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KODAI MATH. J.
3 (1980), 48—55

GEODESIC SYMMETRIES IN SASAKIAN LOCALLY ϕ-SYMMETRIC SPACES

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1. Introduction.

In [2], D'Atri and Nickerson initiated a study of Riemannian manifolds whose local geodesic symmetries are divergence-preserving (volume-preserving up to sign). This class of spaces obviously includes the Riemannian locally symmetric spaces and the harmonic Riemannian spaces.

On the other hand, Takahashi [10] has introduced an interesting notion of Sasakian (locally) ϕ-symmetric space, which is an analogous notion of Hermitian symmetric space, and discussed about its properties.

In § 2, an equation is derived from an infinite sequence of necessary conditions on the curvature tensors (sufficient in the case of an analytic manifold). In § 3, we show that every local geodesic symmetry of a Sasakian locally ϕ-symmetric space is divergence-preserving, and give a necessary and sufficient condition in order that a Sasakian space is locally ϕ-symmetric. In § 4, we show that a 5-dimensional compact, (or $|R|^2$=constant) Sasaki-Einsteinian space of non-negative curvature is locally ϕ-symmetric.

2. Preliminaries.

We shall give some formulas which are used in the subsequent sections. Let (M, g) be a Riemannian space with Levi-Civita connection ∇. By $R=(R_{kji}{}^{h})$, we denote the Riemannian curvature tensor of ∇. Then $R_1=(R_{aij}{}^{a})=(R_{ij})$ and $S=g^{ij}R_{ij}$ are Ricci tensor and scalar curvature respectively. For a tensor field $T=(T_{ijk})$, we put $\nabla T=(\nabla_l T_{ijk})$ and $|T|^2=T_{ijk}T^{ijk}$, where ∇_l denotes the operator of covariant differentiation. We put

$$\beta=R^{abcd}R_{ab}{}^{uv}R_{cduv},$$

$$\gamma=R^{abcd}R_{a}{}^{u}{}_{c}{}^{v}R_{budv}.$$

Then they satisfy the following fundamental formulas.

(2.1) (Bianchi's identities)

Received December 7, 1978

(a) $R_{kjih}+R_{jikh}+R_{ikjh}=0$,

(b) $\nabla_l R_{kjih}+\nabla_k R_{jlih}+\nabla_j R_{lkih}=0$.

The following identities are derived from (2.1)

(2.2) (a) $R^{abcd}\nabla_c R_{madb}=(1/4)R^{abcd}\nabla_m R_{abcd}$,

(b) $R^{abcd}\nabla_d R_{abcm}=(1/4)R^{abcd}\nabla_m R_{abcd}$,

(c) $R^{abcd}\nabla_b R_{dmca}=(1/4)R^{abcd}\nabla_m R_{abcd}$.

(2.3) (Lichnerowicz's formula)

$$(1/2)\Delta|R|^2=|\nabla R|^2-4R^{jihk}\nabla_j\nabla_h R_{ik}$$
$$+2R_{ij}R^{ihkl}R^{j}{}_{hkl}+\beta+4\gamma.$$

Where Δ is the Laplace-Bertrami operator acting on differentiable functions on M.

If M is Einsteinian, then we have

$$(1/2)\Delta|R|^2=|\nabla R|^2+(2/n)S|R|^2+\beta+4\gamma. \tag{2.4}$$

Let (M, g) be a Riemannian manifolds whose local geodesic symmetries are divergence-preserving. D'Atri and Nickerson have found an infinite sequence of necessary conditions on curvature tensors, which are sufficient in the analytic case, as follows [2], [3]: Let X denotes a nonzero vector at a point 0 of a C^∞-Riemannian manifold M and define an endomorphism Π of the tangent space $T_0(M)$ by

$$\Pi(Y)=-R(Y, X)X, \qquad Y\in T_0(M). \tag{2.5}$$

Let $\nabla_X^0\Pi=\Pi$, and define $\nabla_X^i\Pi$, $i=1, 2, \cdots$, by first extending X to the volocity vector field along the geodesic through O determined by X and then extending Π in according with (2.5). Define endomorphisms P^r, $r=2, 3, \cdots$, of $T_0(M)$ by the recurrence formula

$$(r+1)P^r=r(r-1)\nabla_X^{r-2}\Pi-\sum_{q=2}^{r-2}\binom{r}{q}P^q\circ P^{r-q}, \tag{2.6}$$

of Ledger [4]. Our necessary conditions are that

$$\operatorname{trace} P^r=0, \qquad r=3, 5, 7, \cdots, \tag{2.7}$$

for all choices of O and X.

