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ON V-HARMONIC FORMS IN COMPACT LOCALLY CONFORMAL KÄHLER MANIFOLDS WITH THE PARALLEL LEE FORM

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Introduction. A locally conformal Kähler manifold (l. c. K-manifold) has been studied by I. Vaisman [8]. Especially when its Lee form is parallel, the manifold seems to have properties exceedingly similar to that of a Sasakian manifold. In this paper, we consider certain forms which correspond to C(C*)-harmonic forms of a Sasakian manifold and with it we have some informations on the Betti number of the manifold by a decomposition of such forms. The main result is that in a $2m$-dimensional compact l. c. K-manifold with the parallel Lee form, the following relation holds good between the p-th $(p<m)$ Betti number b_p and the dimension a_p of the vector space of certain p-forms which are defined in §2:

$$b_p=a_p-a_{p-2},$$

$$a_p=b_p+b_{p-2}+\cdots+b_{p-2r}, \qquad r=\left[\frac{p}{2}\right].$$

§1. Preliminaries. A locally conformal Kähler manifold is characterized as a Hermitian manifold $M^{2m}(\varphi, g)$, $2m$=the dimension, such that

$$\nabla_k\varphi_{ji}=-\alpha_j\varphi_{ki}+\alpha^r\varphi_{ri}g_{kj}-\alpha_i\varphi_{jk}+\alpha^r\varphi_{jr}g_{ki} \qquad (\varphi_{ji}=\varphi_j{}^r g_{ri})$$

with a closed 1-form α which is called the Lee form, ([2], [8]). Moreover, we assume $\nabla\alpha=0$, $|\alpha|=1$ and M is compact throughout this paper.

In this manifold, the following formulas are valid:

$$\nabla_k\varphi_{ji}=-\beta_j g_{ki}+\beta_i g_{kj}-\alpha_j\varphi_{ki}+\alpha_i\varphi_{kj}, \qquad \beta_j\overset{def}{=}\alpha^r\varphi_{rj},$$

$$J_{ji}\overset{def}{=}\nabla_j\beta_i=-\beta_j\alpha_i+\alpha_j\beta_i-\varphi_{ji} \qquad (=-\nabla_i\beta_j),$$

$$\alpha^r J_{ri}=\beta^r J_{ri}=0, \qquad J_i{}^r J_r{}^l=\beta_i\beta^l+\alpha_i\alpha^l-\delta_i{}^l,$$

$$\nabla_k\nabla_j\beta_i=-\beta^r R_{rkji}$$

$$=\beta_j g_{ki}-\beta_i g_{kj}+(\alpha_j\beta_i-\beta_j\alpha_i)\alpha_k,$$

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$$\nabla^r\nabla_r\beta_i=-2(m-1)\beta_i\,.$$

Furthermore, by virtue of Ricci's identity, we have

$$\alpha^r R_{rijk}=0\,,$$

$$\begin{aligned}&R_{khir}J_j{}^r-R_{khjr}J_i{}^r\\ (1.1)\qquad &=J_{ki}(g_{hj}-\alpha_h\alpha_j)-J_{kj}(g_{hi}-\alpha_h\alpha_i)+J_{hj}(g_{ki}-\alpha_k\alpha_i)-J_{hi}(g_{kj}-\alpha_k\alpha_j)\,,\end{aligned}$$

from which

$$\frac{1}{2}R_{rkhj}J^{rk}=-R_h{}^rJ_{rj}+(2m-3)J_{hj}\,, \tag{1.2}$$

$$R_k{}^rJ_{rj}+R_j{}^rJ_{rk}=0\,, \tag{1.3}$$

$$R_{khi}{}^rJ_{jr}-R_{khj}{}^rJ_{ir}=-(R_{ijk}{}^rJ_{hr}-R_{ijh}{}^rJ_{kr})\,, \tag{1.4}$$

$$R_{khrs}J_j{}^rJ_i{}^s=R_{jirs}J_k{}^rJ_h{}^s\,. \tag{1.5}$$

The exterior product of 1 or 2-form ω and p-form $u\left(=\frac{1}{p!}u_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)$ is given as

$$(\omega\wedge u)_{i_1\cdots i_{p+1}}=\sum_{k=1}(-1)^{k+1}\omega_{i_k}u_{i_1\cdots\hat{i}_k\cdots i_{p+1}}\qquad(\omega:\text{ 1-form})\,,$$

