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# INFINITESIMAL VARIATIONS OF INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

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## § 0. Introduction.

An infinitesimal variation of an invariant (complex) submanifold of a Kaehlerian manifold which carries it into an invariant submanifold is said to be *complex*. An infinitesimal variation is said to be *holomorphic* when it is complex and preserves the complex structure on the invariant submanifold. ([1], [4], [5]). Okumura and two of the present authors [5] proved that an infinitesimal complex conformal variation of a compact orientable submanifold of a Kaehlerian manifold is necessarily isometric and holomorphic and derived a necessary and sufficient condition for a complex variation to be volume-preserving and holomorphic by using an integral formula.

The main purpose of the present paper is to study infinitesimal variations of invariant submanifolds of a Sasakian manifold and to prove theorems analogous to those proved in [4] and [5].

In preliminary §1 we state some properties of invariant submanifolds of a Sasakian manifold.

In §2, we derive fundamental formulas in the theory of infinitesimal variations and study invariant variations, that is, infinitesimal variations which carry an invariant submanifold into an invariant submanifold. In §3 we study $f$-preserving variations, that is, invariant variations which preserve the tensor field $f_b{}^a$ of the Sasakian structure $(f_b{}^a, g_{cb}, f_b)$ induced on an invariant submanifold.

In §4 we study invariant conformal variations and prove that an invariant conformal fiber-preserving variation of a compact orientable invariant submanifold of a Sasakian manifold is necessarily isometric and hence $f$-preserving. In the last §5 we prove an integral formula concerning invariant variation and show some of its applications.

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## § 1. Invariant submanifolds of a Sasakian manifold.

Let $M^{2m+1}$ be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and $(f_i{}^h, g_{ji}, f_i)$ the set of structure tensors of $M^{2m+1}$, where, here and in the sequel, the indices $h, i, j, \cdots$ run over the range $\{1', 2', \cdots, (2m+1)'\}$. Then we have

$$f_i{}^t f_t{}^h = -\delta_i^h + f_i f^h, \quad f_t f_i{}^t = 0, \quad f_t{}^h f^t = 0, \quad f_t f^t = 1 \tag{1.1}$$

and

$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - f_j f_i, \tag{1.2}$$

$f^h$ being the vector field associated with $f_i$, that is, $f^h = f_i g^{ih}$, $g^{ih}$ being contravariant metric tensor. We also have

$$\nabla_i f^h = f_i{}^h \tag{1.3}$$

and

$$\nabla_j f_i{}^h = -g_{ji} f^h + \delta_j^h f_i, \tag{1.4}$$

where $\nabla_i$ denotes the operator of covariant differentiation with respect to $g_{ji}$.

Let $M^{2n+1}$ $(n<m)$ be a $(2n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and isometrically immersed in $M^{2m+1}$ by the immersion $\imath; M^{2n+1} \to M^{2m+1}$, where, here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, (2n+1)\}$. We identify $\imath(M^{2n+1})$ with $M^{2n+1}$ and represent the immersion by $x^h = x^h(y^a)$. If we put $B_b{}^h = \partial_b x^h$ $(\partial_b = \partial/\partial y^b)$, then $B_b{}^h$ are $2n+1$ linearly independent vectors of $M^{2m+1}$ tangent to $M^{2n+1}$. Denoting by $g_{cb}$ the Riemannian metric tensor of $M^{2n+1}$ we have $g_{cb} = g_{ji} B_c{}^j B_c{}^i$ since the immersion is isometric. We denote by $C_y{}^h$ $2(m-n)$ mutually orthogonal unit normals to $M^{2n+1}$, then we have $g_{ji} B_b{}^j C_y{}^i = 0$ and the metric tensor of the normal bundle of $M^{2n+1}$ is given by $g_{zy} = g_{ji} C_z{}^j C_y{}^i = \delta_{zy}$, $\delta_{zy}$ being the Kronecker delta, where, here and in the sequel, the indices $u, v, x, y, z$ run over the range $\{(2n+2), \cdots, (2m+1)\}$.

