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ALMOST HERMITIAN MANIFOLDS SATISFYING SOME CURVATURE CONDITIONS

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§ 1. Introduction.

This paper is devoted to the study of almost Hermitian manifolds satisfying some curvature conditions, for example, $R(X, Y)\cdot R=0$. Let (M, g) be an m-dimensional connected Riemannian manifold with a positive-definite metric tensor $g=(g_{ji})$. Let ∇, $R=(R_{kji}{}^{h})$, $R_1=(R_{ji})$ and $S=g^{ji}R_{ji}$ be the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. In this paper, manifolds and tensor fields are assumed to be of class C^∞ unless otherwise specified. In 1965, Nomizu and Yano (cf. [10]) proved the following

THEOREM 1.1. *Let g be an irreducible locally symmetric Riemannian metric on an m-dimensional manifold M $(m\geqq 3)$. If g' is another Riemannian metric on M whose curvature tensor R' coincides with the curvature tensor R of g, then $g'=cg$, where c is a positive constant and hence, g' is also an irreducible locally symmetric Riemannian metric on M.*

In a locally symmetric space (M, g), at each point of M, we have

$$(*) \qquad R(X, Y)\cdot R=0, \qquad \text{for all tangent vectors } X, Y,$$

where the linear transformation $R(X, Y)$ operates on the curvature tensor R as a derivation defined on the tangent tensor algebra at each point. Conversely, does the algebraic condition (*) on the curvature tensor R imply that $\nabla R=0$? Nomizu gave a conjecture as follows.

CONJECTURE. *Let (M, g) be an m-dimensional complete, irreducible Riemannian manifold with $m\geqq 3$. If (M, g) satisfies the condition (*), then (M, g) is locally symmetric.*

Now, if the conjecture is valid, it must follow that as long as there is an irreducible and locally symmetric Riemannian metric g on M, any metric g' on M such that $R'=R$ is also locally symmetric. This is nothing but Theorem

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1.1. Next, we denote by H the tensor field of type (1, 5) on (M, g) defined by $H(X, Y)=-R(X, Y)\cdot R$. If the conjecture is valid, the tensor field H represents a deviation of a Riemannian metric from a locally symmetric one. However, by examples given by Takagi [19] and the present author [13], the conjecture is negative. In [14], the present author proved that the conjecture is valid in the case where (M, g) is compact and irreducible, provided $\dim M=3$. Thus the following problem will be naturally set.

PROBLEM. Does the algebraic condition (*) on the curvature tensor of a compact and irreducible Riemannian manifold (M, g) with $\dim M>3$ imply the fact that (M, g) is locally symmetric?

On the other hand, it might be interesting to study relations between the Riemannian structure g and the almost complex structure F. For example, how does the Riemannian structure g affect the almost complex structure F in an almost Hermitian manifold (M, F, g)? In § 2, we recall a theorem due to Lichnerowicz for later use. § 3 will be devoted to give some formulas and theorems concerning almost Hermitian manifolds. In § 4, we shall study some K-spaces satisfying the condition (*). In § 5, we shall give by using the tensor fields H, R and etc., a sufficient condition for a 6-dimensional K-space to be a homogeneous almost Hermitian manifold. In § 6, we shall study 4-dimensional F-spaces and H-spaces satisfying the condition (*).

§ 2. A theorem of A. Lichnerowicz.

Let (M, g) be a Riemannian manifold. Lichnerowicz [26] obtained the following formula

$$\frac{1}{2}\Delta f=2H^{p}{}_{kjpih}R^{kjih}-4(\nabla_k\nabla_i R_{hj})R^{kjih}+(\nabla_p R_{kjih})\nabla^p R^{kjih}, \tag{2.1}$$

where $f=R_{kjih}R^{kjih}$, and $H_{pqkjih}=-(\nabla_p\nabla_q R_{kjih}-\nabla_q\nabla_p R_{kjih})$ (cf. § 1).

In each local coordinate neighborhood, (*) is equivalent to

$$H_{pqkjih}=0\,, \quad \text{or} \tag{2.2}$$

$$R_{pqk}{}^{t}R_{tjih}+R_{pqj}{}^{t}R_{ktih}+R_{pqi}{}^{t}R_{kjth}+R_{pqh}{}^{t}R_{kjit}=0\,.$$

From (2.1) and (2.2), Lichnerowicz proved the following

THEOREM 2.1. *Let (M, g) be a compact Riemannian manifold satisfying the condition (*). If $\nabla_k R_{ji}=0$ holds on M, then (M, g) is locally symmetric.*

Fujimura [3], and Sekigawa and Takagi [17] gave some generalizations of Theorem 2.1.

§ 3. Curvature tensors in an almost Hermitian manifold.

Let (M, F, g) be an $m(=2n)$-dimensional almost Hermitian manifold with almost Hermitian structure (F, g). If we now put

$$R^*_{ji}=\frac{1}{2}F_j{}^t R_{itk}{}^h F_h{}^k, \tag{3.1}$$

then, by definition, we have

$$F^{ba}R_{baj}{}^i=-2R^{*i}_t F_j{}^t, \quad \text{and} \quad F^{ba}R_{jba}{}^i=R^{*i}_t F_j{}^t. \tag{3.2}$$

Applying the Ricci identity to $F_i{}^h$, we obtain

$$\nabla_k\nabla_j F_i{}^h-\nabla_j\nabla_k F_i{}^h=R_{kjt}{}^h F_i{}^t-R_{kji}{}^t F_t{}^h. \tag{3.3}$$

From (3.3), contracting with respect to k and h, we get

$$\nabla_t\nabla_j F_i{}^t+\nabla_j F_i=(R_{jt}-R^*_{tj})F_i{}^t, \tag{3.4}$$

where $F_i=-\nabla_k F_i{}^k$.

Now, operating ∇_j to $F^{ji}F_{ih}=-\delta_h{}^j$ gives

$$F^i F_{ih}+F^{ji}\nabla_j F_{ih}=0.$$

Operating $\nabla^h=g^{hi}\nabla_i$ to the both sides of this equation and taking account of (3.4), we have

$$S-S^*=(\nabla^h F^{ji})\nabla_j F_{ih}-F^i F_i-2F^{ji}(\nabla_j F_i), \tag{3.5}$$

where $S^*=g^{ji}R^*_{ji}$.

Next, we shall recall the definitions of special kinds of almost Hermitian manifolds. If (F, g) satisfies

$$\nabla_j F_i{}^h=0, \tag{3.6}$$

then (M, F, g) is called a Kaehlerian space. If (F, g) satisfies

$$\nabla_j F_i{}^h+\nabla_i F_j{}^h=0, \tag{3.7}$$

then (M, F, g) is called a K-space (or a Tachibana space). If (F, g) satisfies

$$\nabla_j F_{ih}+\nabla_i F_{hj}+\nabla_h F_{ji}=0, \tag{3.8}$$

then (M, F, g) is called an H-space (or an almost Kaehlerian space). If (F, g) satisfies

$$\nabla_j\nabla_k F_i{}^h-\nabla_k\nabla_j F_i{}^h=R_{jkt}{}^h F_i{}^t-R_{jki}{}^t F_t{}^h=0, \tag{3.9}$$

then (M, F, g) is called an F-space (or a para-Kaehlerian space). When (M, F, g) is a Kaehlerian space, a K-space or an H-space, the condition $F_i=0$ is satisfied.