$$(\omega\wedge u)_{i_1\cdots i_{p+2}}=\sum_{k<l}(-1)^{k+l+1}\omega_{i_ki_l}u_{i_1\cdots\hat{i}_k\cdots\hat{i}_l\cdots i_{p+2}}\qquad(\omega:\text{ 2-form})\,,$$

where $u_{i_1\cdots\hat{i}_k\cdots i_p}$ means i_k is omitted, and the inner product for p-forms u, v is

$$(u,v)=\frac{1}{p!}\int_M u_{i_1\cdots i_p}v^{i_1\cdots i_p}d\sigma\,.$$

In general, the star operator $*$ in a Hermitian manifold satisfies for a p-forms u, v

$$**u=(-1)^pu\,,\qquad(*u,*v)=(u,v)\,,$$

$$\delta u=-*d*u\,,\qquad\Delta*=*\Delta\,,$$

where

$$(du)_{i_0\cdots i_p}=\sum_{k=0}(-1)^k\nabla_{i_k}u_{i_0\cdots\hat{i}_k\cdots i_p}\,,\qquad(\delta u)_{i_2\cdots i_p}=-\nabla^ru_{ri_2\cdots i_p}\,,$$

$$\begin{aligned}(\Delta u)_{i_1\cdots i_p}&=(\delta du+d\delta u)_{i_1\cdots i_p}\\&=-\nabla^r\nabla_ru_{i_1\cdots i_p}+\sum_k R_{i_k}{}^ru_{i_1\cdots\overset{k}{r}\cdots i_p}+\sum_{k<l}R_{i_ki_l}{}^{rs}u_{i_1\cdots\overset{k}{r}\cdots\overset{l}{s}\cdots i_p}\end{aligned}$$

and $u_{i_1\cdots\overset{k}{r}\cdots i_p}$ means that r appears at the k-th position.

Let operators $e(\omega)$, $i(\omega)$ with respect to a 1-form ω and L, Λ be as follows for a p-form u:

$$e(\omega)u=\omega\wedge u\,,\qquad i(\omega)u=*e(\omega)*u\,,$$

$$Lu=d\beta\wedge u=(e(\beta)d+de(\beta))u\,,$$

$$\Lambda u=(-1)^p*L*u=(i(\beta)\delta+\delta i(\beta))u\,.$$

Explicitly, these are written as

$$(i(\omega)u)_{i_2\cdots i_p}=\omega^r u_{ri_2\cdots i_p}\,,$$

$$(Lu)_{i_1\cdots i_{p+2}}=2\sum_{k<l}(-1)^{k+l+1}\nabla_{i_k}\beta_{i_l}u_{i_1\cdots\hat{i}_k\cdots\hat{i}_l\cdots i_{p+2}}\,,$$

$$(\Lambda u)_{i_3\cdots i_p}=\nabla^r\beta^s u_{rsi_3\cdots i_p}\,.$$

It should be remarked that

$$(e(\omega)u,\,v)=(u,\,i(\omega)v)\,,\qquad (Lu,\,v)=(u,\,\Lambda v)\,.$$

Besides under the condition $\nabla\alpha=0$, it is valid for $\omega=\alpha,\,\beta$,

$$\begin{aligned} Le(\omega)&=e(\omega)L\,, & \Lambda i(\omega)&=i(\omega)\Lambda\,,\\ Li(\omega)&=i(\omega)L\,, & \Lambda e(\omega)&=e(\omega)\Lambda\,. \end{aligned}\tag{1.6}$$

Since $\omega=\alpha,\,\beta$ are Killing, the Lie derivative

$$\theta(\omega)=i(\omega)d+di(\omega)$$

satisfies the relations ([1]):

$$\theta(\omega)=-(e(\omega)\delta+\delta e(\omega))$$

and then $\theta(\omega)$ commutes with $i(\omega)$, $e(\omega)$, d and δ for $\omega=\alpha,\,\beta$ respectively. In the following we often write briefly $e,\,i$ (resp. $e',\,i'$) instead of $e(\beta),\,i(\beta)$ (resp. $e(\alpha),\,i(\alpha)$).