We denote by $\Gamma_{ji}^h$, $\Gamma_{cb}^a$ and $\Gamma_{cy}^x$ the Christoffel symbols formed with $g_{ji}$, those formed with $g_{cb}$ and the components of the connection induced in the normal bundle of $M^{2n+1}$, that is,

$$\Gamma_{cy}^x = (\partial_c C_y{}^h + \Gamma_{ji}^h B_c{}^j C_y{}^i) C^x{}_h, \quad C^x{}_h = C_y{}^i g^{yx} g_{ih} \tag{1.5}$$

respectively, $g^{yx}$ being the contravariant components of the metric tensor of the normal bundle. Then the van der Waerden-Bortolotti covariant derivatives of $B_b{}^h$ and $C_y{}^h$ are respectively given by

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \Gamma_{ji}^h B_c{}^j B_b{}^i - \Gamma_{cb}^a B_a{}^h$$

and

$$\nabla_c C_y{}^h = \partial_c C_y{}^h + \Gamma^h_{ji} B_c{}^j C_y{}^i - \Gamma^x_{cy} C_x{}^h,$$

and the equations of Gauss and Weingarten are respectively

(1.6) $$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h \quad \text{and} \quad \nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h,$$

where $h_{cb}{}^x$ are components of the second fundamental tensors of $M^{2n+1}$ and $h_c{}^a{}_y = h_{cb}{}^z g^{ba} g_{zy}$, $g^{ba}$ being contravariant components of the metric tensor of $M^{2n+1}$.

Denoting by $K_{kji}{}^h$, $K_{dcb}{}^a$ and $K_{dcy}{}^x$ the curvature tensors of $M^{2m+1}$, $M^{2n+1}$ and the normal bundle of $M^{2n+1}$, we have the following equations of Gauss, Codazzi and Ricci respectively:

(1.7) $$K_{dcb}{}^a = K_{kji}{}^h B^{kjia}_{dcbh} + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x,$$

(1.8) $$0 = K_{kji}{}^h B^{kji}_{dcb} C^x{}_h - (\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x)$$

and

(1.9) $$K_{dcy}{}^x = B^{kj}_{dc} C_y{}^i C^x{}_h + (h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y),$$

where $B^{kjia}_{dcbh} = B_d{}^k B_c{}^j B_b{}^i B^a{}_h$, $B^{kji}_{dcb} = B_d{}^k B_c{}^j B_b{}^i$ and $B^{kj}_{dc} = B_d{}^k B_c{}^j$.

A $(2n+1)$-dimensional submanifold $M^{2n+1}$ is called an invariant submanifold of the Sasakian manifold $M^{2m+1}$ if the tangent space at each point of $M^{2n+1}$ is invariant under the action of $f_i{}^h$. Thus for an invariant submanifold $M^{2n+1}$, we have

(1.10) $$f_i{}^h B_b{}^i = f_b{}^a B_a{}^h, \quad f_i{}^h C_y{}^i = f_y{}^x C_x{}^h,$$

$f_b{}^a$ and $f_y{}^x$ being tensor fields of type (1.1) of $M^{2n+1}$ and the normal bundle of $M^{2n+1}$ respectively. Putting $f_{ba} = f_b{}^e g_{ea}$ and $f_{yx} = f_y{}^z g_{zx}$, we have $f_{ba} = f_{ji} B^{ji}_{ba}$ and $f_{yx} = f_{ji} C_y{}^j C_x{}^i$ and consequently

$$f_{ba} = -f_{ab}, \quad f_{yx} = -f_{xy}.$$

On the other hand, we put

(1.11) $$f^h = f^a B_a{}^h + f^x C_x{}^h.$$

Now applying the operator $f_h{}^k$ to the first equation of (1.10) and using (1.1) and (1.11), we find