Suppose that (M, F, g) is an H-space. Then, by (3.8), we have

$$(\nabla^h F^{ji})\nabla_j F_{ih}=\frac{1}{2}(\nabla^h F^{ji})(\nabla_j F_{ih}-\nabla_i F_{jh})=-\frac{1}{2}(\nabla^h F^{ji})\nabla_h F_{ji}\,,$$

and hence, by (3.5)

$$S-S^*=-\frac{1}{2}(\nabla^h F^{ji})\nabla_h F_{ji}\leqq 0\,. \tag{3.10}$$

Thus, we have the following (cf. [26])

THEOREM 3.1. *In an H-space, we have $S\leqq S^*$, and the equality sign occurs if and only if the space is a Kaehlerian space.*

Next, suppose the (M, F, g) is a K-space. Then, by (3.7), we have

$$(\nabla^h F^{ji})\nabla_j F_{ih}=(\nabla^h F^{ji})\nabla_h F_{ji}\geqq 0\,.$$

Thus, we have the following (cf. [26])

THEOREM 3.2. *In a K-space, we have $S\geqq S^*$, and the equality sign occurs if and only if the space is a Kaehlerian space.*

We denote by $A(M, F, g)$ ($I(M, F, g)$, resp.) the group of all automorphisms of (M, F, g) (the group of all isometries of (M, F, g) resp.) which acts effectively on M, and by $A_0(M, F, g)$ ($I_0(M, F, g)$, resp.) the identity component of $A(M, F, g)$ ($I(M, F, g)$, resp.). Then, it is evident that $A(M, F, g)\subset I(M, F, g)$ (and $A_0(M, F, g)\subset I_0(M, F, g)$). Especially, if there exists a subgroup G of $A(M, F, g)$ which acts transitively on M, then (M, F, g) is called a homogeneous almost Hermitian manifold. Recently, concerning with the result of Ambrose and Singer [1], the present author proved the following (cf. [16])

THEOREM 3.3. *Let (M, F, g) be a homogeneous almost Hermitian manifold. Then, there exists a skew-symmetric tensor field $T=(T_{ji}{}^h)$ of type* (1, 2) *on M satisfying the following conditions*

$$\nabla_p R_{kji}{}^h=T_{pt}{}^h R_{kji}{}^t-T_{pk}{}^t R_{tji}{}^h-T_{pj}{}^t R_{kti}{}^h-T_{pi}{}^t R_{kjt}{}^h\,, \tag{A}$$

$$\nabla_p T_{ji}{}^h=T_{pt}{}^h T_{ji}{}^t-T_{pj}{}^t T_{ti}{}^h-T_{pi}{}^t T_{jt}{}^h\,, \tag{B}$$

$$\nabla_p F_j{}^h=T_{pt}{}^h F_j{}^t-T_{pj}{}^t F_t{}^h\,. \tag{C}$$

Conversely, if a complete and simply connected almost Hermitian manifold (M, F, g) admits a skew-symmetric tensor field T of type (1, 2) on M satisfying the conditions (A), (B) and (C), then (M, F, g) is a homogeneous almost Hermitian manifold.

In the above Theorem, the skew-symmetricity of $T=(T_{ji}{}^h)$ means that $T_{jih}=-T_{jhi}$ holds, where $T_{jih}=T_{ji}{}^k g_{kh}$. The rough sketch of the proof of Theorem 3.3 is as follows. Let $O(M, F, g)$ ($U(M, F, g)$, resp.) be the orthonormal frame bundle (the unitary frame bundle, resp.) over M with respect to the Riemannian structure g (the almost Hermitian structure (F, g), resp.). We denote by G the holonomy subbundle of $U(M, F, g)$ with respect to the linear connection $\nabla_X^*=\nabla_X-T(X)$, where $T(X)Y=T(X, Y)$. Then, G acts effectively and transitively on M as a group of automorphisms of (M, F, g).

§4. K-spaces satisfying the condition (*).

In this section, in connection with Theorem 3.3 and the conjecture stated in §1, we shall prove the following main Theorems 4.1 and 4.2.

THEOREM 4.1. *Let (M, F, g) be a complete and irreducible non-Kaehlerian K-space satisfying the condition* (*). *Then (M, F, g) is a compact and locally symmetric space.*

In this paper, when an almost Hermitian manifold (M, F, g) is irreducible with respect to the Riemannian connection ∇, we say that (M, F, g) is a irreducible almost Hermitian manifold.

THEOREM. 4.2. *Let (M, F, g) satisfy the same hypothesis as in Theorem* 4.1. *Assume moreover that M is simply connected. Then (M, F, g) is a compact and irreducible Riemannian symmetric space and furthermore, M admits two actions of compact Lie groups which are effective and transitive on M. But these two actions are not similar to each other* (*cf. Remark below*).

Remark. Let G_1 and G_2 be two compact, connected Lie groups which acts on a manifold M effectively and transitively, K_1 and K_2 the isotropy subgroups of G_1 and G_2 respectively at some point of M. Denote by $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$ the Lie algebras of G_1, G_2, K_1 and K_2, respectively. When there is an isomorphism $\phi: \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi(\mathfrak{k}_1)=\mathfrak{k}_2$, we say that the action of G_1 is similar to that of G_2 (cf. [22]). Let (M, g) be a compact and simply connected, irreducible Riemannian symmetric space, and $M=I_0(M, g)/K$. Let G be a compact Lie group which acts on M transitively and effectively. Then, it is known that, for some kinds of such (M, g), for example,

$$SO(2l+1)/SO(2m)\times SO(2l-2m+1) \ (2<m<l-1), \quad SO(2l+1)/SO(2l) \quad (l\neq 3),$$

$$SO(2l)/SO(2m)\times SO(2l-2m) \quad (1<m<l-1), \quad Sp(l)/Sp(m)\times Sp(l-m),$$

$$E_6/SU(2)\cdot SU(6), \quad E_7/SU^*(8) \ (SU^*(8)=SU(8)/Z_2), \quad E_7/SU(2)\cdot Spin(12),$$

$$E_8/SO(16), \quad E_8/SU(2)\cdot E_7, \quad F_4/SU(2)\cdot Sp(3), \quad F_4/Spin(9), \quad G_2/SO(4) \quad (\text{cf. [22]}),$$

the action of G is always similar to the standard transitive action of $I_0(M, g)$ on M as a Riemannian symmetric space. In [4], Fukami and Ishihara showed that there exists a Tachibana structure (F, g) on a 6-dimensional sphere S^6

with the canonical Riemannian metric g by making use of the properties of the algebra of Cayley numbers and $A_0(S^6, F, g)=G_2$, $S^6=G_2/SU(3)$. Now, let (M, F, g) be a K-space. First of all, we shall write down some fundamental formulas in a K-space (cf. [9], [20], [23], etc.) as follows:

$$F_j{}^b F_i{}^a R_{ba}=R_{ji}, \qquad F_j{}^b F_i{}^a R^*_{ba}=R^*_{ji}, \tag{4.1}$$

$$R^*_{ji}=R^*_{ij}, \tag{4.2}$$

$$(\nabla_j F_{ts})\nabla_i F^{ts}=S_{ji}, \tag{4.3}$$

$$(\nabla_j F_{ts})\nabla^j F^{ts}=S-S^*\geqq 0 \quad \text{(by (4.3) and Theorem 3.2)}, \tag{4.4}$$

$$\nabla^t\nabla_t F_j{}^h=F^{ht}S_{jt}, \tag{4.5}$$

$$R_{jist}-R_{bast}F_j{}^b F_i{}^a=-(\nabla_j F_i{}^r)\nabla_s F_{tr}, \tag{4.6}$$

$$R_{jihk}-R_{abcd}F_j{}^a F_i{}^b F_h{}^c F_k{}^d=0, \tag{4.7}$$

$$\nabla_k\nabla_j F_{ih}=-\frac{1}{2}(R_{kaih}F_j{}^a-R_{akji}F_h{}^a+R_{akjh}F_i{}^a), \tag{4.8}$$

$$(R_{kjih}-R_{kjba}F_i{}^b F_h{}^a)S^{ji}=\frac{1}{4}(3R_{kr}+R^*_{kr})S_h{}^r, \tag{4.9}$$

$$(\nabla_k S_{ji})\nabla^k S^{ji}=\frac{1}{8}(R_{bh}-5R^*_{bh})S^{bi}S_i{}^h, \tag{4.10}$$

where $S_{ji}=R_{ji}-R^*_{ji}$.