We notice here that

$$\begin{aligned} &ei+ie=\text{identity}\,,\\ &ei'=-i'e\,,\qquad e'i=-ie'\,,\qquad ii'=-i'i\,, \end{aligned}\tag{1.7}$$

and

$$\begin{aligned} &\Delta e-e\Delta=\delta L-L\delta\,, && \Delta i-i\Delta=d\Lambda-\Lambda d\,,\\ &\Delta e'-e'\Delta=0\,, && \Delta i'-i'\Delta=0 \end{aligned}\tag{1.8}$$

because, for any Killing vector ω, the following relation holds good:

$$(\Delta e(\omega)-e(\omega)\Delta)u=\delta(d\omega\wedge u)-d\theta(\omega)u+\theta(\omega)du-d\omega\wedge\delta u\,.$$

We remark also $\nabla_\beta J_{ji}=\nabla_\alpha J_{ji}=0$, and from which for $\omega=\alpha,\ \beta$

$$i(\omega)\nabla_\beta=\nabla_\beta i(\omega)\,,\qquad i(\omega)\nabla_\alpha=\nabla_\alpha i(\omega)\,,$$

where we denote $\nabla_\omega u_{i_1\cdots i_p}=\omega^r\nabla_r u_{i_1\cdots i_p}$.

In this paper the following formulas are used frequently:

LEMMA 1.1. *In a l.c. K-manifold M^{2m} with the parallel Lee form, $|\alpha|=1$, the followings hold good for any p-form u.*

(i) $(\Lambda L^k-L^k\Lambda)u=4k(m-p-k)L^{k-1}u+4k(e'i'+ei)L^{k-1}u.$

(i)$'$ *If* $iu=\Lambda u=0,\quad r\geqq 2,$

$$\Lambda^r L^{r+s}=4^r(r+s)\cdots(1+s)\{(m-p-s-r)\cdots(m-p-s-1)L^s u$$
$$+r(m-p-s-(r-1))\cdots(m-p-s-1)e'i'L^s u\}\,.$$

(ii) $(\delta L-L\delta)u=2(d\nabla_\beta-\nabla_\beta d)u+4((m-p)e-ei'e')u\,.$

(iii) $(d\Lambda-\Lambda d)u=2(-\delta\nabla_\beta+\nabla_\beta\delta)u+4((p-m)i-ie'i')u\,.$

Proof. (i) and (i)$'$ are known by the mathematical induction. (ii): Putting

$$(\Gamma u)_{i_0\cdots i_p}=\sum_{k=0}(-1)^k J_{i_k}{}^r\nabla_r u_{i_0\cdots\hat{i}_k\cdots i_p}\,,$$

we get by straightforward computations

$$\frac{1}{2}(\delta Lu-L\delta u)=\Gamma u+ee'i'u+(2m-p-2)eu\,,$$

$$d\nabla_\beta u-\nabla_\beta du=\Gamma u-ee'i'u+peu\,.$$

(iii) is known by the dual of (ii) and the property $*\nabla_\beta u=\nabla_\beta *u$. q.e.d.

§2. V-harmonic forms. At first we get

LEMMA 2.1. *If u is harmonic p-form, then*

(i) *$i(\alpha)u$ and $e(\alpha)u$ are harmonic,*

(ii) $\nabla_\alpha u=0\,,$

(iii) ([3]) *$\Lambda u=0$ (effective) and $i(\beta)u=0$ provided that $p<m$.*

(i) and (ii) are evident if we notice that $\theta(\omega)u=0$ for a Killing vector ω, a harmonic p-form u, and $\nabla_\alpha=\theta(\alpha)$.

Proof of (iii): In general for any p-form u, from (1.8) and Lemma 1.1, it follows that

$$\Delta eiu=2(d\nabla_\beta i-\nabla_\beta di)u+e(\Delta i+4(m-p)ei+4ee'i'i)u$$

and then

$$(\imath eu,\ \Delta e\imath u)=2(ieu,\ d\nabla_\beta iu-\nabla_\beta d\imath u)$$
$$=2(\Lambda eu-i\delta eu,\ \nabla_\beta \imath u)-2(\imath eu,\ \nabla_\beta d\imath u)$$
$$=-2(i\delta eu,\ \nabla_\beta iu)-2(\imath eu,\ \nabla_\beta d\imath u)\,,$$

where we have used $(\Lambda eu,\ \nabla_\beta iu)=(\Lambda u,\ i\nabla_\beta iu)=0$. Hence, for a form u which satisfies

$$d\imath u=\delta eu=0\,, \tag{2.1}$$

the equality

$$(\imath eu,\ \Delta eiu)=0 \tag{2.2}$$

holds good. We notify beforehand that this fact will be used after again in the proof of Lemma 2.5.

