7] proved the following

THEOREM. 2.3. *Let (M, g, J) be a complete, connected and simply connected K-space. Then (M, g, J) is a homogeneous almost Hermitian manifold with homogeneous structure $\hat{T}$ if and only if (M, g) is a Riemannian 3-symmetric space with canonical almost complex structure J.*

We shall remark that the proof of the above theorem in [7] actually yields the following slightly more precise result.

THEOREM 2.4. *Let (M, g, J) be a complete and connected K-space. Then the tensor field $\hat{T}$ is the homogeneous structure of (M, g, J) if and only if (M, g) is a Riemannian locally 3-symmetric space with canonical almost complex structure J.*

3. Proof of Theorem B.

We first prove two lemmas. We put

$$\hat{R}=R^{abcd}R_{ab}{}^{uv}R_{cduv},$$

$$\mathring{R}=R^{abcd}R_a{}^u{}_c{}^vR_{budv}.$$

LEMMA 3.1. *Let (M, g) be a compact orientable Einstein manifold of dimension 6. Then we have*

$$\int_M \mathring{R}\, dM=\frac{1}{4}\int_M\left[|\nabla R|^2+\frac{1}{3}S|R|^2-\hat{R}\right]dM. \tag{3.1}$$

Proof. From the computation, we get the following Lichnerowicz's formula.

$$\begin{aligned}\frac{1}{2}\Delta(|R|^2)=|\nabla R|^2&+4R^{abcd}\nabla_a\nabla_c R_{bd}\\ &+2R^{uv}R_u{}^{abc}R_{vabc}-\hat{R}-4\mathring{R}.\end{aligned} \tag{3.2}$$

If (M, g) is 6-dimensional Einsteinian, (3.2) is reduced to

$$\frac{1}{2}\Delta(|R|^2)=|\nabla R|^2+\frac{1}{3}S|R|^2-\hat{R}-4\mathring{R}. \tag{3.3}$$

Applying Green's theorem to (3.3), we get (3.1).

Making use of Lemma 3.1, we obtain the following formula (3.4) due to Sakai [6].

LEMMA 3.2. *Let (M, g) be a compact orientable Einstein manifold of dimension 6. The Euler-Poincaré characteristic $\chi(M)$ is given by*

$$\chi(M)=\frac{1}{384\pi^{3}}\int_{M}\left[6\hat{R}-2|\nabla R|^{2}+\frac{1}{9}S^{3}-\frac{5}{3}S|R|^{2}\right]dM. \tag{3.4}$$

Proof. In a 6-dimensional compact orientable Riemannian manifold, it is well-known that $\chi(M)$ is given by

$$\chi(M)=\frac{1}{384\pi^{3}}\int_{M}[S^{3}-12S|R_{1}|^{2}+3S|R|^{2}+16R^{ab}R_{a}{}^{c}R_{bc}$$
$$+24R^{ab}R^{cd}R_{abcd}-24R^{uv}R_{u}{}^{abc}R_{vabc}$$
$$+2\hat{R}-8R^{abcd}R_{a}{}^{u}{}_{c}{}^{v}R_{bvdu}]dM.$$

By using the Bianchi's identity repeatedly, we get

$$R^{abcd}R_{a}{}^{u}{}_{c}{}^{v}R_{bvdu}=\mathring{R}-\frac{1}{4}\hat{R}.$$

Thus we have

$$\chi(M)=\frac{1}{384\pi^{3}}\int_{M}[S^{3}-12S|R_{1}|^{2}+3S|R|^{2}+16R^{ab}R_{a}{}^{c}R_{bc}$$
$$+24R^{ab}R^{cd}R_{acbd}-24R^{uv}R_{u}{}^{abc}R_{vabc}$$
$$-8\mathring{R}+4\hat{R}]dM.$$

In Einsteinian case,

$$\chi(M)=\frac{1}{384\pi^{3}}\int_{M}\left[\frac{1}{9}S^{3}-S|R|^{2}-8\mathring{R}+4\hat{R}\right]dM. \tag{3.5}$$

Therefore (3.4) is obtained from (3.5) and (3.1).

We now proceed to prove the theorem. We need the asymptotic expansion of Minakshisundaram-Pleijel for Spec(M, g) given by

$$\sum_{k}\exp(\lambda_{k}t)\sim(4\pi t)^{-m/2}[a_{0}+a_{1}t+a_{2}t^{2}+\cdots],$$

where $m=\dim M$. The coefficients a_0, a_1, a_2 and a_3 have been computed by Berger [1] and Sakai [6]:

$$a_{0}=\mathrm{Vol}(M), \tag{3.6}$$

$$a_{1}=\frac{1}{6}\int_{M}S\,dM, \tag{3.7}$$

$$a_{2}=\frac{1}{360}\int_{M}[5S^{2}-2|R_{1}|^{2}+2|R|^{2}]dM, \tag{3.8}$$

$$a_{3}=\frac{1}{6!}\int_{M}\left[-\frac{142}{63}|\nabla S|^{2}-\frac{26}{63}|\nabla R_{1}|^{2}-\frac{1}{9}|\nabla R|^{2}+\frac{5}{9}S^{3}\right. \tag{3.9}$$