For the sake of later use, we shall establish the following formula (4.11). Taking account of (4.3) and (4.6), we have

$$\begin{aligned}
&R_{kjt}{}^v F^{tu} R_{vubh} F^{jb} - R_{kjt}{}^v F^{tu} R_{vua}{}^j F_h{}^a \\
&=R_k{}^j{}_t{}^v F^{tu}(R_{vubh}F_j{}^b+R_{vujb}F_h{}^b) \\
&=-R_{kjt}{}^v F^{tu} F_h{}^b(\nabla^j F_b{}^r)\nabla_v F_{ur} \\
&=-R_{kjtv}(\nabla^v F^{tu})\nabla^j F_{hu} \\
&=R_{kjtv}(R^{vtj}{}_h-F_b{}^v F_a{}^t R^{baj}{}_h) \\
&=\frac{1}{2}(R^{vtj}{}_h-F_b{}^v F_a{}^t R^{baj}{}_h)(R_{vtjk}-F_v{}^q F_t{}^p R_{qpjk}) \\
&=\frac{1}{2}(\nabla^v F^{tr})(\nabla^j F_{hr})(\nabla_v F_{tq})(\nabla_j F_k{}^q) \\
&=\frac{1}{2}S^{rq}(\nabla^j F_{hr})(\nabla_j F_{kq}) \\
&=-\frac{1}{2}S^{rq}(R_{hrkq}-R_{hrba}F_k{}^b F_q{}^a).
\end{aligned}$$

Thus, from the above equation and (4.9), we have

$$R_{kjt}{}^{v}F^{tu}R_{vubh}F^{jb}-R_{kjt}{}^{v}F^{tu}R_{vua}{}^{j}F_{h}{}^{a}=\frac{1}{8}(3R_{hr}+R^{*}_{hr})S_{k}{}^{r}. \tag{4.11}$$

Now, we define a tensor field $T=(T_{ji}{}^{h})$ of type (1, 2) on M by

$$T_{ji}{}^{h}=-\frac{1}{2}F_{j}{}^{u}\nabla_{u}F_{i}{}^{h}. \tag{4.12}$$

Then, we have

$$T_{ji}{}^{h}=-T_{ij}{}^{h}, \qquad \text{and} \qquad T_{jih}=-T_{jhi}. \tag{4.13}$$

By the second equation of (4.13), T is skew-symmetric. We shall here prove the following

LEMMA 4.3. *In a K-space, T satisfies the conditions* (B) *and* (C) *in Theorem* 3.3.

Proof. From the definition of T and (4.6), we have

$$\nabla_{p}T_{ji}{}^{h}=\frac{1}{2}(R_{pji}{}^{h}-F_{p}{}^{b}F_{j}{}^{a}R_{bai}{}^{h}-F_{j}{}^{t}\nabla_{p}\nabla_{t}F_{i}{}^{h}),$$

$$T_{pt}{}^{h}T_{ji}{}^{t}=\frac{1}{4}F_{p}{}^{u}F_{j}{}^{v}(R_{u}{}^{h}{}_{vi}-R_{u}{}^{h}{}_{ba}F_{v}{}^{b}F_{i}{}^{a}),$$

$$T_{pj}{}^{t}T_{ti}{}^{h}=\frac{1}{4}F_{p}{}^{u}F_{j}{}^{v}(R_{uvi}{}^{h}-R_{bai}{}^{h}F_{u}{}^{b}F_{v}{}^{a}),$$

and

$$T_{pi}{}^{t}T_{jt}{}^{h}=\frac{1}{4}F_{p}{}^{u}F_{j}{}^{v}(R_{uiv}{}^{h}-R_{bav}{}^{h}F_{u}{}^{b}F_{i}{}^{a}).$$

From the above equations, taking account of (4.6), (4.8) and the first Bianchi identity, we have

$$\begin{aligned}
&F_{k}{}^{j}(\nabla_{p}T_{ji}{}^{h}-T_{pt}{}^{h}T_{ji}{}^{t}+T_{pj}{}^{t}T_{ti}{}^{h}+T_{pi}{}^{t}T_{jt}{}^{h})\\
&=\frac{1}{2}(F_{k}{}^{j}R_{pji}{}^{h}+F_{p}{}^{u}R_{uki}{}^{h}+\nabla_{p}\nabla_{k}F_{i}{}^{h})\\
&\quad+\frac{1}{4}(F_{p}{}^{u}R_{kiu}{}^{h}-F_{u}{}^{h}R_{kip}{}^{u}-F_{p}{}^{u}R_{uki}{}^{h}-F_{k}{}^{u}R_{pui}{}^{h}-F_{p}{}^{u}R_{uik}{}^{h}-F_{i}{}^{u}R_{puk}{}^{h})\\
&=\frac{1}{2}\nabla_{p}\nabla_{k}F_{i}{}^{h}+\frac{1}{4}(F^{hu}R_{puki}-F_{i}{}^{u}R_{puk}{}^{h}+F_{k}{}^{u}R_{pui}{}^{h})\\
&\quad+\frac{1}{4}F_{p}{}^{u}(R_{kiu}{}^{h}+R_{uki}{}^{h}+R_{iuk}{}^{h})\\
&=\frac{1}{2}\Big(\nabla_{p}\nabla_{k}F_{i}{}^{h}+\frac{1}{2}(R_{pui}{}^{h}F_{k}{}^{u}-R_{upki}F^{hu}+R_{upk}{}^{h}F_{i}{}^{u})\Big)=0.
\end{aligned}$$

Thus, T satisfies the condition (B). Next, using (3.7), we have

$$F_{p}{}^{u}(\nabla_{u}F_{t}{}^{h})F_{j}{}^{t}=-F_{p}{}^{u}(\nabla_{u}F_{j}{}^{t})F_{t}{}^{h}=F_{p}{}^{u}(\nabla_{j}F_{u}{}^{t})F_{t}{}^{h}=-F_{u}{}^{t}(\nabla_{j}F_{p}{}^{u})F_{t}{}^{h}$$

$$=\nabla_j F_p{}^h,$$

$$F_p{}^u(\nabla_u F_j{}^t)F_t{}^h=-F_p{}^u(\nabla_j F_u{}^t)F_t{}^h=F_u{}^t(\nabla_j F_p{}^u)F_t{}^h=-\nabla_j F_p{}^h.$$

Taking account of the above equations, we have

$$\nabla_p F_j{}^h-T_{pt}{}^h F_j{}^t+T_{pj}{}^t F_t{}^h=\nabla_p F_j{}^h+\frac{1}{2}\nabla_j F_p{}^h+\frac{1}{2}\nabla_j F_p{}^h=0.$$

Thus, T satisfies the condition (C). Q. E. D.

As a consequence of Lemma 4.3, we can now prove Theorem 4.4 given later. Suppose that a K-space (M, F, g) is locally symmetric. Then, by (4.7), we have

$$(\nabla_p F_j{}^u)F_u{}^t R_{tihk}+(\nabla_p F_i{}^u)F_u{}^t R_{jthk}+(\nabla_p F_h{}^u)F_u{}^t R_{jitk}+(\nabla_p F_k{}^u)F_u{}^t R_{jiht}=0.$$

Thus, the above equation together with (3.7), (4.12) and (4.13) implies

$$T_{pj}{}^t R_{tihk}+T_{pi}{}^t R_{jthk}+T_{ph}{}^t R_{jitk}+T_{pk}{}^t R_{jiht}=0,$$

which means that T satisfies the condition (A). Consequently, from Theorem 3.3 and Lemma 4.3, we have the following

THEOREM 4.4. *Let (M, F, g) be a complete, simply connected and locally symmetric K-space. Then (M, F, g) is a homogeneous almost Hermitian manifold.*

