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ON FACTORIZATION OF CERTAIN ENTIRE FUNCTIONS

Dedicated to Professor Eiichi Sakai on his 60th birthday

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1. We say that a meromorphic function $F(z)$ has a non-trivial factorization with left factor $f(z)$ and right factor $g(z)$, if

$$F(z)=f(g(z)), \tag{1}$$

where $f(z)$ is a non-linear meromorphic function and $g(z)$ is a non-linear entire function ($g(z)$ may be meromorphic when $f(z)$ is a rational function). $F(z)$ is said to be *prime*, if it has no non-trivial factorization, i. e. if (1) implies that either $f(z)$ or $g(z)$ is linear. $F(z)$ is said to be *pseudo-prime*, if (1) implies either $f(z)$ or $g(z)$ is not transcendental. Further, we say that an entire function $F(z)$ is *E-prime* (*E-pseudo-prime*), if it is prime (pseudo-prime) for entire functions $f(z)$ and $g(z)$ in (1).

Recently Urabe-Yang [6] proved the E-primeness of $F^{(n)}(z)$ $(n=0, 1, 2, \cdots)$, where

$$F(z)=\int_0^z (e^z-1)e^{z^2}dz$$

and the author [4] proved the E-primeness of $F^{(n)}(z)$,

$$F(z)=\int_0^z (e^z-1)e^{z^k}dz\,, \quad (k\geqq 3:\text{ an integer}).$$

Further in their papers they made a study of the factorization of $F^{(n)}(z)$, where $F(z)$ is an entire function of the form

$$F(z)=\int_0^z \{H_1(z)e^z+H_2(z)\}\,e^{P(z)}dz\,,$$

where $H_j(z)$ $(\not\equiv 0)$, $j=1, 2$, are entire functions of order less than one and $P(z)$ is a polynomial of degree not lower than two.

The purpose of this paper is to improve and to complement their results. And in § 5 we shall prove the primeness of entire functions $F^{(n)}(z)$, $n=0, 1, 2, \cdots$, where

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$$F(z)=\int_0^z \{P_1(z)e^z+P_2(z)\}e^{z^k}dz \quad (k\geqq 3:\text{ an integer})$$

with two polynomials $P_1(z)$ and $P_2(z)$ which are not identically zero.

2. For our purpose we need several lemmas

LEMMA 1 (cf. [3; pp. 117-118]). *Let $a_j(z)$ be entire functions of order at most ρ, $g_j(z)$ also entire functions and $g_j(z)-g_k(z)$ $(j\neq k)$ transcendental entire functions or polynomials of degree higher than ρ. Then the identity relation*

$$\sum_{j=1}^{n} a_j(z)e^{g_j(z)}=a_0(z)$$

holds only when $a_0(z)\equiv a_1(z)\equiv \cdots \equiv a_n(z)\equiv 0$.

From Proposition 2 in Ozawa [5] and the argument in [5, p. 331] we deduce the following Lemma 2 and Lemma 3.

LEMMA 2. *Let $F(z)$ be a transcendental entire function which admits a factorization $f(g(z))$ with a meromorphic (not entire) function $f(z)$ and an entire function $g(z)$. Then we have*

$$f(z)=f_1(z)/(z-\alpha)^n, \qquad f_1(\alpha)\neq 0, \qquad g(z)=\alpha+e^{L(z)},$$

where $f_1(z)$ is an entire function, $L(z)$ a non-constant entire function, α a complex number and n a positive integer.

LEMMA 3. *Let $F(z)$ be a transcendental entire function which admits a factorization $f(g(z))$ with a non-linear rational function (not a polynomial) $f(z)$ and a transcendental meromorphic function $g(z)$. Then we have*

$$F(z)=Q(\beta+e^{M(z)})e^{-mM(z)}, \qquad Q(\beta)\neq 0,$$

where $Q(z)$ is a polynomial, $M(z)$ a non-constant entire function, β a complex number and m a positive integer.

