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MEASURABLE OR CONDENSING MULTIVALUED MAPPINGS AND RANDOM FIXED POINT THEOREMS

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0. Introduction

Various results on random fixed point theorems were given by many authors (cf. Bharucha-Reid [1, 2], Itoh [7, 8], Engl [3, 4] and their references). In [8] almost all known fixed point theorems (e. g. for nonexpansive or condensing mappings) were extended to random cases (except for contraction mappings that is due to Špaček [16] and Hanš [5]) on general measurable spaces. Similar results were obtained by Bharucha-Reid [2] and Engl [3, 4] on measure spaces.

For multivalued mappings, a random fixed point theorem for contraction mappings was proved in [7]. Then in [8], theorems for multivalued condensing or nonexpansive mappings on measurable spaces were treated, where in the former case lower semicontinuity as well as upper semicontinuity are assumed. On measure spaces, Engl [3, 4] gave a theorem which makes possible to derive random fixed point theorems from fixed point theorems for multivalued continuous (in Hausdorff metric) mappings. Moreover, he obtained a complete result of Bohnenblust and Karlin type for upper semicontinuous compact multivalued mappings.

Other results on random equations were treated by Kannan and Salehi [11] and Itoh [9, 10].

In this paper, by adopting the method of Engl [3, 4] we prove random fixed point theorems for upper semicontinuous condensing multivalued mappings. In sections 1 and 2, some results on upper semicontinuity and measurability of multivalued mappings are presented. Then in section 3 random fixed point theorems are given.

1. Upper Semicontinuous Multivalued Mappings

Let X be a metric space. For any $B\subset X$ and $p>0$, let $\mathrm{cl}(B)$ be the closure of B and $B_p=\{x\in X: d(x, B)<p\}$, where $d(x, B)=\inf\{d(x, y): y\in B\}$. Let 2^X be the family of all subsets of X, $CD(X)$ all nonempty closed subsets, and $K(X)$ all nonempty compact subsets of X respectively. If X is a subset of a Banach

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space, denote by $\mathrm{clco}(X)$ the closed convex hull of X and by $CK(X)$ the family of all nonempty compact convex subsets of X. Let Y be another metric space. A mapping $F: X \to CD(Y)$ is said to be *upper semicontinuous* (u.s.c.) if for any closed subset C of Y, $F^{-1}(C)=\{x \in X: F(x) \cap C \neq \emptyset\}$ is a closed subset of X. It is obvious that F is u.s.c. if and only if given $x \in X$, for each open subset V of Y with $V \supset F(x)$, there exists a neighborhood U of x such that $F(y) \subset V$ whenever $y \in U$.

LEMMA 1.1. *Let X be a separable metric space with $\{x_k\}$ a countable dense subset of X and Y be a Banach space. Let $F: X \to CK(Y)$ be an u.s.c. mapping, then the mapping $G: X \to 2^Y$ defined by*

$$G(x)=\bigcap_{n=1}^{\infty} \mathrm{clco}(\cup\{F(x_k): d(x_k, x)<1/n\}) \qquad (x \in X)$$

satisfies the conditions:
(i) *For any $x \in X$, $F(x) \supset G(x) \neq \emptyset$.*
(ii) *G is u.s.c.*

Proof. For each n, define $G_n: X \to CD(Y)$ by

$$G_n(x)=\mathrm{clco}(\cup\{F(x_k): d(x_k, x)<1/n\}) \qquad (x \in X),$$

then

$$G(x)=\bigcap_{n=1}^{\infty} G_n(x).$$

We first show that $G(x)$ is nonempty for every $x \in X$. For any n, take k_n such that $d(x_{k_n}, x)<1/n$. Then

$$G_n(x) \supset \mathrm{clco}(\bigcup_{i \geqq n} F(x_{k_i})).$$

Since $\{x_{k_n}\}_{n=1}^{\infty} \cup \{x\}$ is compact and F is u.s.c., $\bigcup_{n=1}^{\infty} F(x_{k_n}) \cup F(x)$ is compact, hence $\mathrm{clco}(\bigcup_{i \geqq n} F(x_{k_i}))$ is compact. Thus

$$G(x)=\bigcap_{n=1}^{\infty} G_n(x) \supset \bigcap_{n=1}^{\infty} \mathrm{clco}(\bigcup_{i \geqq n} F(x_{k_i})) \neq \emptyset.$$