Proof of Theorem 4.1. Let (M, F, g) be a complete and irreducible non-Kaehlerian K-space satisfying the condition (*). Then the condition (*) implies in particular

$$R_{kjt}{}^h R_i{}^t-R_{kji}{}^t R_t{}^h=0. \tag{**}$$

By the definition of R^*_{ji} and the condition (*), we have

$$R^{*i}_t R_h{}^t-R^{*t}_h R_t{}^i=0. \tag{4.14}$$

By a straightforward calculation, we get

$$\begin{aligned}\nabla_p\nabla_k R^*_{ji}=\frac{1}{2}((\nabla_p\nabla_k F^{ba})R_{bati}F_j{}^t&+(\nabla_k F^{ba})(\nabla_p R_{bati})F_j{}^t\\ &+(\nabla_k F^{ba})R_{bati}\nabla_p F_j{}^t+(\nabla_p F^{ba})(\nabla_k R_{bati})F_j{}^t\\ &+F^{ba}(\nabla_p\nabla_k R_{bati})F_j{}^t+F^{ba}(\nabla_k R_{bati})\nabla_p F_j{}^t\\ &+(\nabla_p F^{ba})R_{bati}(\nabla_k F_j{}^t)+F^{ba}(\nabla_p R_{bati})\nabla_k F_j{}^t\\ &+F^{ba}R_{bati}\nabla_p\nabla_k F_j{}^t).\end{aligned}$$

Taking account of (3.2), (3.3) and (*), we have, from the above equation

$$\nabla_p\nabla_k R^*_{ji}-\nabla_k\nabla_p R^*_{ji}=-R_{pkj}{}^t R^*_{ti}-R_{pki}{}^t R^*_{jt} \tag{4.15}$$

$$=R_{pks}{}^{b}F^{sa}R_{bati}F_{j}{}^{t}-R^{*}_{ai}F_{t}{}^{a}(R_{pks}{}^{t}F_{j}{}^{s}-R_{pkj}{}^{s}F_{s}{}^{t})$$

$$=R_{pks}{}^{b}F^{sa}R_{bati}F_{j}{}^{t}-R^{*}_{ai}R_{pks}{}^{t}F_{t}{}^{a}F_{j}{}^{s}-R^{*}_{ai}R_{pkj}{}^{a}\,.$$

Now, by making use of the definition of S_{ji}, we have easily

$$(R_{kjih}-R_{kjba}F_{i}{}^{b}F_{h}{}^{a})S^{ji}$$
$$=R_{kjih}R^{ji}-R_{kjih}R^{*ji}-R_{kjba}F_{i}{}^{b}F_{h}{}^{a}R^{ji}+R_{kjba}F_{i}{}^{b}F_{h}{}^{a}R^{*ji}\,. \tag{4.16}$$

To obtain a new formula (4.21), we shall compute the right hand side of (4.16). Taking account of the formula (**) given above, we have

$$R_{kjih}R^{ji}=-R_{kjh}{}^{i}R_{i}{}^{j}=-R_{kji}{}^{j}R_{h}{}^{i}=R_{ki}R_{h}{}^{i}. \tag{4.17}$$

By using (3.2), (4.1) and (**), we have

$$R_{kjba}F_{i}{}^{b}F_{h}{}^{a}R^{ji}=-R_{kjba}F_{i}{}^{j}F_{h}{}^{a}R^{ib}=-R_{kji}{}^{b}R_{b}{}^{a}F^{ij}F_{ha}$$
$$=R^{*b}_{t}F_{k}{}^{t}R_{b}{}^{a}F_{ha}=F_{t}{}^{b}R^{*t}_{k}R_{ba}F_{h}{}^{a}=R^{*}_{kt}R_{h}{}^{t}\,. \tag{4.18}$$

Because of (3.2) and (4.15), we have

$$R_{kjih}R^{*ji}=R_{k}{}^{j}{}_{j}{}^{i}R^{*}_{ih}+R_{k}{}^{j}{}_{s}{}^{b}F^{sa}R_{bath}F_{j}{}^{t}$$
$$-R^{*}_{ah}R_{kjs}{}^{t}F_{t}{}^{a}F^{js}-R^{*}_{ah}R_{k}{}^{j}{}_{j}{}^{a}$$
$$=R_{kjs}{}^{b}F^{sa}R_{bath}F^{jt}+R^{*}_{ah}R^{*a}_{k}\,. \tag{4.19}$$

Taking account of (3.2), (4.1) and (4.15), we have

$$R_{kjba}F_{i}{}^{b}F_{h}{}^{a}R^{*ji}=-R_{kjba}F_{i}{}^{j}R^{*ib}F_{h}{}^{a}$$
$$=-(R_{kji}{}^{b}R^{*a}_{b}+R_{kjs}{}^{b}F^{su}R_{but}{}^{a}F_{i}{}^{t}-R^{*a}_{u}R_{kjs}{}^{t}F_{t}{}^{u}F_{i}{}^{s}-R^{*a}_{t}R_{kji}{}^{t})F^{ij}F_{ha}$$
$$=-R_{k}{}^{t}{}_{s}{}^{v}F^{su}R_{vut}{}^{a}F_{ha}+R^{*}_{th}R_{k}{}^{t}\,. \tag{4.20}$$

Substituting (4.17)$\sim$(4.20) into the right hand side of (4.16), we have

$$(R_{kjih}-R_{kjba}F_{i}{}^{b}F_{h}{}^{a})S^{ji}$$
$$=S_{ht}S_{k}{}^{t}-(R_{kjt}{}^{v}F^{tu}R_{vubh}F^{jb}-R_{kjt}{}^{v}F^{tu}R_{vua}{}^{j}F_{h}{}^{a})+2R^{*}_{th}S_{k}{}^{t}\,. \tag{4.21}$$

This equation (4.21) together with (4.9) and (4.11) implies

$$\frac{1}{4}(3R_{kt}+R^{*}_{kt})S_{h}{}^{t}$$
$$=S_{kt}S_{h}{}^{t}-\frac{1}{8}(3R_{kt}+R^{*}_{kt})S_{h}{}^{t}+2R^{*}_{ht}S_{k}{}^{t}\,. \tag{4.22}$$

Taking account of (4.14), we have, from (4.22)

$$S_{h}{}^{t}(R_{kt}-5R^{*}_{kt})=0\,. \tag{4.23}$$

Transvecting (4.23) with S^{hk} and using (4.10), we have finally the formula

$$\nabla_k S_{ji}=0. \tag{4.24}$$

Since (M, F, g) is irreducible, as a consequence of (4.24), there exists a constant c such that

$$S_{ji}=cg_{ji}. \tag{4.25}$$

Thus, from (4.23) and (4.25), we have

$$c(R_{ji}-5R^*_{ji})=0. \tag{4.26}$$

Since (M, F, g) is non-Kaehlerian, we have $c\neq 0$. Thus, from (4.26), we have

$$R_{ji}=5R^*_{ji}. \tag{4.27}$$

From (4.25) and (4.27), we have

$$R_{ji}=\frac{5}{4}cg_{ji}, \qquad R^*_{ji}=\frac{1}{4}cg_{ji}. \tag{4.28}$$

Using (4.4) and (4.25), we have $c>0$. Thus, (M, F, g) is an Einstein space with positive scalar curvature $S=(5/4)mc$. Since (M, F, g) is complete, by Myers' theorem, we see that M is compact, and its diameter $d(M)$ satisfies $d(M)\leqq 2(\sqrt{(m-1)/5c})\pi$. Consequently, because of Theorem 2.1, Theorem 4.1 is proved completely. Q. E. D.

Proof of Theorem 4.2. We assume furthermore that M is simply connected. Then, by Theorem 4.1, (M, F, g) is a compact, simply connected and irreducible Riemannian symmetric space, and furthermore, by Theorem 4.4, (M, F, g) is a homogeneous almost Hermitian manifold. Since (M, F, g) is a Riemannian symmetric space, of course, the tensor field $T_1=0$ of type (1, 2) on M satisfies the conditions (A) and (B) in Theorem 3.3. Let G_1 and G be the holonomy subbundles over M of $O(M, F, g)$ through a point of $U(M, F, g)$ with respect to the linear connections ∇_X and $\nabla^*_X=\nabla_X-T(X)$, respectively, where T is the tensor field on M defined by (4.12). Then, G_1 (G, resp.) acts on M effectively and transitively as a group of isometries of (M, F, g) (automorphisms of (M, F, g), resp.). Since M is compact, $A_0(M, F, g)$ and $I_0(M, F, g)$ are both compact Lie groups (cf. [7]). Now, we assume that $A_0(M, F, g)=I_0(M, F, g)$ holds. Then, $G_1\subset A_0(M, F, g)$ and hence, $G_1\subset U(M, F, g)$. However, since (M, F, g) is non-Kaehlerian, this is a contradiction. Thus, $A_0(M, F, g)$ is a proper subgroup of $I_0(M, F, g)$. Thus, $\dim A_0(M, F, g)<\dim I_0(M, F, g)$ holds, and hence, the actions of $A_0(M, F, g)$ and $I_0(M, F, g)$ on M are not similar to each other. Consequently, we have Theorem 4.2. Q. E. D.