LEMMA 4 (Clunie [1]). *Let $f(z)$ and $g(z)$ be two transcendental entire functions. Then*

$$\lim_{r\to\infty}\frac{T(r, f(g))}{T(r, g)}=\infty .$$

LEMMA 5 (Goldstein [2]). *Let $F(z)$ be an entire function of finite order such that $\delta(a, F)=1$ for some $a\neq\infty$. Then $F(z)$ is E-pseudo-prime.*

From the reasoning in the proof of Theorem 3 in [4] we deduce that

LEMMA 6. *Let $\{a_\nu\}$ be non-vanishing zeros of e^z-az^μ, where a is a non-zero constant and μ is an integer. Then there is an integer N such that $\{a_\nu\}$ satisfy*

$$\begin{cases} \operatorname{Re} a_\nu = r_\nu \cos\theta_\nu = \log|a| + \mu \log r_\nu , \\ \operatorname{Im} a_\nu = r_\nu \sin\theta_\nu = \alpha + \mu\theta_\nu + (-1)^\nu 2\pi[\nu/2] \end{cases}$$

for all $\nu \geqq N$, *where* $a_\nu = r_\nu e^{i\theta_\nu}$, $a = |a| e^{i\alpha} (0 \leqq \theta_\nu < 2\pi,\ 0 \leqq \alpha < 2\pi)$ *and* $[x]$ *is the greatest integer not exceeding* x.

3. Now we shall prove

THEOREM 1. *Let* $P(z)$ *be a polynomial of degree* k $(k \geqq 2)$ *and* $H_1(z)$ *and* $H_2(z)$ *two entire functions which are of order less than one and are not identically zero. The entire function* $F(z) = \{H_1(z)e^z + H_2(z)\} e^{P(z)}$ *has a non-trivial factorization if and only if there are a complex number* a *and an integer* m *such that the following identities*

$$P(z+a) - P(-z+a) = -z + m\pi i \tag{3.1}$$

and

$$H_1(z+a)e^a - (-1)^m H_2(-z+a) = 0 \tag{3.2}$$

hold. Then the non-trivial factorization of $F(z)$ *is* $F(z) = f(A(z-a)^2 + B)$, *where* $f(z)$ *is an entire function and* A $(\neq 0)$ *and* B *are constants.*

Proof. Assume that $F(z)$ has a non-trivial factorization $f(g(z))$.

(I) Suppose that $f(z)$ and $g(z)$ are entire. Then from the reasoning in proof of Theorem 2 in [4] we deduce that $g(z)$ is a polynomial of degree two. Put $g(z) = A(z-a)^2 + B$ $(A \neq 0)$ and $w = z - a$. Then $F(w+a) = f(Aw^2 + B)$ and consequently $F(w+a) = F(-w+a)$. Hence we have

$$\begin{aligned} &\{H_1(w+a)e^{w+a} + H_2(w+a)\} e^{P(w+a) - P(-w+a)} \\ &\qquad = \{H_1(-w+a)e^{-w+a} + H_2(-w+a)\}. \end{aligned} \tag{3.3}$$

If $\deg \{P(w+a) - P(-w+a)\} \geqq 2$, then Lemma 1 implies that $H_1(w+a) \equiv H_2(w+a) \equiv 0$, which contradicts our assumptions $H_1(z) \not\equiv 0$ and $H_2(z) \not\equiv 0$. Hence we put $P(w+a) - P(-w+a) = \alpha w + \beta$. If $\alpha \neq -1$, then Lemma 1 yields that $H_1 \equiv 0$ or $H_2 \equiv 0$, which is a contradiction. Hence we have $\alpha = -1$. Then (3.3) reduces to

$$\{H_2(w+a)e^\beta - H_1(-w+a)e^a\} e^{-w} + H_1(w+a)e^{a+\beta} - H_2(-w+a) = 0 .$$

Again using Lemma 1 we have

$$H_2(w+a)e^\beta - H_1(-w+a)e^a = 0 \quad \text{and} \quad H_1(w+a)e^{a+\beta} - H_2(-w+a) = 0 .$$

Therefore, using $H_2 \not\equiv 0$, we obtain (3.1) and (3.2).

(II) Suppose that $f(z)$ is meromorphic (not entire) and $g(z)$ is entire. Then Lemma 2 implies that

$$F(z)=e^{-nL(z)}f_1(\alpha+e^{L(z)}), \qquad f_1(\alpha)\neq 0,$$

where $f_1(z)$ is an entire function, $L(z)$ a non-constant entire function, α a complex number and n a positive integer. Since $F(z)=\{H_1(z)e^z+H_2(z)\}e^{P(z)}$ has an infinite number of zeros the exponent of convergence of which is one, $f_1(z)$ has a zero and $L(z)$ is linear, that is, $L(z)=\gamma z+\delta\,(\gamma\neq 0)$. Hence we have

$$\{H_1(z)e^z+H_2(z)\}e^{P(z)+n\gamma z+n\delta}=f_1(\alpha+e^{\gamma z+\delta}).$$

Since $\deg\{P(z)+n\gamma z+n\delta\}\geqq 2$ and $\alpha+e^{\gamma z+\delta}$ is transcendental, the argument in (I) implies that $f_1(z)$ is linear. Hence it follows from Lemma 1 that $H_1(z)e^z+H_2(z)\equiv 0$, that is, $H_1\equiv H_2\equiv 0$, which is a contradiction.