For $r=3$, the condition trace $P^3=0$ gives trace $\nabla_X\Pi=0$, or $(\nabla_X R_1)(X, X)=0$ ([2]); in terms of the local components,

$$\nabla_k R_{ji}+\nabla_j R_{ik}+\nabla_i R_{kj}=0, \tag{2.8}$$

from which S is a constant.

For $r=5$, the condition trace $P^5=0$ gives trace $\nabla_X(\Pi\circ\Pi)=0$ ([1]); in terms of the local components

$$P_{(ijklm)}(R_{pij}{}^q\nabla_m R_{qkl}{}^p)=0\,, \tag{2.9}$$

where $P_{(ijklm)}$ denotes the sum of terms obtained by permuting the given free indices, i, j, k, l, m.

Now, to derive an information from (2.9), we need the following

LEMMA 2.1. *For a tensor field $T=(T_{ijklm})$, we have*

$$\begin{aligned} g^{ij}g^{kl}P_{(ijklm)}(T_{ijklm})=8(&T_i{}^i{}_j{}^j{}_m+T_{ij}{}^{ij}{}_m+T_{ij}{}^{ji}{}_m\\ &+T_i{}^i{}_{jm}{}^j+T_{ij}{}^i{}_m{}^j+T_{ij}{}^j{}_{mi}\\ &+T_i{}^i{}_{mj}{}^i+T_{ijm}{}^{ij}+T_{ijm}{}^{ji}\\ &+T_{im}{}^i{}_j{}^j+T_{imj}{}^{ij}+T_{imj}{}^{ji}\\ &+T_{mi}{}^i{}_j{}^j+T_{mij}{}^{ij}+T_{mij}{}^{ji})\,. \end{aligned}$$

Proof. We see that

$$P_{(ijklm)}(T_{ijklm})=P_{(ijkl)}(T_{ijklm}+T_{ijkml}+T_{ijmkl}+T_{imjkl}+T_{mijkl})\,.$$

The conclusion follows by a similary argument to Lemma 3.1 in [13].

If we put $A_{ijklm}=R_{pij}{}^q\nabla_k R_{qlm}{}^p$, then we obtain

LEMMA 2.2. *Let M be a Riemannian space satisfying* (2.9). *Then we have*

$$\begin{aligned} g^{ij}g^{kl}P_{(ijklm)}(A_{ijklm})=8[&4R^{ab}\nabla_m R_{ab}+(3/2)R^{abcd}\nabla_m R_{abcd}\\ &+8R_{abcm}\nabla^a R^{bc}]\,. \end{aligned} \tag{2.10}$$

Proof. By straightforward calculations using (2.8), we have

$$\begin{aligned} &g^{ij}g^{kl}P_{(ijkl)}(R_{pij}{}^q\nabla_m R_{qkl}{}^p)\\ &\qquad=8[R^{pq}\nabla_m R_{pq}+(1/2)R^{abcd}\nabla_m R_{abcd}]\,,\\ &g^{ij}g^{kl}P_{(ijkl)}(R_{pij}{}^q\nabla_l R_{qkm}{}^p)\\ &\qquad=8[(3/2)R^{ab}\nabla_m R_{ab}+R^{abcd}\nabla_c R_{madb}+R^{abcd}\nabla_b R_{dcma}]\,,\\ &g^{ij}g^{kl}P_{(ijkl)}(R_{pij}{}^q\nabla_l R_{qmk}{}^p)\\ &\qquad=8[(3/2)R^{ab}\nabla_m R_{ab}+R^{abcd}\nabla_d R_{abcm}-R^{abcd}\nabla_b R_{acdm}]\,,\\ &g^{ij}g^{kl}P_{(ijkl)}(R_{pim}{}^q\nabla_l R_{qjk}{}^p)=32R^{abc}{}_m\nabla_a R_{bc}\,,\\ &g^{ij}g^{kl}P_{(ijkl)}(R_{pmi}{}^q\nabla_l R_{qjk}{}^p)=32R^{abc}{}_m\nabla_a R_{bc}\,. \end{aligned}$$

Thus, by Lemma 2.1 and (2.2), we get (2.10).