Now, let u be harmonic. By virtue of $\theta(\beta)u=0$, (2.1) is satisfied and then from (2.2) it follows $deiu(=Liu)=0$. So, making use of Lemma 1.1, we can obtain

$$(-L\Lambda iu,\ iu)=4((m-p)\imath u+e'i'\imath u,\ \imath u)\,,$$

which implies $\imath u=0$ under $p<m$, and then $\Lambda u=(\delta\imath+i\delta)u=0$. q. e. d.

Definition. A form u is called V-harmonic if it satisfies

$$du=0 \qquad \text{and} \quad \delta u=e(\beta)\Lambda u\,.$$

As a harmonic p-form $(p<m)$ is effective, the following is trivial:

Proposition 2.2. *A p-form $(p<m)$ is harmonic if and only if it is effective V-harmonic.*

Corresponding to a well known property between a harmonic form and a Killing vector, we can get

Proposition 2.3. *For any V-harmonic form u,*

$$\theta(\beta)u=0$$

holds.

This property follows immediately from the Lemma:

Lemma 2.4. *For any V-harmonic form u, $di(\beta)u=0$ is valid.*

Proof. If u is V-harmonic, taking account of (1.8), we get

$$\delta diu=\Delta iu-d\delta iu=i\Delta u+di\delta u$$

$$=id\delta u+di\delta u=\theta(\beta)e\Lambda u\,.$$

Then it follows that

$$(d\imath u,\ d\imath u)=(iu,\ \delta d\imath u)=(iu,\ \theta(\beta)e\Lambda u)=0$$

because of $e\theta(\beta)=\theta(\beta)e$. q. e. d.

Next we shall consider orthogonal property to β of V-harmonic form. For it, we provide

LEMMA 2.5. *For any V-harmonic form u, it is valid that*

(i) $\delta e(\beta)u=0$,

(ii) $i(\beta)u$ *is V-harmonic*,

(iii) $Li(\beta)u=0$.

Proof. (i) follows from $\delta eu=-\theta(\beta)u-e\delta u=0$.
(ii): $d\imath u=0$ is Lemma 2.4. Next, taking account of (1.6), we have

$$\delta iu=\Lambda u-i\delta u=\Lambda u-\imath e\Lambda u=e\Lambda\imath u\,.$$

(iii): By virtue of (i) and (ii), the equality (2.1) holds good, and then (2.2) as mentioned before. Hence on account of $\delta eiu=-\theta(\beta)\imath u-e\delta iu=0$ ((ii)), we can get

$$(deiu,\ deiu)=-(dieu,\ deiu)=-(\imath eu,\ \Delta eiu)=0\,,$$

which implies $Liu=0$. q. e. d.

THEOREM 2.6. *In a compact l. c. K-manifold $M^{2m}(\varphi, g, \alpha)$ with the parallel Lee form, a V-harmonic p-form u $(p<m)$ is orthogonal to β, i. e., $i(\beta)u=0$.*

Proof. By virtue of Lemma 2.5 and Lemma 1.1, we get

$$-L\Lambda\imath u=4((m-p)i+e'i'i)u\,,$$

and then

$$\begin{aligned}(iu,\ -L\Lambda iu)&=-(\Lambda\imath u,\ \Lambda\imath u)\\&=4(m-p)(iu,\ iu)+4(i'\imath u,\ i'iu)\,.\end{aligned}$$

This equality implies $iu=0$ for $m>p$. q. e. d.

PROPOSITION 2.7. *If a p-form u $(p<m)$ is V-harmonic, then so is Λu.*

Proof. Let u be V-harmonic. About the codifferential, we know obviously $\delta\Lambda u=\Lambda\delta u=e\Lambda(\Lambda u)$.