(III) Suppose that $f(z)$ is non-linear rational and $g(z)$ is meromorphic. Then Lemma 3 implies that

$$F(z)=e^{-mM(z)}Q(\beta+e^{M(z)}), \qquad Q(\beta)\neq 0,$$

where $Q(z)$ is a polynomial, $M(z)$ a non-constant entire function, β a complex number and m a positive integer. Hence we can apply the same reasoning as in (II) and we arrive at a contradiction.

Thus if $F(z)$ has a non-trivial factorization, then (3.1) and (3.2) hold.

Conversely, assume that (3.1) and (3.2) hold. Then we have

$$\{H_1(w+a)e^{w+a}+H_2(w+a)\}e^{P(w+a)}$$
$$=\{H_1(-w+a)e^{-w+a}+H_2(-w+a)\}e^{P(-w+a)}.$$

Putting $G(w)=\{H_1(w+a)e^{w+a}+H_2(w+a)\}e^{P(w+a)}$, we have $G(w)=G(-w)$. And so $G(w)$ is an even function. Hence there is an entire function $f(w)$ such that $G(w)=f(w^2)$. Since $F(w+a)=G(w)$, we obtain $F(z)=f((z-a)^2)$, which is a desired non-trivial factorization of $F(z)$.

Thus the proof of Theorem 1 is complete.

4. Next we shall prove

THEOREM 2. *Let $P(z)$ be a polynomial of degree k $(k\geqq 3)$ and $H_1(z)$ and $H_2(z)$ two entire functions which are of order less than one and are not identically zero. Suppose that all but a finite number of zeros of $H_1(z)e^z+H_2(z)$ are simple and there are two positive numbers K and N_0 satisfying*

$$\sum_{j\neq l} 1/|a_j(a_j-a_l)|\leqq K \tag{4.1}$$

for all $l\geqq N_0$, where $\{a_j\}_{j=1}^{\infty}$ are non-vanishing zeros of $H_1(z)e^z+H_2(z)$. The entire function

$$F(z)=\int_0^z \{H_1(z)e^z+H_2(z)\}e^{P(z)}dz$$

has a non-trivial factorization if and only if there are a complex number a and an integer m such that the following indentities

$$P(z+a)-P(-z+a)=-z+m\pi i \tag{4.2}$$

and

$$H_1(z+a)e^a+(-1)^m H_2(-z+a)=0 \tag{4.3}$$

hold. Then the non-trivial factorization of $F(z)$ is $F(z)=f(A(z-a)^2+B)$, where $f(z)$ is an entire function and $A(\neq 0)$ and B are constants.

Proof. Assume that $F(z)$ has a non-trivial factorization $f(g(z))$.

(I) Suppose that $f(z)$ and $g(z)$ are entire. Then from the reasoning in proof of Theorem 1 in [4] we deduce that $g(z)$ is a polynomial of degree two. Put $g(z)=A(z-a)^2+B(A\neq 0)$ and $w=z-a$. Then $F(w+a)=f(Aw^2+B)$ and consequently $F'(w+a)=-F'(-w+a)$. Hence we have

$$\{H_1(w+a)e^{w+a}+H_2(w+a)\}\,e^{P(w+a)-P(-w+a)} \tag{4.4}$$
$$=-\{H_1(-w+a)e^{-w+a}+H_2(-w+a)\}.$$

Hence as in (I) of the proof of our Theorem 1 we deduce from (4.4) that (4.2) and (4.3) hold.

(II) Suppose that $f(z)$ is meromorphic (not entire) and $g(z)$ is entire. Then Lemma 2 implies that

$$F(z)=e^{-nL(z)}f_1(\alpha+e^{L(z)}), \qquad f_1(\alpha)\neq 0,$$

where $f_1(z)$ is an entire function, $L(z)$ a non-constant entire function, α a complex number and n a positive integer. Since $F(z)$ is of finite order, we deduce from Lemma 4 that $\alpha+e^{L(z)}$ is of finite order, that is, $L(z)$ is a polynomial. It follows that

$$\{H_1(z)e^z+H_2(z)\}\,e^{P(z)}=F'(z)=L'(z)e^{-nL(z)}f_2(e^{L(z)}), \tag{4.5}$$

where $f_2(z)=-nf_1(\alpha+z)+zf_1'(\alpha+z)$. Since $F'(z)$ has an infinite number of zeros, $f_2(z)$ is a non-constant entire function. (4.5) yields that