$$-B_b{}^k + f_b(f^a B_a{}^k + f^x C_x{}^k) = f_b{}^e f_e{}^a B_a{}^k,$$

where $f_b = f^c g_{cb}$, from which

(1.12) $$f_b{}^e f_e{}^a = -\delta^a_b + f_b f^a, \quad f_b f^x = 0.$$

Since $M^{2n+1}$ is odd dimensional, we see from the first equation of (1.12) that $f_b$ never vanishes and consequently from the second equation of (1.12) we see that $f^x=0$ and consequently (1.11) becomes

$$f^h=f^a B_a{}^h, \tag{1.13}$$

which shows that $f^h$ is tangent to the invariant submanifold. From (1.1), (1.10) and (1.13) we find

$$f_b{}^a f^b=0, \quad f_a f^a=1. \tag{1.14}$$

Also transvecting (1.2) with $B_c{}^j B_b{}^i$, we find

$$f_c{}^e f_b{}^d g_{ed}=g_{cb}-f_c f_b. \tag{1.15}$$

Equations (1.12), (1.14) and (1.15) show that the invariant submanifold $M^{2n+1}$ admits an almost contact metric structure.

Applying the operator $f_h{}^k$ to the second equation of (1.10) and using (1.1) and (1.13), we find

$$-C_y{}^k=f_y{}^z f_z{}^x C_x{}^k,$$

from which

$$f_y{}^z f_z{}^x=-\delta_y^x. \tag{1.16}$$

From (1.2), we have

$$f_z{}^v f_y{}^u g_{vu}=g_{zy} \tag{1.17}$$

and consequently $f_y{}^x$ defines an almost Hermitian structure in the normal bundle.

Now differentiating (1.13) covariantly along $M^{2n+1}$ and using (1.3) and (1.6), we find

$$f_i{}^h B_b{}^i=(\nabla_b f^a)B_a{}^h+f^a h_{ba}{}^x C_x{}^h,$$

from which

$$\nabla_b f^a=f_b{}^a \tag{1.18}$$

and

$$h_{cb}{}^x f^b=0. \tag{1.19}$$

Also differentiating the first equation of (1.10) covariantly and substituting (1.4) and (1.6), we find

$$(-g_{ji}f^h+\delta_j^h f_i)B_{cb}^{ji}+f_i{}^h h_{cb}{}^x C_x{}^i=(\nabla_c f_b{}^a)B_a{}^h+f_b{}^e h_{ce}{}^x C_x{}^h,$$

from which, taking account of (1.10) and (1.13)

$$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c^a f_b, \tag{1.20}$$

$$h_{cb}{}^y f_y{}^x = h_{ce}{}^x f_b{}^e. \tag{1.21}$$

From (1.21) we find $g^{cb} h_{cb}{}^y f_y{}^x = 0$, from which, $f_y{}^x$ defining an almost complex structure,

$$g^{cb} h_{cb}{}^y = 0, \tag{1.22}$$

which shows that an invariant submanifold $M^{2n+1}$ is minimal.

Equations (1.18) and (1.20) show that the almost contact metric structure induced on the invariant submanifold is Sasakian.

Differentiating the second equation of (1.10) covariantly, we find

$$(-g_{ji} f^h + \delta_j^h f_i) B_c{}^j C_y{}^i - f_i{}^h h_c{}^a{}_y B_a{}^i = (\nabla_c f_y{}^x) C_x{}^h - f_y{}^x h_c{}^a{}_x B_a{}^h,$$

from which, using (1.10) and (1.21),

$$\nabla_c f_y{}^x = 0, \tag{1.23}$$

which shows that the almost Hermitian structure in the normal bundle is covariantly constant.