The relation $F(x) \supset G(x)$ is an easy consequence of F being u.s.c. Indeed, for any $p>0$, take n for sufficiently large, then $d(x_k, x)<1/n$ implies $F(x_k) \subset (F(x))_p$. Since $(F(x))_p$ is convex, $G(x) \subset G_n(x) \subset \mathrm{cl}((F(x))_p)$, which yields $G(x) \subset F(x)$. Now we prove that G is u.s.c. Let C be any closed subset of Y and $\{z_i\}$ be a sequence of $G^{-1}(C)$ converging to some $z \in X$. For each n, choose z_i such that $d(z_i, z)<1/2n$. If $d(x_k, z_i)<1/2n$, then $d(x_k, z)<1/n$, hence $G_{2n}(z_i) \subset G_n(z)$ and $G_n(z) \cap C \supset G_{2n}(z_i) \cap C \neq \emptyset$. Since F is u.s.c., there exists $j_n>n$ for which $d(x_k, z) <1/j_n$ implies $F(x_k) \subset (F(z))_{1/n}$. Thus $\emptyset \neq G_{j_n}(z) \cap C \subset \mathrm{cl}((F(z))_{1/n})$. There exists $y_n \in G_{j_n}(z) \cap C$ such that $d(y_n, F(z)) \leqq 1/n$. Since $F(z)$ is compact, some subsequence $\{y_m\}$ of $\{y_n\}$ converges to an element y of C. If $j_m>n$, then $y_m \in$

$G_{j_m}(z)\cap C\subset G_n(z)\cap C$. It follows that $y\in G_n(z)\cap C$ for all n. This implies

$$y\in \bigcap_{n=1}^{\infty} G_n(z)\cap C=G(z)\cap C,$$

and $z\in G^{-1}(C)$. Hence $G^{-1}(C)$ is closed and G is u.s.c.

Remark 1.2. Almost the same proof as above also establishes the following: Let X be a separable metric space with $\{x_k\}$ a countable dense subset of X, Y be a metric space, and $F: X\to K(Y)$ be u.s.c. Then the mapping $G: X\to 2^Y$ by

$$G(x)=\bigcap_{n=1}^{\infty} \mathrm{cl}(\cup\{F(x_k): d(x_k, x)<1/n\}) \qquad (x\in X)$$

has the properties:

(i) For any $x\in X$, $F(x)\supset G(x)\neq\emptyset$.

(ii) G is upper semicontinuous.

§ 2. Measurable Multivalued Mappings

In the sequel, let $(T, \mathcal{A})$ be a measurable space. A mapping $F: T\to 2^X$ is said to be $(\mathcal{A})$*-measurable* if for each closed subset C of X, $F^{-1}(C)=\{t\in T: F(t)\cap C\neq\emptyset\}\in\mathcal{A}$. F is said to be $(\mathcal{A}$-)*weakly measurable* if for each open subset B of X, $F^{-1}(B)\in\mathcal{A}$. It is obvious that if F is measurable, then F is weakly measurable. If $F(t)\in K(X)$ for all $t\in T$, then the converse is valid by Himmelberg [6, Theorem 3.1]. See also Wagner [17]. Denote by $\mathcal{B}$ the Borel field of X and by $\mathcal{A}\times\mathcal{B}$ the product σ-algebra of $\mathcal{A}$ and $\mathcal{B}$ on $T\times X$.

PROPOSITION 2.1. *Let X be a separable metric space with $\{x_k\}$ a countable dense subset of X and Y be a separable Banach space. Let $F: T\times X\to CK(Y)$ be a mapping having the properties:*

(a) *For each $t\in T$, $F(t, \cdot)$ is u.s.c.*

(b) *For each $x\in X$, $F(\cdot, x)$ is weakly measurable.*

Then the mapping $G: T\times X\to 2^Y$ defined by

$$G(t, x)=\bigcap_{n=1}^{\infty} \mathrm{clco}(\cup\{F(t, x_k): d(x_k, x)<1/n\})$$

$(t\in T, x\in X)$ satisfies the following conditions:

(i) *For each $t\in T$ and $x\in X$, $F(t, x)\supset G(t, x)\neq\emptyset$.*

(ii) *For each $t\in T$, $G(t, \cdot)$ is u.s.c.*

(iii) *G is $\mathcal{A}\times\mathcal{B}$-measurable.*

Proof. (i) and (ii) is clear from Lemma 1.1.