$$\{(H_1(z)e^z+H_2(z))/L'(z)\}\,e^{P(z)+nL(z)}=f_2(e^{L(z)}). \tag{4.6}$$

If $\deg\{P(z)+nL(z)\}\leqq 1$, then $\deg P(z)=\deg L(z)\geqq 3$. Hence the function in left hand side of (4.6) is of order not greater than one and the function in right hand side of (4.6) is of order not less than three, which is untenable. If $\deg\{P(z)+nL(z)\}\geqq 2$, then the function has a maximal deficiency at zero. Hence Lemma 5 implies that $f_2(z)$ is a polynomial. Put $f_2(z)=a_m z^m+\cdots+a_1 z+a_0$ $(a_m\neq 0)$. Then we have

$$\{H_1(z)e^z+H_2(z)\}\,e^{P(z)+nL(z)}=L'(z)\{a_m e^{mL(z)}+\cdots+a_1 e^{L(z)}+a_0\}.$$

However using Lemma 1 we have $H_1(z)\equiv 0$, which is a contradiction.

(III) Suppose that $f(z)$ is non-linear rational and $g(z)$ is meromorphic. Then Lemma 3 implies that

$$F(z)=e^{-mM(z)}Q(\beta+e^{M(z)}), \qquad Q(\beta)\neq 0,$$

where $Q(z)$ is a polynomial, $M(z)$ a non-constant entire function, β a complex number and m a positive integer. Hence as in (II) we arrive at a contradiction.

Thus if $F(z)$ has a non-trivial factorization, then (4.2) and (4.3) hold.

Conversely, assume that (4.2) and (4.3) hold. Then we have (4.4). Hence $G(w)=\{H_1(w+a)e^{w+a}+H_2(w+a)\}e^{P(w+a)}$ is an odd function. Therefore there is an entire function $G^*(w)$ such that $G(w)=wG^*(w^2)$. Put

$$f(z)=\frac{1}{2}\int_{a^2}^{z} G^*(z)dz.$$

Then we have $2f'(z)=G^*(z)$ and $f(a^2)=0$. Since $F'(w+a)=G(w)=wG^*(w^2)$, we obtain $F(z)=f((z-a)^2)$, which is a desired non-trivial factorization of $F(z)$.

Thus the proof of Theorem 2 is complete.

5. Finally, as an application of our Theorem 1 and Theorem 2, we shall give prime entire functions.

THEOREM 3. *Let $P(z)$ be a polynomial of degree $k\,(k\geqq 3)$ such that if k is odd, $P(z)$ is arbitrary and if k is even, $P(z)=\alpha_k z^k\ (\alpha_k\neq 0)$. Let $P_1(z)$ and $P_2(z)$ be two polynomials which are not identically zero. Then all $F^{(n)}(z)$, $n=0, 1, 2, \cdots$, are prime, where*

$$F(z)=\int_0^z \{P_1(z)e^z+P_2(z)\}e^{P(z)}dz.$$

Proof. Since $P(z)$ is a polynomial of degree $k\,(k\geqq 3)$ and $P_1(z)$ and $P_2(z)$ are two polynomials which are not identically zero, we have $F^{(n)}(z)=\{P_{1n}(z)e^z+P_{2n}(z)\}e^{P(z)}$, where $P_{1n}(z)$ and $P_{2n}(z)$ are polynomials which are not identically zero. Further it is clear that $P(z)$ does not satisfy (3.1). Hence we deduce from our Theorem 1 that all $F^{(n)}(z)$, $n=1, 2, \cdots$, are prime.

It is clear that all but a finite number of zeros of $P_1(z)e^z+P_2(z)$ are simple and $P(z)$ does not satisfy (4.2). Hence the primeness of $F(z)$ follows from our Theorem 2 if we show that the zeros $\{b_j\}$ of $P_1(z)e^z+P_2(z)$ satisfy (4.1).

Now we prove that $\{b_j\}$ satisfy (4.1). It follows that there is a positive number R_0 such that

$$\frac{P_2(z)}{P_1(z)}=-az^{\mu}(1+A(z)), \qquad |A(z)|\leqq\frac{M}{|z|}$$

hold for all $z\in\Omega=\{z\,;\ |z|>R_0\}$, where a is a non-zero complex number, μ an integer and M a positive number. Let $\{a_\nu\}$ be zeros of e^z-az^{μ}. Since Lemma 6 valids for $\{a_\nu\}$ and we have

$$\operatorname{Re} b_j = \rho_j \cos\phi_j = \log|a| + \mu \log \rho_j + \log|1 + A(\rho_j e^{i\phi_j})|$$