PROPOSITION 2.3. *Let M be a Riemannian space satisfying* (2.8) *and* (2.9). *Then we have*

$$8R^{ab}\nabla_m R_{ab}+3R^{abcd}\nabla_m R_{abcd}+16R_{abcm}\nabla^a R^{bc}=0\,. \tag{2.11}$$

COROLLARY 2.4. *If a Riemannian space with the parallel Ricci tensor satisfies* (2.9), *then* $|R|^2$ *is constant.*

3. Sasakian locally ϕ-symmetric space.

Let M be an n-dimensional Sasakian space with structure tensors ϕ, ξ, η and g (cf. [6], [7], [9]):

$$\begin{aligned}
(3.1)\quad &(a)\quad \xi^i\eta_i=1\,, \qquad g_{ji}\xi^i=\eta_j\,,\\
&(b)\quad \phi_j{}^i\phi_k{}^j=-\delta_k{}^i+\eta_k\xi^i\,,\\
&(c)\quad g_{ji}\phi_k{}^j\phi_h{}^i=g_{kh}-\eta_h\eta_k\,,\\
&(d)\quad \nabla_j\eta_i=\phi_{ji}\,,\\
&(e)\quad \nabla_k\phi_{ji}=\eta_j g_{ki}-\eta_i g_{kj}\,,
\end{aligned}$$

from which we have

$$\begin{aligned}
(3.2)\quad &(a)\quad \eta_r R_{kji}{}^r=\eta_k g_{ji}-\eta_j g_{ki}\,,\\
&(b)\quad \eta_r R_j{}^r=(n-1)\eta_j\,,\\
&(c)\quad R_{ji}\eta^j\eta^i=n-1\,,\\
&(d)\quad \eta_r\nabla_l R_{kji}{}^r=-\phi_l{}^s R_{kjis}+\phi_{lk}g_{ji}-\phi_{lj}g_{ki}\,,\\
&(e)\quad \eta_r\nabla_l R_k{}^r=-\phi_l{}^r R_{kr}+(n-1)\phi_{lk}\,,\\
&(f)\quad \phi_a{}^r R_{rbcd}=\phi_b{}^r R_{racd}+\phi_{ca}g_{db}-\phi_{cb}g_{da}+\phi_{db}g_{ca}-\phi_{da}g_{cb}\,,\\
&(g)\quad \phi_a{}^r\phi_b{}^s\phi_{rscd}=R_{abcd}+\phi_{ad}\phi_{bc}-\phi_{ac}\phi_{bd}+g_{da}g_{bd}-g_{ad}g_{bc}\,,\\
&(h)\quad \phi^{rk}R_{lkrj}=(n-2)\phi_{lj}+\phi_{jr}R_l{}^r\,,\\
&(i)\quad \phi^{rk}R_{rkjl}=2(n-2)\phi_{jl}-2\phi_{jr}R_l{}^r\,.
\end{aligned}$$

Let M be a Sasakian space satisfying (2.9) and

$$\nabla_l R_{ji}=\{(n-1)\phi_{li}+R_{rl}\phi_i{}^r\}\eta_j+\{\phi_j{}^r R_{rl}+(n-1)\phi_{lj}\}\eta_i\,. \tag{3.3}$$

It is easily seen that the Ricci tensor satisfies (2.8), $R_{ljim}\nabla^l R^{ji}=0$, and $R^{ab}\nabla_m R_{ab}=0$, taking account of (3.1) and (3.2). Thus, by (2.11) we have the following

PROPOSITION 3.1. *If a Sasakian space satisfies* (2.9) *and* (3.3), *then* $|R|^2$ *is constant.*