We close this section by preparing some formulas for later use. It is known that on a Sasakian manifold the following identities are valid:

$$\frac{1}{2} K_{dcba} f^{dc} = K_{bc} f_a{}^c + (2n-1) f_{ba}, \tag{1.24}$$

$$K_{be} f_a{}^e + K_{ae} f_b{}^e = 0, \tag{1.25}$$

$$K_{be} f^e = 2n f_b, \tag{1.26}$$

where $K_{dcba} = K_{dcb}{}^e g_{ea}$, $K_{cb} = K_{ecb}{}^e$ and $f^{dc} = g^{de} f_e{}^c$. Transvecting (1.24) with $f_e{}^b$ and using (1.1), we find

$$\frac{1}{2} K_{dcba} f^{dc} f_e{}^b = K_{bc} f_e{}^b f_a{}^c + (2n-1)(-g_{ea} + f_e f_a),$$

from which, using (1.25),

$$\frac{1}{2} K_{dcba} f^{dc} f_e{}^b = K_{ea} - (K_{ed} f^d) f_a + (2n-1)(-g_{ea} + f_e f_a)$$

or

$$\frac{1}{2} K_{dcbe} f^{dc} f_a{}^e = K_{ba} - (2n-1) g_{ba} - f_b f_a. \tag{1.27}$$

## § 2. Infinitesimal variations of invariant submanifolds.

Let $M^{2n+1}$ be a $(2n+1)$-dimensional invariant submanifold of a $(2m+1)$-dimensional Sasakian manifold $M^{2m+1}$. We consider an infinitesimal variation of $M^{2n+1}$ of $M^{2m+1}$ given by

$$\bar{x}^h = x^h + v^h(y)\varepsilon, \tag{2.1}$$

where $\varepsilon$ is an infinitesimal. Putting $\bar{B}_b{}^h = \partial_b \bar{x}^h$, we have

$$\bar{B}_b{}^h = B_b{}^h + (\partial_b v^h)\varepsilon, \tag{2.2}$$

which are $2n+1$ linearly independent vectors tangent to the varied submanifold at $(\bar{x}^h)$. We displace $\bar{B}_b{}^h$ back parallelly from $(\bar{x}^h)$ to $(x^h)$, then we have

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}^h(x+v\varepsilon)v^j \bar{B}_b{}^i \varepsilon.$$

Thus putting $\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h$, we obtain

$$\delta B_b{}^h = (\nabla_b v^h)\varepsilon, \tag{2.3}$$

neglecting terms of order higher than one with respect to $\varepsilon$, where

$$\nabla_b v^h = \partial_b v^h + \Gamma_{ji}^h B_b{}^j v^i. \tag{2.4}$$

In the sequel we always neglect terms of order higher than one with respect to $\varepsilon$.

On the other hand, putting

$$v^h = v^a B_a{}^h + v^x C_x{}^h, \tag{2.5}$$

we have

$$\nabla_b v^h = (\nabla_b v^a - h_b{}^a{}_x v^x) B_a{}^h + (\nabla_b v^x + h_{ba}{}^x v^a) C_x{}^h. \tag{2.6}$$

Thus (2.3) can be written as

$$\delta B_b{}^h = [(\nabla_b v^a - h_b{}^a{}_x v^x) B_a{}^h + (\nabla_b v^x + h_{ba}{}^x v^a) C_x{}^h]\varepsilon. \tag{2.7}$$

When the tangent space at a point $(x^h)$ of the submanifold and that at the corresponding point $(\bar{x}^h)$ of the varied submanifold are always parallel, the variation is said to be *parallel* [3]. Hence (2.7) implies

LEMMA 2.1. ([3]) *In order for an infinitesimal variation of a submanifold to be parallel, it is necessary and sufficient that*

$$\nabla_b v^x + h_{ba}{}^x v^a = 0. \tag{2.8}$$

We now assume that the infinitesimal variation (2.1) carries an invariant submanifold into an invariant submanifold and call such a variation an *invariant variation*. For an invariant variation, $f_i{}^h(x+v\varepsilon)\bar{B}_b{}^i$ are linear combinations of $\bar{B}_b{}^h$ and vice versa.