We shall prove now $d\Lambda u=0$. On account of Lemma 1.1 and (1.7), we have

$$d\Lambda u=2(-\delta\nabla_\beta u+\nabla_\beta\delta u),$$

from which

$$\begin{aligned}(d\Lambda u,\, d\Lambda u)&=2(\Lambda u,\, \delta\nabla_\beta e\Lambda u)\\ &=2(\Lambda u,\, -\theta(\beta)\nabla_\beta\Lambda u-e\delta\nabla_\beta\Lambda u)\\ &=2(\theta(\beta)\Lambda u,\, \nabla_\beta\Lambda u)-2(\iota\Lambda u,\, \delta\nabla_\beta\Lambda u)\\ &=0,\end{aligned}$$

where we have used $\nabla_\beta e=e\nabla_\beta$ and the properties of Lie derivative: $\theta(\beta)\Lambda=\Lambda\theta(\beta)$, $(\theta(\beta)v,\, w)=-(v,\, \theta(\beta)w)$ for any forms v, w. q. e. d.

We can also state a V-harmonic form with the Laplacian as follow:

PROPOSITION 2.8. *A p-form u ($p<m$) is V-harmonic if and only if $i(\beta)u=0$ and $\Delta u=L\Lambda u$.*

Proof. Necessity follows from $\Delta u=d\delta u=de\Lambda u=L\Lambda u$ (Prop. 2.7) and Theorem 2.6.

Now we prove the sufficiency. Since

$$\begin{aligned}&(du,\, du)+(\delta u-e\Lambda u,\, \delta u-e\Lambda u)\\ &\quad=(du,\, du)+(\delta u,\, \delta u)-2(e\Lambda u,\, \delta u)+(e\Lambda u,\, e\Lambda u)\\ &\quad=(u,\, \Delta u)-2(\Lambda u,\, \Lambda u-\delta iu)+(\Lambda u,\, \Lambda u-e\iota\Lambda u),\end{aligned}$$

we know, under the assumption $\iota u=0$ and $\Delta u=L\Lambda u$, the right hand side is zero. Hence $du=0$, $\delta u=e\Lambda u$, which prove our Theorem. q. e. d.

The following Proposition provids examples of V-harmonic forms actually.

PROPOSITION 2.9. *The $2k$-form $L^k\cdot 1$ is V-harmonic for any k.*

Proof. $d(L^k\cdot i)=0$ is trivial. So we shall prove $\delta L^k\cdot 1=e\Lambda L^k\cdot 1$ by induction.

For $k=1$, as $\delta L\cdot 1=4(m-1)\beta$, $e\Lambda L\cdot 1=4(m-1)\beta$, it is satisfied. Now we assume $\delta L^{k-1}\cdot 1=e\Lambda L^{k-1}\cdot 1$. Taking account of Lemma 1.1 and $\nabla_\beta d\beta=0$, $i^r L^r\cdot 1=0$, we can obtain

$$\begin{aligned}\delta L^k\cdot 1&=\delta L\cdot L^{k-1}\cdot 1\\ &=L\delta L^{k-1}\cdot 1+4(m-2k+1)eL^{k-1}\cdot 1\end{aligned}$$

$$=Le\Lambda L^{k-1}\cdot 1+4(m-2k+1)eL^{k-1}\cdot 1$$

$$=e\Lambda L^{k}\cdot 1\,. \qquad \text{q. e. d.}$$

§ 3. **Decomposition to harmonic forms.** The purpose of this section is to study the Betti number relating with V-harmonic forms.

At first we consider a relation between $\Delta\Lambda$ and $\Lambda\Delta$. On account of Lemma 1.1 and $\Lambda\delta=\delta\Lambda$, we have for any p-form u

$$\begin{aligned}\Lambda\Delta u&=\Lambda(\delta d+d\delta)u\\ &=\delta d\Lambda u-2\delta(\nabla_{\beta}\delta+2(m-p)\imath-2\imath e'i')u\\ &\quad+\delta d\Lambda u-2(-\delta\nabla_{\beta}\delta+2(p-1-m)i\delta-2\imath e'i'\delta)u\\ &=\Delta\Lambda u-4(p-m)\Lambda u+4(i\delta+\delta ie'i'+e'i'i\delta)u\,.\end{aligned}$$

Since

$$\begin{aligned}-\delta e'ii'u&=\theta(\alpha)ii'u+e'\delta ii'u\\ &=\nabla_{\alpha}ii'u+e'i'\Lambda u-e'i'i\delta u\,,\end{aligned}$$

taking account of $\delta i'=-i'\delta$ and $\theta(\alpha)=\nabla_{\alpha}$, we can get finally

$$\Lambda\Delta u-\Delta\Lambda u=4\{(m-p)\Lambda+i\delta+\nabla_{\alpha}ii'+e'i'\Lambda\}\,u\,.$$