for $b_j = \rho_j e^{i\phi_j} \in \Omega$, we may assume, without loss of generality, that

$$|\operatorname{Re} a_\nu| < \tau \log r_\nu \quad \text{and} \quad |\operatorname{Re} b_j| < \tau \log \rho_j, \quad a_\nu,\ b_j \in \Omega \tag{5.1}$$

with $|\mu| < \tau$. Put $\eta(r) = (\sigma \log r)/r$ with $\pi\tau < 2\sigma$ and

$$C_1 = \{z = r \exp i((\pi/2) - \eta(r))\,;\ r > R_0\},$$

$$C_2 = \{z = r \exp i((\pi/2) + \eta(r))\,;\ r > R_0\},$$

$$C_3 = \{z = r \exp i((3\pi/2) - \eta(r))\,;\ r > R_0\},$$

$$C_4 = \{z = r \exp i((3\pi/2) + \eta(r))\,;\ r > R_0\}$$

and

$$\Gamma_{y\nu} = \{z = a_\nu + x + iy\,;\ |x| \leqq 3\sigma \log|a_\nu|\}.$$

We claim that for any $\delta > 0$, if R_0 is sufficiently large, the equation $P_1(z)e^z + P_2(z) = 0$, that is, $e^z - az^\mu(1 + A(z)) = 0$ has only one solution in $\Omega_\nu \subset \Omega$ and has no solution in $\Omega - \cup \Omega_\nu$, where Ω_ν is a domain bounded by four curves C_1, C_2, $\Gamma_{\delta\nu}$ and $\Gamma_{-\delta\nu}$ when ν is even, that is, $\operatorname{Im} a_\nu > 0$ and by C_3, C_4, $\Gamma_{\delta\nu}$ and $\Gamma_{-\delta\nu}$ when ν is odd, that is, $\operatorname{Im} a_\nu < 0$. We may assume that $\operatorname{Im} a_\nu > 0$. On C_1 we have

$$\lim_{z\to\infty} |az^\mu/e^z| = \lim_{z\to\infty} |az^\mu A(z)/e^z| = 0,$$

and consequently, if R_0 is sufficiently large,

$$|e^z - az^\mu| > |az^\mu A(z)| \qquad \text{for all} \quad z \in C_1 \cap \Omega. \tag{5.2}$$

Since $\tau + \mu > 0$, we also have on C_2

$$\lim_{z\to\infty} |e^z/(az^\mu)| = \lim_{z\to\infty} |az^\mu A(z)/(az^\mu)| = 0,$$

and consequently

$$|e^z - az^\mu| > |az^\mu A(z)| \qquad \text{for all} \quad z \in C_2 \cap \Omega. \tag{5.3}$$

We choose y arbitrarily such that $\delta \leqq |y| \leqq 2\pi - \delta$ and fix it. For $z = a_\nu + x + iy \in \Gamma_{y\nu}$, we have

$$|(x + iy)/a_\nu| < |3\sigma \log|a_\nu| + 2\pi|/|a_\nu| \to 0 \qquad (\nu \to \infty)$$

and $|1 - e^{x+iy}| \geqq \sin\delta$. Further since $e^{a_\nu} - aa_\nu{}^\mu = 0$, if R_0 is sufficiently large, we have

$$|e^z - az^\mu| \geqq (\sin\delta)|a|\,|a_\nu|^{\mu/2} \qquad \text{for} \quad z \in \Gamma_{y\nu} \cap \Omega.$$

On the other hand we have

$$|az^\mu A(z)| \leqq 2M|a|\,|a_\nu|^{\mu-1} \qquad \text{for} \quad z \in \Gamma_{y\nu} \cap \Omega.$$

Hence, if R_0 is sufficiently large and $\delta \leqq |y| \leqq 2\pi-\delta$, we have

$$|e^z - az^\mu| > |az^\mu A(z)| \qquad \text{for all} \quad z \in \Gamma_{y_\nu} \cap \Omega . \tag{5.4}$$

Therefore using Rouché's theorem and taking (5.1) into account, we conclude from (5.2), (5.3) and (5.4) that the equation $P_1(z)e^z + P_2(z) = 0$, that is, $e^z - az^\mu(1+A(z)) = 0$ has only one solution in $\Omega_\nu \subset \Omega$ and no solution in $\Omega - \cup \Omega_\nu$. Hence, since $\{a_\nu\}$ satisfy (4.1), the zeros $\{b_j\}$ of $P_1(z)e^z + P_2(z)$ satisfy (4.1).

Thus the proof of Theorem 3 is complete.

Remark. An example in Remark 2 of [4] implies that our Theorem 3 is not true in the case when $k=2$.

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