Now we have

$$
\begin{aligned}
&f_i{}^h(x+v\varepsilon)\bar{B}_b{}^i\\
&\quad=[f_i{}^h+v^j\partial_j f_i{}^h\varepsilon][B_b{}^i+\partial_b v^i\varepsilon]\\
&\quad=f_i{}^hB_b{}^i+v^j[-\Gamma^h_{jt}f_i{}^t+\Gamma^t_{ji}f_t{}^h-g_{ji}f^h+\delta^h_jf_i]B_b{}^i\varepsilon+f_i{}^h\partial_b v^i\varepsilon\\
&\quad=f_b{}^a\bar{B}_a{}^h-f_b{}^a(\partial_a v^h)\varepsilon+f_i{}^h(\partial_b v^i+\Gamma^i_{jt}B_b{}^jv^t)\varepsilon-f_b{}^a\Gamma^h_{ji}v^jB_a{}^i\varepsilon\\
&\quad\quad+f_b(v^aB_a{}^h+v^xC_x{}^h)\varepsilon-v_bf^aB_a{}^h\varepsilon\\
&\quad=f_b{}^a\bar{B}_a{}^h+f_i{}^h\nabla_bv^i\varepsilon-f_b{}^a\nabla_av^h\varepsilon+(f_bv^a-v_bf^a)B_a{}^h\varepsilon+f_bv^xC_x{}^h\varepsilon,
\end{aligned}
$$

where we have used (1.4), (1.13), (2.2), (2.4) and (2.5), and consequently from (2.6)

$$
\begin{aligned}
f_i{}^h(x+v\varepsilon)\bar{B}_b{}^i
&=[f_b{}^a+\{f_e{}^a(\nabla_bv^e-h_b{}^e{}_xx^x)-f_b{}^e(\nabla_ev^a-h_e{}^a{}_xv^x)\\
&\quad+f_bv^a-v_bf^a\}\varepsilon]\bar{B}_a{}^h\\
&\quad+[(\nabla_bv^y+h_{ba}{}^yv^a)f_y{}^x-f_b{}^e(\nabla_ev^x+h_{ea}{}^xv^a)+f_bv^x]\bar{C}_x{}^h\varepsilon,
\end{aligned}
\tag{2.9}
$$

where $\bar{C}_x{}^h$ denote $2(m-n)$ mutually orthogonal unit normals to the varied submanifold and $v_b=g_{ba}v^a$. Hence we have, using (1.21),

PROPOSITION 2.2. *In order for an infinitesimal variation of an invariant submanifold to be invariant it is necessary and sufficient that*

$$(\nabla_bv^y+h_{ba}{}^yv^a)f_y{}^x-f_b{}^e(\nabla_ev^x+h_{ea}{}^xv^a)+f_bv^x=0 \tag{2.10}$$

or

$$(\nabla_bv^y)f_y{}^x-f_b{}^e\nabla_ev^x+f_bv^x=0. \tag{2.11}$$

When $v^x=0$, that is, when the variation vector $v^h$ is tangent to the submanifold, the variation is said to be *tangential* and when $v^a=0$, that is, when the variation vector $v^h$ is normal to the submanifold, the variation is said to be *normal* [3]. From Proposition 2.2, we have

COROLLARY 2.3. *A tangential variation of an invariant submanifold is invariant.*

Combining Proposition 2.2 and Lemma 2.1, we also have

COROLLARY 2.4. *In order for a parallel variation of an invariant submanifold to be invariant, it is necessary and sufficient that the variation is tangential.*

Now applying the operator $\delta$ to $g_{cb}=g_{ji}B_c{}^jB_b{}^i$ and using $\delta g_{ji}=0$ and (2.7), we find

$$\delta g_{cb}=[\nabla_c v_b+\nabla_b v_c-2h_{cbx}v^x]\,\varepsilon, \tag{2.12}$$

from which

$$\delta g^{ba}=-[\nabla^b v^a+\nabla^a v^b-2h^{ba}{}_x v^x]\,\varepsilon, \tag{2.13}$$

where $\nabla^b=g^{ba}\nabla_a$ and $h^{ba}{}_x=g^{be}g^{ad}h_{edx}$.

An infinitesimal variation for which $\delta g_{cb}=0$ is said to be *isometric* [1], [3].