Recently, Ogawa [11] proved the following

THEOREM 4.5. *Let (M, F, g) be a compact Kaehlerian space satisfying the* *condition* (*). *If the scalar curvature of* (M, F, g) *is constant, then* (M, F, g) *is locally symmetric.*

From Theorems 4.1 and 4.5, in connection with the problem stated in §1, we have immediately the following

COROLLARY 4.6. *Let* (M, F, g) *be a compact and irreducible K-space satisfying the condition* (*). *If the scalar curvature of* (M, F, g) *is constant, then* (M, F, g) *is locally symmetric.*

§5. 6-dimensional K-spaces.

Let (M, F, g) be a 6-dimensional complete non-Kaehlerian K-space. Then, besides the formulas (4.1)~(4.10), the following identities hold (cf. [9], [20]).

$$\nabla_k\nabla_j F_{ih}=-\frac{S}{30}(g_{kj}F_{ih}+g_{ki}F_{hj}+g_{kh}F_{ji}), \tag{5.1}$$

$$\begin{aligned}(\nabla_t F_{kj})\nabla^t F_{ih}&=F_i{}^t F_h{}^s R_{kjts}-R_{kjih}\\ &=-\frac{S}{30}(g_{ji}g_{kh}-g_{ki}g_{jh}-F_{ji}F_{kh}+F_{ki}F_{jh}).\end{aligned} \tag{5.2}$$

From (5.2), we have

$$F_p{}^t R_{kjth}=-F_h{}^t R_{kjpt}+\frac{S}{30}(F_{hk}g_{jp}-F_{hj}g_{kp}-g_{hk}F_{jp}+g_{hj}F_{kp}). \tag{5.3}$$

Transvecting (5.2) with g^{kh}, we get

$$R_{ji}-R^*_{ji}=\frac{2S}{15}g_{ji},$$

from which and (4.9), taking account of Theorem 3.2,

$$R_{ji}-5R^*_{ji}=0.$$

From the above equations, we have

$$R_{ji}=\frac{S}{6}g_{ji}, \qquad R^*_{ji}=\frac{S}{30}g_{ji}. \tag{5.4}$$

From (3.2) and (5.4), we have

$$F^{ba}R_{baji}=-\frac{S}{15}F_{ji}, \qquad F^{ba}R_{jbai}=\frac{S}{30}F_{ji}. \tag{5.5}$$

Now, we shall establish the integral formula (5.26) given later, which implies the following

THEOREM 5.1. *Let* (M, F, g) *be a* 6*-dimensional complete and simply connected non-Kaehlerian K-space satisfying the condition*

$$2H^p{}_{kjpih}R^{kjih}+\frac{S}{15}\left(R_{kjih}R^{kjih}-\frac{S^2}{15}\right)\geqq 0.$$

Then, (M, F, g) is a homogeneous almost Hermitian manifold.

For the sake of later use, we shall prepare the following formulas (5.6)$\sim$(5.9), (5.11) and (5.13). By making use of (5.2)$\sim$(5.5), we have

$$(5.6)\qquad \begin{aligned} R^{jpih}F_i{}^kF_p{}^tR_{kjth} &= -F_i{}^kF_h{}^tR^{jpih}R_{kjpt} \\ &\quad +\frac{S}{30}R^{jpih}F_i{}^k(F_{hk}g_{jp}-F_{hj}g_{kp}-g_{hk}F_{jp}+g_{hj}F_{kp}) \\ &= -\Big(R^{ktjp}-\frac{S}{30}(g^{jt}g^{pk}-g^{jk}g^{pt}-F^{jt}F^{pk}+F^{jk}F^{pt})\Big)R_{kjpt}-\frac{2}{75}S^2 \\ &= -R^{ktjp}R_{kjpt}-\frac{4}{75}S^2 . \end{aligned}$$

The formulas (5.4) and (5.5) imply

$$(5.7)\qquad R^{jpih}F_{pi}F^{kt}R_{kjth} = -\Big(\frac{S}{30}\Big)^2F^{jh}F_{jh} = -\frac{S^2}{150}.$$

Now, making use of (5.2)$\sim$(5.5) and taking account of (5.6), we have

$$(5.8)\qquad \begin{aligned} R^{jpih}F_p{}^kF_i{}^tR_{kjth} &= -F_p{}^kF_i{}^tR^{jpih}R_{kthj}-F_p{}^kF_i{}^tR^{jpih}R_{khjt} \\ &= -R^{jpih}\Big(R_{pihj}-\frac{S}{30}(g_{pj}g_{ih}-g_{ph}g_{ij}-F_{pj}F_{ih}+F_{ph}F_{ij})\Big) \\ &\quad -R^{ktjp}R_{kjpt}-\frac{4}{15}S^2 \\ &= -R^{jpih}R_{pihj}+\frac{2}{75}S^2-R^{ktjp}R_{kjpt}-\frac{4}{75}S^2 \\ &= -\frac{2}{75}S^2 . \end{aligned}$$

By making use of (5.2), (5.4) and (5.5), we find

$$(5.9)\qquad \begin{aligned} &R^k{}_v{}^{ih}F_k{}^jF^{uv}R_{ujih} \\ &= -R^{kvih}\Big(R_{vkih}-\frac{S}{30}(g_{ki}g_{vh}-g_{vi}g_{kh}-F_{ki}F_{vh}+F_{vi}F_{kh})\Big) \\ &= R_{kjih}R^{kjih}-\frac{4}{75}S^2 . \end{aligned}$$

Furthermore, from (5.2) and (5.4), we have

$$(5.10)\qquad \begin{aligned} &R^{jpih}(\nabla^kF_p{}^u)(\nabla_iF_u{}^t)R_{kjth} \\ &= -\frac{S}{30}\Big(R^{jpih}R_{ijph}+\frac{S^2}{6}+R^{jpih}F_i{}^kF_p{}^tR_{kjth}+R^{jpih}F_{pi}F^{kt}R_{kjth}\Big). \end{aligned}$$

Substituting (5.6) and (5.7) respectively into the third term and the forth term in the parenthesis of the right hand side of (5.10), we have

$$R^{jpih}(\nabla^k F_p{}^u)(\nabla_i F_u{}^t)R_{kjth}=-\frac{4}{5\cdot 15^2}S^3 . \tag{5.11}$$

From (5.1) and (5.4), we have

$$\begin{aligned} & R^{jpih}F_p{}^a(\nabla^k\nabla_i F_a{}^t)R_{kjth} \\ &\quad =\frac{S}{30}\Big(R^{jpih}R_{ijph}+F_p{}^k F_i{}^t R^{jpih}R_{kjth}+\frac{S^2}{6}\Big). \end{aligned} \tag{5.12}$$

Substituting (5.8) into the second term in the parenthesis of the right hand side of (5.12), we have

$$R^{jpih}F_p{}^a(\nabla^k\nabla_i F_a{}^t)R_{kjth}=\frac{S}{30}\Big(R^{jpih}R_{ijph}+\frac{7}{50}S^2\Big). \tag{5.13}$$