By definition (T. Takahashi [10]), a Sasakian space is said to be locally ϕ-symmetric if the Riemannian curvature tensor R satisfies

$$
\begin{aligned}
(3.4)\qquad 0=\nabla_l R_{kjih} &-(g_{kl}\phi_{ji}-g_{jl}\phi_{ki}-R_{kjls}\phi_i{}^s)\eta_h \\
&-(-\phi_{li}g_{jh}+g_{ji}\phi_{lh}+R_{jsih}\phi_l{}^s)\eta_k \\
&-(\phi_{li}g_{kh}-g_{ki}\phi_{lh}-R_{ksih}\phi_l{}^s)\eta_j \\
&-(g_{jl}\phi_{kh}-g_{kl}\phi_{jh}-R_{kjl}{}^s\phi_{sh})\eta_i\,.
\end{aligned}
$$

Let T_{lkjih} put the right hand side of (3.4). Then we can calculate the square $|T|^2$ by using (3.1) and (3.2) and get the following

PROPOSITION 3.2.

$$
(3.5)\qquad |\nabla R|^2-4(|R|^2-4S+2n(n-1))\geqq 0\,.
$$

The equality sign is valid if and only if M is locally ϕ-symmetric.

COROLLARY 3.3. *In an n-dimensional Sasaki-Einsteinian space M, we have*

$$
(3.6)\qquad |\nabla R|^2-4(|R|^2-2n(n-1))\geqq 0\,.
$$

The equality sign is valid if and only if M is locally ϕ-symmetric.

Now putting $T_{ji}{}^h=\eta_j\phi_i{}^h-\eta_i\phi_j{}^h+\phi_{ji}\xi^h$, then (3.4) is rewritten as follows (Takahashi [10], p. 108).

$$
(3.7)\qquad \nabla_l R_{kji}{}^h=-T_{ls}{}^h R_{kji}{}^s+T_{lk}{}^s R_{sji}{}^h+T_{lj}{}^s R_{ksi}{}^h+T_{li}{}^s R_{kjs}{}^h\,.
$$

On the other hand, it is easily seen that

$$
(3.8)\qquad \nabla_k T_{ji}{}^h=\phi_{kj}\phi_i{}^h-\phi_{ki}\phi_j{}^h+\phi_{ji}\phi_k{}^h\,.
$$

Thus, by the result of Ambrose and Singer ([1]), we can see that a locally ϕ-symmetric Sasakian space is locally homogeneous and analytic. Moreover, from (3.7) and (3.8) they satisfy the following sufficient conditions of D'Atri and Nickerson [2] in order that every local geodesic symmetry of an analytic Riemannian space is divergence-preserving,

$$
(3.9)\qquad (\nabla_X R)(Y, X)X=T(X, R(Y, X)X)-R(T(X, Y)X)X\,,
$$

$$
(3.10)\qquad (\nabla_X T)(X, Y)=0\,,
$$

for any tangent vectors X, Y. Thus we have

THEOREM 3.4. *Let M be a Sasakian locally ϕ-symmetric space. Then every local geodesic symmetry of M is divergence-preserving.*

4. Sasakian spaces of dimensions 3 and 5.

First we consider a 3-dimensional Sasakian space with constant scalar curvature. It is known (cf. Tanno [11]) that the Riemannian curvature tensor is given by

$$(4.1)\qquad \begin{aligned} R_{kji}{}^h = &[(1/2)S-2](\delta_k{}^h g_{ji} - \delta_j{}^h g_{ki}) \\ &+[3-(1/2)S](\delta_k{}^h \eta_j \eta_i - \delta_j{}^h \eta_k \eta_i \\ &+\eta_k \xi^h g_{ji} - \eta_j \xi^h g_{ki}). \end{aligned}$$

Applying ∇_l to (4.1), we have

$$(4.2)\qquad \begin{aligned} \nabla_l R_{kji}{}^h = &[3-(1/2)S](\delta_k{}^h \phi_{lj} \eta_i + \delta_k{}^h \eta_j \phi_{li} \\ &-\delta_j{}^h \phi_{lk} \eta_i - \delta_j{}^h \eta_k \phi_{li} + \delta_{lk} \xi^h g_{ji} \\ &+\eta_k \phi_l{}^h g_{ji} - \phi_{lj} \xi^h g_{ki} - \eta_j \phi_l{}^h g_{ki}), \end{aligned}$$

taking account of (3.1) and the constancy of scalar curvature.