(iii) For each n, define $H_n: T\times X\to 2^Y$ by $H_n(t, x)=\cup\{F(t, x_k): d(x_k, x)<1/n\}$ $(t\in T, x\in X)$, then H_n is $\mathcal{A}\times\mathcal{B}$-weakly measurable. Indeed, for any open subset B of Y,

$$\begin{aligned} H_n^{-1}(B) &= \{(t, x) \in T \times X : H_n(t, x) \cap B \neq \emptyset\} \\ &= \bigcup_{k=1}^{\infty} \{t \in T : F(t, x_k) \cap B \neq \emptyset\} \times \{x \in X : d(x, x_k) < 1/n\} \in \mathcal{A} \times \mathcal{B}\,. \end{aligned}$$

Then the mapping $G_n : T \times X \to CD(Y)$ defined by $G_n(t, x) = \mathrm{clco}(H_n(t, x))$ is $\mathcal{A} \times \mathcal{B}$-weakly measurable by Himmelberg [6, Theorem 9.1]. If we show that

$$G^{-1}(C) = \bigcap_{n=1}^{\infty} G_n^{-1}(C_{1/n})$$

for every closed subset C of Y, then we can conclude that G is $\mathcal{A} \times \mathcal{B}$-measurable. It is obvious that

$$G^{-1}(C) \subset \bigcap_{n=1}^{\infty} G_n^{-1}(C_{1/n})\,.$$

Conversely, if

$$(t, x) \in \bigcap_{n=1}^{\infty} G_n^{-1}(C_{1/n})\,,$$

then $G_n(t, x) \cap C_{1/n} \neq \emptyset$ for all n. Since $F(t, \cdot)$ is u.s.c., by the same way as in the proof of Lemma 1.1 we have

$$\emptyset \neq \bigcap_{n=1}^{\infty} G_n(t, x) \cap \mathrm{cl}(C_{1/n}) = G(t, x) \cap C\,.$$

Hence

$$\bigcap_{n=1}^{\infty} G_n^{-1}(C_{1/n}) \subset G^{-1}(C)\,.$$

Remark 2.2. Let F be as in Proposition 2.1, then F itself is not necessarily $\mathcal{A} \times \mathcal{B}$-measurable (cf. Engl [3, 4]).

Remark 2.3. By a slight modification of the above proof we can prove the following: Let X be a separable metric space with $\{x_k\}$ a countable dense subset of X and Y be a metric space. Let $F: T \times X \to K(Y)$ be a mapping with the properties:

(a) For each $t \in T$, $F(t, \cdot)$ is u.s.c.

(b) For each $x \in X$, $F(\cdot, x)$ is weakly measurable. Define $G: T \times X \to K(Y)$ by

$$G(t, x) = \bigcap_{n=1}^{\infty} \mathrm{cl}(\cup \{F(t, x_k) : d(x_k, x) < 1/n\})$$

($t \in T$, $x \in X$), then G fulfills the conditions:

(i) For each $t \in T$, $x \in X$, $F(t, x) \supset G(t, x) \neq \emptyset$.

(ii) For each $t \in T$, $G(t, \cdot)$ is u.s.c.

(iii) G is $\mathcal{A} \times \mathcal{B}$-measurable.

If X is also complete, the proof of the following lemma is essentially contained in [7, Proposition 4]. If X is a separable Banach space, the same is obtained in Engl [4] by a different method.

LEMMA 2.4. *Let X be a separable metric space, $F: T \to CD(X)$ be a weakly measurable mapping, and $u: T \to X$ be a measurable mapping. Then $d(u(\cdot), F(\cdot))$ is a measurable function on T.*

Proof. Define $f: T \times X \to R$ (the real numbers) by $f(t, x) = d(x, F(t))$ ($t \in T$, $x \in X$), then f is measurable in t by Himmelberg [6, Theorem 3.3] and continuous in x. Hence the function $f(\cdot, u(\cdot)) = d(u(\cdot), F(\cdot))$ on T is measurable (cf. Himmelberg [6, Theorem 6.5]).

§3. Random Fixed Point Theorems

Let Σ and Σ^* be the respective sets of infinite and finite sequences of positive integers. For $\sigma \in \Sigma$, denote $(\sigma_1, \cdots, \sigma_n)$ by $\sigma|n$ and let $A: \Sigma^* \to \mathcal{A}$. Then

$$\bigcup_{\sigma \in \Sigma} \bigcap_{n=1}^{\infty} A_{\sigma|n}$$

is said to be obtained from $\mathcal{A}$ by the Souslin operation. $\mathcal{A}$ is called a *Souslin family* if every set obtained from $\mathcal{A}$ in this way is also in $\mathcal{A}$. If there exists a complete σ-finite measure on $(T, \mathcal{A})$, then $\mathcal{A}$ is a Souslin family (cf. Wagner [17, p. 864] and the references cited there).