We now put

$$\bar{\Gamma}^a_{cb}=(\partial_c\bar{B}_b{}^h+\Gamma^h_{ji}(\bar{x})\bar{B}_c{}^j\bar{B}_b{}^i)\bar{B}^a{}_h \tag{2.14}$$

and

$$\delta\Gamma^a_{cb}=\bar{\Gamma}^a_{cb}-\Gamma^a_{cb}, \tag{2.15}$$

where $\bar{\Gamma}^a_{cb}$ are Christoffel symbols of the varied submanifold.

Substituting (2.2) into (2.14), we find by a straightforward computation

$$\delta\Gamma^a_{cb}=[(\nabla_c\nabla_b v^a+K_{kji}{}^h v^k B^{ji}_{cb})B^a{}_h+h_{cb}{}^x(\nabla^a v_x+h_d{}^a{}_x v^d)]\,\varepsilon, \tag{2.16}$$

from which, using equations of Gauss and Codazzi [3],

$$\begin{aligned}\delta\Gamma^a_{cb}=&(\nabla_c\nabla_b v^a+K_{dcb}{}^a v^d)\,\varepsilon-[\nabla_c(h_{bex}v^x)+\nabla_b(h_{cex}v^x)\\&-\nabla_e(h_{cbx}v^x)]\,g^{ea}\varepsilon.\end{aligned} \tag{2.17}$$

An infinitesimal variation of a submanifold for which $\delta\Gamma^a_{cb}=0$ is said to be *affine*.

We now prove

LEMMA 2.5. *If an infinitesimal variation of an invariant submanifold is isometric, then we have*

$$\begin{aligned}g^{cb}\nabla_c\nabla_b v_a+\frac{1}{2}K_{dceb}f_a{}^e f^{dc}v^b-2\nabla^e(h_{ea}{}^x v_x)&\\=-(2n-1)v_a-(f_b v^b)f_a.&\end{aligned} \tag{2.18}$$

*Proof.* Since an isometric variation is affine, we have from (2.17)

$$\nabla_c\nabla_b v_a+K_{dcba}v^d-\nabla_c(h_{bax}v^x)-\nabla_b(h_{cax}v^x)+\nabla_a(h_{cbx}v^x)=0, \tag{2.19}$$

from which

$$g^{cb}\nabla_c\nabla_b v_a+K_{da}v^d-2\nabla^e(h_{eax}v^x)=0,$$

which, together with (1.27), proves the lemma.

## § 3. Infinitesimal *f*-preserving variations.

Suppose that an infinitesimal variation of an invariant submanifold is invariant. Then putting

$$f_i{}^h(x+v\varepsilon)\bar{B}_b{}^i=(f_b{}^a+\delta f_b{}^a)\bar{B}_a{}^h, \tag{3.1}$$

we have, from (2.9),

$$\delta f_b{}^a=[f_e{}^a(\nabla_b v^e-h_b{}^e{}_x v^x)-f_b{}^e(\nabla_e v^a-h_e{}^a{}_x v^x)+(f_b v^a-v_b f^a)]\varepsilon. \tag{3.2}$$

When an infinitesimal invariant variation satisfies $\delta f_b{}^a=0$, we say that the invariant variation is *f-preserving*. From (1.20), (1.21) and (3.2) we have

LEMMA 3.1. *In order for an infinitesimal variation to be f-preserving, it is necessary and sufficient that the variation satisfies* (2.10) *or* (2.11) *and*

$$(\nabla_b v^e)f_e{}^a-f_b{}^e(\nabla_e v^a)-2h_b{}^e{}_x v^x f_e{}^a+(f_b v^a-v_b f^a)=0 \tag{3.3}$$

*or equivalently*

$$\mathcal{L}f_b{}^a-2h_b{}^e{}_x v^x f_e{}^a=0, \tag{3.4}$$

$\mathcal{L}$ *denoting the Lie derivation with respect to* $v^a$.