To obtain the integral formula (5.26), we define a tensor field $L=(L_{pkjih})$ by

$$L_{pkjih}=\nabla_p R_{kjih}+T_{pk}{}^t R_{tjih}+T_{pj}{}^t R_{ktih}+T_{pi}{}^t R_{kjth}+T_{ph}{}^t R_{kjit}\,, \tag{5.14}$$

where $T=(T_{ji}{}^h)$ is the tensor field defined by (4.12). We here compute the square of the length $\|L\|$ of the tensor field L, that is,

$$\begin{aligned} \|L\|^2 &= L_{pkjih}L^{pkjih} \\ &=(\nabla_p R_{kjih})\nabla^p R^{kjih} \\ &\quad +2(\nabla^p R^{kjih})(T_{pk}{}^t R_{tjih}+T_{pj}{}^t R_{ktih}+T_{pi}{}^t R_{kjth}+T_{ph}{}^t R_{kjit}) \\ &\quad +T^{pku}R_u{}^{jih}T_{pk}{}^v R_{vjih}+T^{pju}R^k{}_u{}^{ih}T_{pj}{}^v R_{kvih} \\ &\quad +T^{piu}R^{kj}{}_u{}^h T_{pi}{}^v R_{kjvh}+T^{phu}R^{kji}{}_u T_{ph}{}^v R_{kjvh} \\ &\quad +2(T^{pku}R_u{}^{jih}T_{pj}{}^v R_{kvih}+T^{pku}R_u{}^{jih}T_{pi}{}^v R_{kjvh} \\ &\quad +T^{pku}R_u{}^{jih}T_{ph}{}^v R_{kjiv}+T^{pju}R^k{}_u{}^{ih}T_{pi}{}^v R_{kjvh} \\ &\quad +T^{pju}R^k{}_u{}^{ih}T_{ph}{}^v R_{kjiv}+T^{piu}R^{kj}{}_u{}^h T_{ph}{}^v R_{kjiv}). \end{aligned}$$

Thus we have

$$\begin{aligned} \|L\|^2 &=(\nabla_p R_{kjih})\nabla^p R^{kjih}+8(\nabla^p R^{kjih})T_{pi}{}^t R_{kjth} \\ &\quad +4T_{pk}{}^u R_{ujih}T^{pkv}R_v{}^{jih} \\ &\quad +4(T^{pku}R_u{}^{jih}T_{pj}{}^v R_{kvih}+2T^{pku}R_u{}^{jih}T_{pi}{}^v R_{kjvh}). \end{aligned} \tag{5.15}$$

We now compute each of four terms appearing in the expression (5.15) above. In the first step, we compute the second term of (5.15). From the definition of T and the second Bianchi identity, we have

$$(\nabla^p R^{kjih})T_{pi}{}^t R_{kjth}=\frac{1}{2}(\nabla^p R^{kjih})(F_p{}^u\nabla_i F_u{}^t)R_{kjth} \tag{5.16}$$

$$=-\frac{1}{2}(\nabla^k R^{jpih})(F_p{}^u\nabla_i F_u{}^t)R_{kjth}-\frac{1}{2}(\nabla^j R^{pkih})(F_p{}^u\nabla_i F_u{}^t)R_{kjth}$$

$$=-\nabla^k(R^{jpih}(F_p{}^u\nabla_i F_u{}^t)R_{kjth})$$

$$+R^{jpih}(\nabla^k F_p{}^u)(\nabla_i F_u{}^t)R_{kjth}+R^{jpih}F_p{}^u(\nabla^k\nabla_i F_u{}^t)R_{kjth}\,.$$

Substituting (5.11) and (5.13) into the second term and the third term of the right hand side of (5.16) respectively, we have

$$(5.17)\qquad (\nabla^p R^{kjih})T_{pi}{}^t R_{kjth}$$

$$=-\nabla^k(R^{jpih}(F_p{}^u\nabla_i F_u{}^t)R_{kjth})-\frac{S}{60}\Big(R_{kjih}R^{kjih}-\frac{S^2}{15}\Big)\,.$$

In the second step, we compute the third term in the right hand side of (5.15). By making use of (4.3) and (5.4), we have

$$(5.18)\qquad T_{pk}{}^u R_{ujih}T^{pkv}R_v{}^{jih}$$

$$=\frac{1}{4}(F_p{}^a\nabla_k F_a{}^u)(F^{pb}\nabla^k F_b{}^v)R_{ujih}R_v{}^{jih}$$

$$=\frac{1}{4}(\nabla_k F_a{}^u)(\nabla^k F^{av})R_{ujih}R_v{}^{jih}$$

$$=\frac{S}{30}R_{kjih}R^{kjih}\,.$$

In the third step, we compute the forth term of the right hand side of (5.15). By making use of (5.2) and (5.5), we have

$$(5.19)\qquad T_{pk}{}^u R_{ujih}T^{pjv}R^k{}_v{}^{ih}$$

$$=\frac{1}{4}(F_p{}^a\nabla_k F_a{}^u)(F^{pb}\nabla^j F_b{}^v)R_{ujih}R^k{}_v{}^{ih}$$

$$=-\frac{S}{120}\Big(R_{kjih}R^{kjih}-\frac{2}{75}S^2+F_k{}^j F^{uv}R^k{}_v{}^{ih}R_{ujih}\Big)\,.$$

Substituting (5.9) into the last term in the parenthesis of the right hand side of (5.19), we have

$$(5.20)\qquad T_{pk}{}^u R_{ujih}T^{pjv}R^k{}_v{}^{ih}=-\frac{S}{60}\Big(R_{kjih}R^{kjih}-\frac{S^2}{25}\Big)\,.$$

Similarly, by making use of (5.2), (5.4) and (5.5), we have

$$(5.21)\qquad T_{pk}{}^u R_{ujih}T^{piv}R^{kj}{}_v{}^h$$

$$=\frac{1}{4}(F_p{}^a\nabla_k F_a{}^u)(F^{pb}\nabla^i F_b{}^v)R_{ujih}R^{kj}{}_v{}^h$$

$$=\frac{1}{4}(\nabla_a F_k{}^u)(\nabla^a F^{iv})R_{ujih}R^{kj}{}_v{}^h$$

$$=-\frac{S}{120}\left(\frac{4}{25}S^2-R_{ujih}R^{ijuh}+R_{ujih}F^{uv}F_k{}^iR^{kj}{}_v{}^h\right).$$

Substituting (5.6) into the last term in the parenthesis of the right hand side of (5.21), we have

$$T_{pk}{}^uR_{ujih}T^{piv}R^{kj}{}_v{}^h=-\frac{1}{5\cdot 15^2}S^3. \tag{5.22}$$

Finally, substituting (5.17), (5.18), (5.20) and (5.22) into the right hand side of (5.15), we have

$$\begin{aligned}\|L\|^2=&(\nabla_pR_{kjih})\nabla^pR^{kjih}-8\nabla^k(R^{jpih}(F_p{}^u\nabla_iF_u{}^t)R_{kjth})\\&-\frac{2S}{15}\left(R_{kjih}R^{kjih}-\frac{S^2}{15}\right)+\frac{2S}{15}R_{kjih}R^{kjih}\\&-\frac{S}{15}\left(R_{kjih}R^{kjih}-\frac{S^2}{25}\right)-\frac{8S^3}{5\cdot 15^2}\\=&(\nabla_pR_{kjih})\nabla^pR^{kjih}-8\nabla^k(R^{jpih}(F_p{}^u\nabla_iF_u{}^t)R_{kjth})\\&-\frac{S}{15}\left(R_{kjih}R^{kjih}-\frac{S^2}{15}\right).\end{aligned} \tag{5.23}$$

On the other hand, because of (4.4) and (5.4), (M, F, g) is an Einstein space with positive scalar curvature. Thus, by Myers' theorem, M is compact, and the formula (2.1) implies in particular

$$(\nabla_pR_{kjih})\nabla^pR^{kjih}=\frac{1}{2}\Delta(R_{kjih}R^{kjih})-2H^p{}_{kjpih}R^{kjih}. \tag{5.24}$$

Substituting (5.24) into (5.23), we have

$$\begin{aligned}\|L\|^2=&\frac{1}{2}\Delta(R_{kjih}R^{kjih})-8\nabla^k(R^{jpih}(F_p{}^u\nabla_iF_u{}^t)R_{kjth})\\&-2H^p{}_{kjpih}R^{kjih}-\frac{S}{15}\left(R_{kjih}R^{kjih}-\frac{S^2}{15}\right).\end{aligned} \tag{5.25}$$

From (5.25), by taking account of Green's theorem, we have finally the integral formula

$$\int_M\|L\|^2dM=-\int_M\left(2H^p{}_{kjpih}R^{kjih}+\frac{S}{15}\left(R_{kjih}R^{kjih}-\frac{S^2}{15}\right)\right)dM, \tag{5.26}$$

where dM denotes the volume element of (M, F, g).

