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ON A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION AND THE COSINE FUNCTION BY FACTORIZATION II

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1. In our previous paper [1] we proved the following

THEOREM A. *Let $F(z)$ be an entire function for which there exist polynomials $P_m(z)$ of degree m and entire functions $f_m(z)$ so that $F(z)=P_m(f_m(z))$ for $m=2^j$, $j=1, 2, \cdots$ and for $m=3$. Then $F(z)$ is either $Ae^{H(z)}+B$ or $A\cos\sqrt{H(z)}+B$ with constants A, B and an entire function $H(z)$.*

In this paper we shall give an application of this theorem.

THEOREM 1. *Let $F(z)$ be an entire function for which*

$$F(z)=P_2\left(F\left(\frac{z}{n}\right)\right)=P_3\left(F\left(\frac{z}{m}\right)\right)$$

with polynomials P_k of degree k and positive integers n, m. Then $F(z)$ is either $Ae^{az}+B$ or $A\cos az+B$ or $A\cos\sqrt{az}+B$ with constants A, B and a.

This theorem gives again a characterization of exp and cos. It seems to the present author that there is another proof depending on the power series expansion. If we omit the condition $F(z)=P_3\left(F\left(\frac{z}{m}\right)\right)$ in our theorem, we cannot say that theorem 1 holds.

In this theorem we may put m, n as non-zero constants and we have the same conclusion.

2. *Proof of Theorem* 1. Evidently

$$F(z)=P_{2^j}\left(F\left(\frac{z}{n^j}\right)\right)=P_3\left(F\left(\frac{z}{m}\right)\right)$$

for $j=1, 2, 3, \cdots$. Hence Theorem A implies that $F(z)$ is either $Ae^{H(z)}+B$ or $A\cos\sqrt{H(z)}+B$. By $F(z)=P_2(F(z/n))$ we have further $m(r, F)\sim 2m(r/n, F)$ as $r\to\infty$. For $r\geqq r_0$

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$$(1-\varepsilon)2m\left(\frac{r}{n}, F\right) \leqq m(r, F) \leqq (1+\varepsilon)2m\left(\frac{r}{n}, F\right).$$

For $r_0 \leqq r_2 \leqq nr_0 = r_1$ and $r = n^p r_2$

$$\varlimsup_{p\to\infty} \frac{\log m(n^p r_2, F)}{\log n^p r_2} \leqq \frac{\log 2 + \log(1+\varepsilon)}{\log n}$$

and

$$\varliminf_{p\to\infty} \frac{\log m(n^p r_2, F)}{\log n^p r_2} \geqq \frac{\log 2 + \log(1-\varepsilon)}{\log n}.$$

Hence the order of F is equal to $\log 2/\log n$. Evidently $n \geqq 2$. If $F(z) = Ae^{H(z)} + B$, its order $\geqq 1$ and so n should be equal to 2. Therefore $F(z) = A_1 e^{az} + B$. If $F(z) = A\cos\sqrt{H(z)} + B$, its order should be a half integer and hence $H(z)$ is either $az + a_0$ if $n = 4$ or $az^2 + a_1 z + a_0$ if $n = 2$. Let $P_2(w)$ be $C_2 w^2 + C_1 w + C_0$. Then by $F(z) = P_2(F(z/2))$ in the latter case

$$\begin{aligned} &A\cos\sqrt{az^2 + a_1 z + a_0} + B \\ &= \frac{C_2 A^2}{2}\cos\sqrt{az^2 + 2a_1 z + 4a_0} + (2C_2 AB + C_1 A)\cos\sqrt{\frac{a}{4}z^2 + \frac{a_1}{2}z + a_0} \\ &\quad + C_2 B^2 + C_1 B + C_0 + \frac{C_2}{2}A^2. \end{aligned}$$

Without appealing to the impossibility of Borel's identity we can proceed in the following manner. The left hand side term is symmetric with respect to $-a_1/2a$ and the right hand side term is symmetric with respect to $-a_1/a$. Hence $a_1 = 0$. Next consider the asymptotic behavior of both sides for $\sqrt{az^2} = iy$ with real y for $y \to \infty$. Then $2A = C_2 A^2$, $2C_2 AB + C_1 A = 0$. Hence

$$\begin{aligned} &A\cos\sqrt{az^2 + a_0} + B \\ &\qquad = A\cos\sqrt{az^2 + 4a_0} + C_2 B^2 + C_1 B + C_0 + \frac{C_2}{2}A^2. \end{aligned}$$

Let us put $az^2 + a_0 = \left(2n\pi + \frac{\pi}{2}\right)^2$. Then

$$B = A\sin\frac{3a_0}{2n\pi + \frac{\pi}{2}}\left(\frac{1}{2} + O\left(\frac{1}{n^2}\right)\right) + C_2 B^2 + C_1 B + C_0 + \frac{C_2}{2}A^2.$$

Hence $a_0 = 0$ and $B = C_2 B^2 + C_1 B + C_0 + C_2 A^2/2$. Therefore

$$F(z) = A\cos\sqrt{a}z + B.$$

In the former case

$$A\cos\sqrt{az+a_0}+B$$
$$=C_2\frac{A^2}{2}\cos\sqrt{az+4a_0}+(2ABC_2+C_1A)\cos\sqrt{\frac{a}{4}z+a_0}$$
$$+\frac{C_2}{2}A^2+C_2B^2+C_1B+C_0\,.$$

Quite similarly we have

$$a_0=0\,,\qquad A=\frac{C_2}{2}A^2\,,\qquad 2ABC_2+C_1A=0\,,$$
$$B=\frac{C_2}{2}A^2+C_2B^2+C_1B+C_0\,.$$

Hence

$$F(z)=A\cos\sqrt{az}+B\,.$$

This completes the proof of our theorem.

BIBLIOGRAPHY

[1] OZAWA, M., On a characterization of the exponential function and the cosine function by factorization. Kodai Math. J. 1 (1978), 45-74.

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