LEMMA 3.1. *For any p-form u, we have*

$$\begin{aligned}(\Lambda\Delta-\Delta\Lambda)u&=4\{(m-p)\Lambda+e(\alpha)i(\alpha)\Lambda-\nabla_{\alpha}i(\alpha)i(\beta)+i(\beta)\delta\}\,u\\ &=4\{(m-p+1)\Lambda-i(\alpha)e(\alpha)\Lambda-\nabla_{\alpha}i(\alpha)i(\beta)+i(\beta)\delta\}\,u\,.\end{aligned}$$

Taking the dual of above formula and on account of $*\nabla_{\alpha}=\nabla_{\alpha}*$ we can get

LEMMA 3.2. *For any p-form u,*

$$(L\Delta-\Delta L)u=4\{(p-m+1)L-e(\alpha)i(\alpha)L+e(\beta)d+\nabla_{\alpha}e(\alpha)e(\beta)\}\,u$$

holds good.

Especially if u is a V-harmonic p-form $(p<m)$, Lemma 3.2 implies

$$\begin{aligned}\Delta Lu&=LL\Lambda u-4((p-m+1)Lu-e'i'Lu+\nabla_{\alpha}e'eu)\\ &=L(\Lambda Lu-4(m-p-1)u-4e'i'u)-4((p-m+1)L-e'i'L+\nabla_{\alpha}e'e)u\\ &=L\Lambda(Lu)-4\nabla_{\alpha}e'eu\,,\end{aligned}$$

which means Lu is also V-harmonic if $\nabla_{\alpha}e'eu=0$.

From this fact, we know that the $(2p+1)$-form $(2p-1<m)\alpha_{\wedge}d\beta_{\wedge\cdots\wedge}d\beta$ is V-harmonic, because $e'\alpha_{\wedge}d\beta_{\wedge\cdots\wedge}d\beta=0$ and α is V-harmonic.

PROPOSITION 3.3. *If u is a harmonic p-form, then $L^k u$ is V-harmonic, where $2+p\leqq 2k+p\leqq m+2$.*

Proof. It is sufficient to notice

$$\nabla_\alpha e'eL^r u=0\,,$$

which follows from Lemma 2.1 (ii), $\nabla_\alpha e'e=e'e\nabla_\alpha$ and $\nabla_\alpha d\beta=0$. q.e.d.

THEOREM 3.4. *In a compact l.c. K-manifold $M^{2m}(\varphi, g, \alpha)$ with the parallel Lee form, any V-harmonic p-form u $(p<m)$ can be represented uniquely as*

$$u=\sum_{k=0}^{r} L^k\phi_{p-2k}\,,\qquad r=\left[\frac{p}{2}\right],$$

where ϕ_{p-2k} is harmonic $(p-2k)$-form.

Conversely, p-forms $(p<m)$ of the type in the right hand side are V-harmonic.

Proof. We shall prove it by the mathematical induction. At first the case $p=0$ and 1 are trivial because a V-harmonic form is harmonic necessarily. We assume now its validity for $(p-2)$-form. Let u be a V-harmonic p-form $(p<m)$. Since Λu_p is V-harmonic by virtue of Proposition 2.7, there exist harmonic $(p-2-2k)$-forms ψ_{p-2-2k} such that

$$\Lambda u_p=\sum_k L^k\psi_{p-2-2k}\,.$$

Now we put

$$v_{p-2}=\sum_k L^k\phi_{p-2-2k}\,,$$

where

$$\phi_{p-2-2k}=\frac{\psi_{p-2-2k}}{4(k+1)(m-p+k+1)}-\frac{e'i'\psi_{p-2-2k}}{4(k+1)(m-p+k+1)(m-p+2+k)}\,.$$

From Lemma 2.1, ϕ_{p-2-2k} are also harmonic. By virtue of Lemma 1.1 and Lemma 2.1, it follows

$$\begin{aligned}\Lambda Lv_{p-2}&=\sum_k \Lambda L^{k+1}\Big(\psi_{p-2-2k}-\frac{e'i'\psi_{p-2-2k}}{m-p+2+k}\Big)/4(k+1)(m-p+k+1)\\&=\sum_k [4(k+1)(m-p+k+1)L^k\psi_{p-2-2k}+4(k+1)e'i'L^k\psi_{p-2-2k}\\&\quad -e'i'\{4(k+1)(m-p+k+1)L^k\psi_{p-2-2k}+4(k+1)e'i'L\}/(m-p+2+k)]\\&\quad\times\frac{1}{4(k+1)(m-p+k+1)}\\&=\sum_k L^k\psi_{p-2-2k}\end{aligned}$$