Proof of Theorem 5.1. If (M, F, g) satisfies the inequality

$$2H^p{}_{kjpih}R^{kjih}+\frac{S}{15}\left(R_{kjih}R^{kjih}-\frac{S^2}{15}\right)\geqq 0,$$

then, we see from (5.26) that $L_{pkjih}=0$ on M, and hence, the tensor field T satisfies the condition (A) in Theorem 3.3. Consequently, from Lemma 4.3 and Theorem 3.3, we have Theorem 5.1. Q.E.D.

By direct computation, we have

$$\Big(R_{kjih}-\frac{S}{30}(g_{ji}g_{kh}-g_{ki}g_{jh})\Big)\Big(R^{kjih}-\frac{S}{30}(g^{ji}g^{kh}-g^{ki}g^{jh})\Big)=R_{kjih}R^{kjih}-\frac{S^2}{15}.$$

Thus, from (5.26) and the above identity, if a 6-dimensional complete and non-Kaehlerian K-space is simply connected and satisfies the condition (*), then it is isometric with a 6-dimensional sphere S^6, in which the tensor field T defined by (4.12) satisfies the conditions (A), (B) and (C) in Theorem 3.3. However, we have other examples of K-spaces in which the tensor field T defined by (4.12) satisfies the conditions (A), (B) and (C) in Theorem 3.3. The rest of this section will be devoted to show the fact. Let $M=G/K$ be a compact homogeneous space with $\chi(M)\neq 0$, where G is a compact simple Lie group acting effectively on M and K is the isotropy subgroup at some point $x\in M$. Then, $\chi(M)\neq 0$ implies $\operatorname{rank} G=\operatorname{rank} K$. We now assume that the Riemannian metric g of M is determined by a biinvariant metric $\langle,\rangle$ on G. Furthermore, we assume that G admits an automorphism θ of order 3 and K is the fixed point set of θ. Since $\operatorname{rank} G=\operatorname{rank} K$, θ is an inner automorphism (cf. [6], [24]). Then $M=G/K$ is a reductive homogeneous space. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be the corresponding orthogonal direct sum decomposition of $\mathfrak{g}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively. The subspace $\mathfrak{m}$ can be naturally identified with the tangent space at $x=K\in G/K$. Let Θ be the automorphism of $\mathfrak{g}$ determined by θ. And we put $\Theta|_{\mathfrak{m}}=-\frac{1}{2}I+\frac{\sqrt{3}}{2}F$, where I denotes the identity. Then F: $\mathfrak{m}\to\mathfrak{m}$ gives rise to a G-invariant almost complex structure (also denoted by F) on M. This almost complex structure F is called the canonical almost complex structure determined by θ. The almost complex structure F satisfies

$$\langle FX, FY\rangle=\langle X, Y\rangle$$

and

$$[FX, Y]_{\mathfrak{m}}=-F[X, Y]_{\mathfrak{m}}, \qquad [X, Y]_{\mathfrak{k}}=[FX, FY]_{\mathfrak{k}}, \tag{5.27}$$

for $X, Y\in\mathfrak{m}$ (cf. [5], [24]). On the other hand, since $(G/K, g)$ is a homogeneous Riemannian manifold, the Riemannian connection ∇ of $(G/K, g)$ is given at $x=K$ by the formula:

$$2\langle\nabla_X Y, Z\rangle=-\langle X, [Y, Z]_{\mathfrak{m}}\rangle-\langle Y, [X, Z]_{\mathfrak{m}}\rangle+\langle Z, [X, Y]_{\mathfrak{m}}\rangle, \tag{5.28}$$

for $X, Y, Z\in\mathfrak{m}$. Since $\langle,\rangle$ is biinvariant, $(G/K, g)$ is naturally reductive with respect to the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$, i.e., $\langle X, [Y, Z]_{\mathfrak{m}}\rangle+\langle [Y, X]_{\mathfrak{m}}, Z\rangle=0$ holds for $X, Y, Z\in\mathfrak{m}$, and furthermore, because of (5.28),

$$\nabla_X Y=\frac{1}{2}[X, Y]_{\mathfrak{m}}, \qquad \text{for } X, Y\in\mathfrak{m}, \tag{5.29}$$

holds at $x=K$. Taking account of (5.27) and (5.29), we have at $x=K$

$$(\nabla_X F)Y=\nabla_X(FY)-F(\nabla_X Y) \tag{5.30}$$

$$=\frac{1}{2}[X, FY]_{\mathfrak{m}}-\frac{1}{2}F[X, Y]_{\mathfrak{m}}$$
$$=-F[X, Y]_{\mathfrak{m}},$$

for $X, Y\in\mathfrak{m}$. Thus, $(G/K, F, g)$ is a K-space (cf. [5], [24]). It is well known that the curvature tensor R of $(G/K, g)$ is given at $x=K$ by the formula (cf. [8]):

$$R(X, Y)Z=-[[X, Y]_{\mathfrak{k}}, Z]-\frac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} +\frac{1}{4}[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}}-\frac{1}{4}[Y, [X, Z]_{\mathfrak{m}}]_{\mathfrak{m}}, \quad \text{for} \quad X, Y, Z\in\mathfrak{m}. \tag{5.31}$$

Now, let T be a G-invariant tensor field of type (1, 2) on $M=G/K$ determined by

$$T(X, Y)=\frac{1}{2}[X, Y]_{\mathfrak{m}}, \quad \text{for} \quad X, Y\in\mathfrak{m}. \tag{5.32}$$

Since $(G/K, g)$ is naturally reductive, T is skew-symmetric. Taking account of (5.30), we obtain at $x=K$

$$T(X, Y)=\frac{1}{2}[X, Y]_{\mathfrak{m}}$$
$$=-\frac{1}{2}F(F[X, Y]_{\mathfrak{m}})$$
$$=\frac{1}{2}F(\nabla_X F)Y, \quad \text{for} \quad X, Y\in\mathfrak{m}.$$

Thus, the tensor field T coincides with the tensor field given by (4.12) and hence, satisfies by Lemma 4.3 the conditions (B) and (C) in Theorem 3.3. Furthermore, taking account of (5.29), (5.31) and (5.32), we can easily show that T satisfies the condition (A) in Theorem 3.3. The following spaces are examples of 6-dimensional compact, simply connected and non-Kaehlerian K-spaces (cf. [5], [24]):

$$SU(3)/S(U(1)\times U(1)\times U(1)), \qquad SO(5)/U(2), \qquad G_2/SU(3)=S^6.$$

§6. Some 4-dimensional F-spaces and H-spaces.

In [12], Sawaki and the present author proved that any F-space with non-vanishing pointwise constant holomorphic sectional curvature is a Kaehlerian space. On the other hand, in [21], Tricerri and Vanhecke gave an example of non-Kaehlerian locally flat almost Hermitian manifolds of dimension 4. Obviously, any locally flat almost Hermitian manifold is an F-space. Therefore, it might be interesting to give some sufficient conditions for an almost Hermitian manifold to be Kaehlerian in terms of the curvature tensor and others. In connection with this, we shall prove the following

THEOREM 6.1. *Let (M, F, g) be a 4-dimensional H-space satisfying the condition (*). If the scalar curvature S of (M, F, g) is non-negative on M, then (M, F, g) is a Kaehlerian space.*

Taking account of Theorem 3.1, Theorem 6.1 follows immediately from the following

LEMMA 6.2. *Let (M, F, g) be a 4-dimensional almost Hermitian manifold satisfying the condition (*). If the scalar curvature S of (M, F, g) is non-negative on M, then $S \geqq S^*$ holds on M.*