On the other hand, substituting (4.1) into (3.4) and using (3.2) (f), we can get (4.2). Thus we have.

THEOREM 4.1.*) *A 3-dimensional Sasakian space with constant scalar curvature is locally ϕ-symmetric.*

COROLLARY 4.2. *Let M be a 3-dimensional Sasakian space M whose every geodesic symmetry is divergence-peserving. Then M is locally ϕ-symmetric.*

In the following, let M be a 5-dimensional Sasaki-Einsteinian space. We denote by $T_p(M)$ its tangent space at p. Then we have the following lemma given by Ogawa [5].

LEMMA 4.3. *Let M be a 5-dimensional Sasaki-Einsteinian space. Then we can take for any $p \in M$ an adapted basis $(X_1, X_2=\phi X_1, X_3, X_4=\phi X_3, X_5=\xi)$ of $T_p(M)$ such that*

$$(4.3)\qquad \text{(a)}\quad \begin{aligned} &R_{1212}=R_{4343}(=a),\ R_{1313}=R_{2424}(=b), \\ &R_{1414}=R_{2323}(=c),\ R_{1234}=b+1,\ R_{2341}=c+1, \\ &R_{1234}=b+c+2,\ R_{5i5i}=-1, \\ &\textit{and all the other } R_{ijkl}=0, \end{aligned}$$

*) Prof. Tanno has suggested me to be able to prove the same fact by mean of local fibring. I wish to express my thanks to him.

$$\text{(b)}\quad \rho(X_1, X_2) \geqq 2\{\rho(X_1, X_2)+\rho(X_1, X_4)\}-3,$$

where $\rho(X, Y)=-R_{ijkl}X^iY^jX^kY^l$ *is the sectional curvature at* p *with respect to a* 2-*plane spanned by the orthonormal vectors* $X, Y \in T_p(M)$.

By this lemma, he calculated the following

$$(S/5)|R|^2=32\{a^2+3(b^2+c^2)+6(b+c)+2bc+8\}, \tag{4.4}$$

$$(2/5)S|R|^2+\beta+4\gamma=-64\{5(b+c)^2+15(b+c)+(9(b+c)+22)bc+6\}. \tag{4.5}$$

Now substituting (4.5) into (2.4), we get

$$|\nabla R|^2=(1/2)\Delta|R|^2+32(10b^2+20bc+10c^2+30b+30c +18b^2c+18bc^2+44bc+12). \tag{4.6}$$

Moreover substituting (4.4) and (4.6) into the left hand side of (3.5), we get

$$|\nabla R|^2-4(|R|^2-2n(n-1)) =(1/2)\Delta|R|^2+6(b^2+c^2+10bc+3b+3c+3b^2c+3bc^2), \tag{4.7}$$

by using $a+b+c=-3$. By virtue of (b) of Lemma 4.3, we have

$$b+c \geqq -2.$$

If the Sasakian space in consideration has sectional curvature $\geqq 0$, then it satisfies that $b+c \leqq 0$. So putting $x=b+c$ and $y=bc$, the range on which (x, y) exists is

$$D=\{-2 \leqq x \leqq 0,\ 0 \leqq y \leqq (1/4)x^2\}.$$

If we put $f(x, y)=x^2+(8x+8)y+3x$, then for $(x, y) \in D$ we have

$$f(x, y) \leqq x^2+(1/4)x^2(3x+8)+3x=3x[(1/2)x+1]^2.$$

Hence if all sectional curvatures are non-negative, then we see that $f(x, y) \leqq 0$ for all $(x, y) \in D$. Thus we get the following results.

THEOREM 4.4. *If a* 5-*dimensional compact Sasaki-Einsteinian space has non-negative sectional curvature, then it is locally* ϕ-*symmetric.*

THEOREM 4.5. *Let* M *be a* 5-*dimensional Sasaki-Einsteinian space. If* M *is of non-negative sectional curvature and* $|R|^2$ *is constant, then* M *is locally* ϕ-*symmetric.*

THEOREM 4.6. *Let* M *be a* 5-*dimensional Sasaki-Einsteinian space. If* M *is of non-negative sectional curvature and satisfies* (2.9), *then* M *is locally* ϕ-*symmetric.*

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