namely, $\Lambda Lv_{p-2}=\Lambda u_p$. Now we define a p-form ϕ_p as

$$\phi_p=u_p-Lv_{p-2}.$$

Since $Lv_{p-2}=\sum_k L^{k+1}\phi_{p-2-2k}$ is V-harmonic because of Proposition 3.3, ϕ_p is V-harmonic. Moreover as $\Lambda\phi_p=\Lambda u_p-\Lambda Lv_{p-2}=0$, ϕ_p is harmonic. Then $u_p=\phi_p+Lv_{p-2}=\phi_p+\sum L^{k+1}\phi_{p-2-2k}$ is the desired representation.

The uniqueness comes from the following Lemma:

LEMMA 3.5. *For harmonic p, q-form ω, ζ ($p, q<m$), we have*

$$(L^k\omega, L^h\zeta)=0 \qquad (k\neq h).$$

Proof. As $i\omega=\Lambda\omega=0$, making use of Lemma 1.1 and (1.6), we know for $h<k$

$$\Lambda^h L^k\omega=\lambda L^{k-h}\omega+\mu L^{k-h}e'i'\omega, \qquad (\lambda, \mu=\text{const.}).$$

Then from the property $(L\omega, \zeta)=(\omega, \Lambda\zeta)$, the Lemma is proved. q.e.d.

From Proposition 3.3, Theorem 3.4 and Lemma 3.5, we can get

COROLLARY 3.6. *If u is a V-harmonic p-form ($p<m$), then so is Lu. Moreover the operator L is injective.*

By virtue of Theorem 3.4 and Corollary 3.6, we can now obtain the desired result:

THEOREM 3.7. *In a compact $2m$-dimensional l.c. K-manifold with the parallel Lee form, we have for $p<m$,*

$$a_p=b_p+b_{p-2}+\cdots+b_{p-2r}, \qquad r=\left[\frac{p}{2}\right]$$

$$b_p=a_p-a_{p-2},$$

where a_p is the dimension of the vector space V_p of all V-harmonic p-forms and b_p is the p-th Betti number,

§4. V^*-harmonic forms.

In this section we shall consider a dual form of a V-harmonic form.

DEFINITION. A form u is called V^*-harmonic if it satisfies

$$du=i(\beta)Lu, \qquad \delta u=0.$$

For example, $\beta\wedge d\beta\wedge\ldots\wedge d\beta$ is V^*-harmonic. From the definition, we know easily

PROPOSITION 4.1. *A p-form u is V^*-harmonic if and only if the $(2m-p)$-form $*u$ is V-harmonic.*

Since $\beta_\wedge d\beta_\wedge\dots{}_\wedge d\beta$ is V^*-harmonic $(2p+1)$ for any p, by virtue of Proposition 4.1, $a_{2m-2p-1}\geqq 1$ is valid for any p. Hence combining with Proposition 2.9, we can say

THEOREM 4.2. *In a compact $2m$-dimensional l. c. K-manifold with the parallel Lee form, we have $a_k\geqq 1$ for any $k=0, 1, \cdots 2m$.*

LEMMA 4.3. *For a V^*-harmonic p-form u, we have*

(i) $e(\beta)u=0 \qquad (p>m)$,

(ii) $\theta(\beta)u=0 \qquad (\forall p)$,

(iii) $\Lambda e(\beta)u=0 \qquad (\forall p)$.

Proof. (i) follows from $i*u=0$, $(2m-p<m)$.

(ii) follows from Proposition 2.3, i. e., $*\theta(\beta)u=\theta(\beta)*u=0$ for any V-harmonic $(2m-p)$-form $*u$.

(iii) follows from Lemma 2.5 (iii), i. e., $*\Lambda eu=(-1)^p Li*u=0$ for any V-harmonic $(2m-p)$-form $*u$.

Next we shall consider a decomposition of V^*-harmonic forms. For it, we provide some Lemmas.

LEMMA 4.4. *For any $p(\neq m)$, we have*

$$H_p=V_p\cap V_p^*,$$

where V_p^* is the vector space of V^*-harmonic p-forms.