Proof. We now put $B_{kjih}=g(R(e_k, e_j)e_i, e_h)$, $1 \leqq k, j, i, h \leqq 4$, for an orthonormal basis $(e_i)=(e_1, e_2, e_3, e_4)$ in $T_x(M)$ at each point $x \in M$. Taking account of the arguments developed in [15], at each point $x \in M$, we may choose an orthonormal basis (e_i) in $T_x(M)$ in such a way that one of the following conditions (I)-(i), (I)-(ii), (I)-(ii)′, (I)-(iii), (II), (III), (IV) and (V) holds:

(I)-(i) $\qquad B_{1212}=B_{1313}=B_{1414}=B_{2323}=B_{2424}=B_{3434}=-\lambda/3 \qquad (\lambda \neq 0)$,

otherwise, B_{kjih} being zero, and $4\lambda=S$;

(I)-(ii) $\qquad B_{1212}=B_{3434}=-2\lambda/3, \quad B_{1313}=B_{2424}=B_{1414}=B_{2323}=-\lambda/6,$

$\qquad B_{1234}=\lambda/3, \quad B_{1423}=-\lambda/6, \quad B_{1342}=-\lambda/6 \qquad (\lambda \neq 0)$,

otherwise, B_{kjih} being zero, and $4\lambda=S$;

(I)-(ii)′ $\qquad B_{1212}=B_{3434}=-2\lambda/3, \quad B_{1313}=B_{2424}=B_{1414}=B_{2323}=-\lambda/6,$

$\qquad B_{1234}=-\lambda/3, \quad B_{1423}=\lambda/6, \quad B_{1342}=\lambda/6 \qquad (\lambda \neq 0)$,

otherwise, B_{kjih} being zero, and $4\lambda=S$;

(I)-(iii) $\qquad B_{1212}=B_{3434}=-\lambda \qquad (\lambda \neq 0)$,

otherwise, B_{kjih} being zero, and $4\lambda=S$;

(II) $\qquad B_{1212}=-\lambda, \quad B_{3434}=-\mu \qquad (\lambda \neq \mu, \ \lambda, \mu \neq 0)$,

otherwise, B_{kjih} being zero, and $2(\lambda+\mu)=S$;

(III) $\qquad B_{1212}=B_{1313}=B_{2323}=-\lambda/2 \qquad (\lambda \neq 0)$,

otherwise, B_{kjih} being zero, and $3\lambda=S$;

(IV) $\qquad B_{1212}=-\lambda \qquad (\lambda \neq 0)$,

otherwise, B_{kjih} being zero, and $2\lambda=S$;

(V) $\qquad B_{kjih}=0, \qquad 1 \leqq k, j, i, h \leqq 4.$

We now put $f_{ji}=g(Fe_j, e_i)$, $1\leqq j, i\leqq 4$. Then, taking account of $g(Fe_j, Fe_i)=g(e_j, e_i)=\delta_{ji}$, we have

$$(f_{ji})=\begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & -f_{14} & f_{13} \\ -f_{13} & f_{14} & 0 & -f_{12} \\ -f_{14} & -f_{13} & f_{12} & 0 \end{pmatrix}, \tag{6.1}$$

or

$$(f_{ji})=\begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{14} & -f_{13} \\ -f_{13} & -f_{14} & 0 & f_{12} \\ -f_{14} & f_{13} & -f_{12} & 0 \end{pmatrix}, \tag{6.2}$$

where $(f_{12})^2+(f_{13})^2+(f_{14})^2=1$.

We here consider eight cases (A)$\sim$(H) as followings. By the definition of S^*, we have

$$-2S^*=\sum_{a,b,j,k} f_{ba}B_{bakj}f_{kj}\,. \tag{6.3}$$

(A) In the case (I)-(ii), we have from (6.3)

$$\begin{aligned} S^*=&-2(f_{12}B_{1212}f_{12}+f_{12}B_{1234}f_{34}+f_{13}B_{1313}f_{13}+f_{13}B_{1324}f_{24}\\ &+f_{14}B_{1414}f_{14}+f_{14}B_{1423}f_{23}+f_{23}B_{2314}f_{14}+f_{23}B_{2323}f_{23}\\ &+f_{24}B_{2413}f_{13}+f_{24}B_{2424}f_{24}+f_{34}B_{3412}f_{12}+f_{34}B_{3434}f_{34})\\ =&-(2\lambda/3)(2f_{12}f_{34}+f_{13}f_{24}-f_{14}f_{23}-4(f_{12})^2-(f_{13})^2-(f_{14})^2)\,. \end{aligned}$$

Thus, when (f_{ji}) has the form (6.1), we have

$$S^*=4\lambda(f_{12})^2\,,$$

and hence

$$S^*=S(f_{12})^2\,. \tag{6.4}$$

When (f_{ji}) has the form (6.2), we have

$$S^*=4\lambda/3\,,$$

and hence

$$S^*=S/3\,. \tag{6.5}$$

(B) In the case (I)-(ii)′, we have similarly

$$S^*=S/3\,, \tag{6.6}$$

when (f_{ji}) has the form (6.1), and

$$S^* = S(f_{12})^2, \tag{6.7}$$

when (f_{ji}) has the form (6.2).

(C) In the case (I)-(i), we have

$$\begin{aligned} S^* &= (2\lambda/3)((f_{12})^2+(f_{13})^2+(f_{14})^2+(f_{23})^2+(f_{24})^2+(f_{34})^2) \\ &= 4\lambda/3 = S/3. \end{aligned} \tag{6.8}$$

(D) In the case (I)-(iii), we have

$$\begin{aligned} S^* &= (2\lambda)((f_{12})^2+(f_{34})^2) = (4\lambda)(f_{12})^2 \\ &= S(f_{12})^2. \end{aligned} \tag{6.9}$$

(E) In the case (II), we have

$$\begin{aligned} S^* &= 2(\lambda(f_{12})^2+\mu(f_{34})^2) \\ &= S(f_{12})^2 \end{aligned} \tag{6.10}$$

(F) In the case (III), we have

$$S^* = \lambda = S/3. \tag{6.11}$$

(G) In the case (IV), we have

$$S^* = 2\lambda(f_{12})^2 = S(f_{12})^2. \tag{6.12}$$

(H) In the case (V), it is evident that

$$S^* = S = 0. \tag{6.13}$$

Taking account of (6.4)$\sim$(6.13), we can prove Lemma 6.2. Q. E. D.

Taking account of the arguments developed in the proof of Lemma 6.2, we can prove the following Propositions 6.3 and 6.4.

PROPOSITION 6.3. *Let (M, F_0, g_0) be a Kaehlerian space of constant holomorphic sectional curvature $c(>0)$ of complex dimension 2 and F be an almost complex structure on M such that (M, F, g_0) is an H-space. Then, $F=F_0$ or $F=-F_0$.*

Proof. From the hypothesis for (M, F_0, g_0), at each point $x \in M$, only the case (I)-(ii)$'$ with $\lambda=3c/2$ occurs with respect to any orthonormal basis $(e_i)=(e_1, e_2=F_0e_1, e_3, e_4=F_0e_3)$ in $T_x(M)$. Since the scalar curvature of (M, F, g_0) is positive and constant on M, from Theorem 6.1, (F, g_0) is a Kaehlerian structure on M. Thus, from Theorem 3.1, and (6.6), (6.7), we have finally

$$Fe_1=e_2=F_0e_1\,,\qquad Fe_3=e_4=F_0e_3\,,$$

or

$$Fe_1=-e_2=-F_0e_1\,,\quad Fe_3=-e_4=-F_0e_3\,.$$

Q. E. D.

PROPOSITION 6.4. *Let (M, F_0, g_0) be a Kaehlerian space of constant holomorphic sectional curvature c $(\neq 0)$ and F be an almost complex structure on M such that (M, F, g_0) is an F-space. Then, $F=F_0$ or $F=-F_0$.*

Proof. In general, in an F-space (M, F, g), from (3.3) and (3.4), it follows that $S=S^*$. Thus, taking account of (6.6) and (6.7), we have finally

$$Fe_1=e_2=F_0e_1\,,\qquad Fe_3=e_4=F_0e_3\,,$$

or

$$Fe_1=-e_2=-F_0e_1\,,\quad Fe_3=-e_4=-F_0e_3\,.$$

Q. E. D.

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