$$-\frac{2}{3}S|R_1|^2+\frac{2}{3}S|R|^2-\frac{4}{7}R^{ab}R_b{}^cR_{ac}$$

$$+\frac{20}{63}R^{ab}R^{cd}R_{acbd}-\frac{8}{63}R^{uv}R_u{}^{abc}R_{vabc}$$

$$+\frac{8}{21}\hat{R}\Big]dM.$$

It may be noticed that instead of Spec$(M, g)=$Spec(M', g'), we mainly use $a_i=a_i'$ for $i=0, 1, 2, 3$.

Since the 6-dimensional non-Kähler K-space is an Einsteinian, the coefficients a_i are rewritten

$$a_0=\mathrm{Vol}(M), \tag{3.6$'$}$$

$$a_1=\frac{1}{6}S\ \mathrm{Vol}(M), \tag{3.7$'$}$$

$$a_2=\frac{7}{3\cdot 180}S^2\ \mathrm{Vol}(M)+\frac{1}{180}\int_M |R|^2 dM, \tag{3.8$'$}$$

$$a_3=\frac{248}{7!\cdot 3^4}S^3\ \mathrm{Vol}(M)+\frac{122}{7!\cdot 3^3}S\int_M |R|^2 dM + \frac{1}{5!\cdot 3^2}\int_M\Big[\frac{4}{7}\hat{R}-\frac{1}{6}|\nabla R|^2\Big]dM. \tag{3.9$'$}$$

From these, $a_i=a_i'$ imply

$$\mathrm{Vol}(M)=\mathrm{Vol}(M'), \tag{3.10}$$

$$S=S', \tag{3.11}$$

$$\int_M |R|^2 dM=\int_{M'}|R'|^2 dM', \tag{3.12}$$

$$\int_M\Big[\frac{24}{7}\hat{R}-|\nabla R|^2\Big]dM=\int_{M'}\Big[\frac{24}{7}\hat{R}'-|\nabla' R'|^2\Big]dM'. \tag{3.13}$$

By Lemma 3.2, we have

$$\int_M\Big[\frac{24}{7}\hat{R}-|\nabla R|^2\Big]dM$$

$$=\frac{384\cdot 4}{7}\pi^3\chi(M)+\frac{1}{7}\int_M\Big[|\nabla R|^2-\frac{1}{15}S\Big(|R|^2-\frac{1}{15}S^2\Big)\Big]dM$$

$$+\frac{101}{3\cdot 5\cdot 7}S\int_M |R|^2 dM-\frac{101}{3^2\cdot 5^2\cdot 7}S^3\ \mathrm{Vol}(M).$$

Considering (3.10)$\sim$(3.12) and Theorem 2.2, $\chi(M)=\chi(M')$ implies

(3.14) $$\int_M |L|^2 dM = \int_{M'} |L'|^2 dM'.$$

Theorem B now follows from Theorem 2.4.

In the course of the proof, we established the following

COROLLARY 3.3. *Let* (M, g, J) *and* (M', g', J') *be* 6-*dimensional complete and connected K-spaces which are non-Kählerian. We assume that* Spec$(M, g)=$ Spec(M', g'). *If*

$$\int_M \hat{R}\, dM = \int_{M'} \hat{R}' dM' \quad or \quad \int_M \mathring{R}\, dM = \int_{M'} \mathring{R}' dM'$$

is satisfied, then (M, g) *is Riemannian locally* 3-*symmetric if and only if* (M', g') *is Riemannian locally* 3-*symmetric.*

We shall conclude this paper by noticing the following

PROPOSITION 3.4. *Let* (M, g, J) *be a* 6-*dimensional complete and connected K-space which is non-Kählerian. Then we have*

$$\chi(M) \leqq \frac{1}{64\pi^3} \int_M \left[\hat{R} - \frac{3}{10} S\left(|R|^2 - \frac{1}{15} S^2\right)\right] dM$$

with equality holding if and only if (M, g) *is a Riemannian locally* 3-*symmetric space.*

Proof. By Lemma 3.2,

$$\int_M |\nabla R|^2 dM = -192\pi^3 \chi(M) + \frac{1}{2} \int_M \left[6\hat{R} + \frac{1}{9} S^3 - \frac{5}{3} S |R|^2\right] dM.$$

From this and Theorem 2.2, we have

$$\int_M |L|^2 dM = -192\pi^3 \chi(M) + 3\int_M \left[\hat{R} - \frac{3}{10} S\left(|R|^2 - \frac{1}{15} S^2\right)\right] dM$$
$$\geqq 0.$$

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