Proof. From the definition, $H_p\supset V_p\cap V_p^*$ is trivial. We shall prove $H_p\subset V_p\cap V_p^*$. For $p<m$, it holds good evidently because of $iu=\Lambda u=0$ $(u\in H_p)$. As for $p>m$, taking account of that $e\Lambda u=-*iL*u$ and $*u$ is harmonic for a harmonic form u, we have also $e\Lambda u=0$ and $iLu=0$. Hence the Lemma is proved. q. e. d.

LEMMA 4.5. (i) *$e(\beta)$ is a homomorphism of $V_p\cup V_p^*\to V_{p+1}^*$. Especially, $e(\beta)_{|V_p}$ is injective for $p<m$.*

(ii) *$i(\beta)$ is a homomorphism of $V_p\cup V_p^*\to V_{p-1}$. Especially, $i(\beta)_{|V_p^*}$ is surjective for $p<m+1$.*

Proof. For $u\in V_p$, $deu=Lu=iLeu+eLiu=iLeu$ because $Liu=0$ for any p (Lemma 2.5), and $\delta eu=-\theta(\beta)u-e\delta u=0$ because $\theta(\beta)u=0$ (Prop. 2.3). Then $eu\in V_{p+1}^*$.

For $u\in V_p^*$, $deu=Lu-edu=Lu-eiLu=iLeu$, and as above, by virtue of Lemma 4.2, $\delta eu=0$. Then $eu\in V_{p+1}^*$.

Especially, for $u \in V_p$ $(p<m)$, if $eu=0$, then $0=ieu=u-eiu=u$, namely, $e(\mathring{\beta})$ is 1 : 1.

In a similar way, (ii) can be verified. Especially for non-zero $u \in V_{p-1}$, from (i), $eu \in V_p^*$, and $ieu=u-eiu=u$ for $p-1<m$. q. e. d.

LEMMA 4.6. *It is valid that for* $p<m$

$$H_p=i(\beta)e(\beta)V_p^*.$$

Therefore $b_p=0$ *if any only if* $e(\beta)V_p^*=\{0\}$ *for* $p<m$.

Proof. It is sufficient to notice that for $u \in V_p^*$, $V_p \ni ieu=u-eiu \in V_p^*$ and for $u \in H_p$ $(\subset V_p^*)(p<m)$, $u=ieu+eiu=ieu$. q. e. d.

LEMMA 4.7. *If* $p<m$, *we have* $V_p^*=H_p \oplus e(\beta)V_{p-1}$.

Hence $a_p^*=b_p+a_{p-1}=b_p+\sum_{k=0}^{r} b_{p-1-2k}$ $\left(r=\left[\frac{p-1}{2}\right]\right)$ *holds good where* a_p^* *is the dimension of* V_p^*.

Proof. $H_p \cap eV_{p-1}=\{0\}$ follows from $ieu=u$ for $u \in V_{p-1}$ $(p \leqq m)$ which oppose to $iH_p=\{0\}$ $(p<m)$. Next, for $u \in V_p^*$, from the previous Lemmas, $u=ieu+eiu \in H_p \oplus eV_{p-1}$ is valid. Moreover from $V_p^* \supset eV_{p-1}$, $V_p^* \supset H_p \oplus eV_{p-1}$ is valid also, which completes the proof. q. e. d.

Making use of Lemma 4.7 and Theorem 3.4, we can obtain the following:

THEOREM 4.8. *In a compact l. c. K-manifold* $M^{2m}(\varphi, g, \alpha)$ *with the parallel Lee form, any V^*-harmonic p-form* u $(p<m)$ *is decomposed uniquely in the following form.*

$$u=\phi_p+\sum_{k=0}^{r} e(\beta)L^k\phi_{p-1-2k}, \qquad r=\left[\frac{p-1}{2}\right],$$

where ϕ_k *is harmonic k-form.*

Conversely, p-forms $(p<m)$ *of the type in the right hand side are V^*-harmonic.*

Remark. Recently, Ogawa and Tachibana [6] obtain the fact that if a connected compact orientable Riemannian manifold admits a parallel vector field, then $\sum_{k=0}^{p}(-1)^k b_{p-k} \geqq 0$ holds good. Hence in our manifold now, as a_p-a_{p-1} $=\sum_{k=0}^{p}(-1)^k b_{p-k}$ because of Theorem 3.7, we can see the relation $a_p \geqq a_{p-1}$.

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