For any bounded subset B of X, let $\gamma(B) = \inf\{c > 0 : B$ can be covered by a finite number of subsets of diameters less than or equal to $c\}$. A mapping $F: X \to CD(X)$ is said to be *condensing* if for any bounded subset B of X with $\gamma(B) > 0$, $\gamma(F(B)) < \gamma(B)$, where $F(B) = \cup\{F(x) : x \in B\}$.

Now we prove the following random fixed point theorem by using the results in section 2.

THEOREM 3.1. *Let $\mathcal{A}$ be a Souslin family and X be a nonempty closed convex subset of a separable Banach space Y. Let $F: T \times X \to CK(Y)$ be a mapping satisfying the conditions:*

(i) *For any $t \in T$, $F(t, X)$ is bounded and $F(t, bdX) \subset X$, where bdX is the boundary of X.*

(ii) *For any $t \in T$, $F(t, \cdot)$ is u. s. c. and condensing.*

(iii) *For any $x \in X$, $F(\cdot, x)$ is weakly measurable.*

Then there exists a measurable mapping $u: T \to X$ such that $u(t) \in F(t, u(t))$ for all $t \in T$.

Proof. Choose countable dense elements $\{x_k\}$ of X and define $G: T \times X \to CK(Y)$ as in Proposition 2.1, then G is $\mathcal{A} \times \mathcal{B}$-measurable. The mapping $v: T \times X \to X$ by $v(t, x) = x$ ($t \in T$, $x \in X$) is $\mathcal{A} \times \mathcal{B}$-measurable. By Lemma 1.3, $f(t, x) = d(v(t, x), G(t, x))$ ($t \in T$, $x \in X$) is a ($\mathcal{A} \times \mathcal{B}$-)measurable function on $T \times X$. Define $H: T \to 2^X$ by $H(t) = \{x \in X : x \in G(t, x)\}$ ($t \in T$), then for any $t \in T$, $H(t)$ is nonempty and compact by Petryshyn and Fitzpatrick [14] and the method of the proof of Smart [15, Theorem 9.2.4]. Moreover we have

$$\begin{aligned}\operatorname{Gr} H &= \{(t, x) \in T \times X : x \in H(t)\} \\ &= \{(t, x) \in T \times X : f(t, x) = 0\} \\ &\in \mathcal{A} \times \mathcal{B}\,.\end{aligned}$$

By Leese [12] (cf. Wagner [17, Theorem 4.2]) there exists a measurable mapping $u: T \to X$ such that for each $t \in T$, $u(t) \in H(t)$, hence $u(t) \in G(t, u(t)) \subset F(t, u(t))$.

COROLLARY 3.2. *Let $(T, \mathcal{A}, m)$ be a (complete) σ-finite measure space and Y, X, and $F: T \times X \to CK(Y)$ be as in Theorem 3.1. Then there exists a measurable mapping $u: T \to X$ such that $u(t) \in F(t, u(t))$ for m-a.e. (all) $t \in T$.*

Proof. If m is complete, then $\mathcal{A}$ is a Souslin family and the conclusion follows from Theorem 3.1.

If m is not complete, the usual method of considering the completion of $(T, \mathcal{A}, m)$ easily yields the conclusion. We include the proof for completeness. Let $(T, \mathcal{A}^*, m^*)$ be the completion of $(T, \mathcal{A}, m)$. Then by Theorem 3.1 there exists a $\mathcal{A}^*$-measurable mapping $v: T \to X$ for which $v(t) \in F(t, v(t))$ for all $t \in T$. Since X is separable, we may take a countable open base $\{B_n\}$ of X. For each n, $v^{-1}(B_n) = A_n \cup N_n$, where $A_n \in \mathcal{A}$ and N_n is contained in some $D_n \in \mathcal{A}$ with $m(D_n) = 0$. Then

$$D = \bigcup_{n=1}^{\infty} D_n \in \mathcal{A}$$

and $m(D) = 0$. Let $u: T \to X$ be a mapping defined by

$$u(t) = \begin{cases} v(t) & \text{if} \quad t \in T - D, \quad \text{or} \\ y & \text{if} \quad t \in D, \end{cases}$$

where y is any fixed element of X. It is easy to observe that u is $\mathcal{A}$-measurable and $u(t) \in F(t, u(t))$ for every $t \in T - D$.

Remark 3.3. We can also state and prove similar results as above in the case that the domain of $F(t, \cdot)$ is dependent on $t \in T$ as in Engl [3, 4]. The proofs are almost the same as those given above. We omit the details.

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