Now applying the operator $\delta$ to $f_b{}^a f^b=0$, we find

$$(\delta f_b{}^a)f^b+f_b{}^a(\delta f^b)=0,$$

from which, substituting (3.2) and taking account of (1.19),

$$[f_e{}^a(f^b\nabla_b v^e)+v^a-(v_b f^b)f^a]\varepsilon+f_b{}^a(\delta f^b)=0$$

or, using (1.12) and (1.18)

$$\delta f^a=[\mathcal{L}f^a+\alpha f^a]\varepsilon$$

for a certain function $\alpha$. On the other hand, applying the operator $\delta$ to $g_{cb}f^c f^b=1$ and using (2.12)

$$[\mathcal{L}g_{cb}-2h_{cbx}v^x]f^c f^b\varepsilon+2g_{cb}(\delta f^c)f^b=0,$$

from which, using (1.19) and the above equation, $\alpha=0$ and consequently we have

(3.5) $$\delta f^a=(\mathcal{L}f^a)\,\varepsilon.$$

We now define a tensor field $T_{cb}$ by

(3.6) $$T_{cb}=\nabla_c v_b-(\nabla_e v_d)f_c{}^e f_b{}^d-2h_{cb}{}^x v_x+f_c f_b(\nabla_e v_d)f^e f^d +f_c(f_b{}^e v_e)-f_b(f_c{}^e v_e)$$

and prove

LEMMA 3.2. *In order for an infinitesimal isometric invariant variation of an invariant submanifold to be f-preserving, it is necessary and sufficient that* $T_{cb}=0$.

*Proof*. Suppose that an infinitesimal invariant variation of an invariant submanifold is $f$-preserving. Then by Lemma 3.1, we have

$$(\nabla_b v^e)f_e{}^a-f_b{}^e(\nabla_e v^a)-2h_b{}^e{}_x v^x f_e{}^a+(f_b v^a-v_b f^a)=0.$$

Transvecting this with $f_a{}^c$ and taking account of (1.12) and (1.19), we find

(3.7) $$\nabla_b v^c-(\nabla_e v_d)f_b{}^e f^{cd}-2h_b{}^c{}_x v^x-f_e(\nabla_b v^e)f^c-f_b(f_e{}^c v^e)=0,$$

from which, transvecting with $f^b$,

$$f^e\nabla_e v^c=f^c(\nabla_e v_d)f^e f^d+f_e{}^c v^e,$$

and consequently, using $\nabla_c v_b+\nabla_b v_c-2h_{cbx}v^x=0$ which means that the variation is isometric,

$$f_e\nabla_c v^e=-f_c(\nabla_e v_d)f^e f^d+f_c{}^e v_e.$$

Thus (3.7) becomes

$$\nabla_b v_c-(\nabla_e v_d)f_b{}^e f_c{}^d-2h_{bc}{}^x v_x+f_b f_c(\nabla_e v_d)f^e f^d -(f_b{}^e v_e)f_c+(f_c{}^e v_e)f_b=0,$$

which means $T_{cb}=0$.

Conversely we suppose that $T_{cb}=0$. Then we have

$$\nabla_c v^b+(\nabla_e v^d)f_c{}^e f_d{}^b-2h_c{}^b{}_x v^x+f_c f^b(\nabla_e v_d)f^e f^d -(f_c{}^e v_e)f^b-(f_e{}^b v^e)f_c=0.$$

Transvecting this with $f_b{}^a$ and using (1.12), we find

(3.8) $$(\nabla_c v^e)f_e{}^a-(\nabla_e v^a)f_c{}^e-2h_c{}^e{}_x v^x f_e{}^a+v^a f_c+f_d(\nabla_e v^d)f_c{}^e f^a -f_c f^a(f_e v^e)=0.$$

Transvecting the same equation with $f_b$, we have

$$(\nabla_c v^b) f_b + f_c (\nabla_e v_d) f^e f^d - f_c^{\ e} v_e = 0. \tag{3.9}$$

Therefore substituting (3.9) into (3.8), we obtain

$$(\nabla_c v^e) f_e^{\ a} - (\nabla_e v^a) f_c^{\ e} - 2h_c^{\ e}{}_x v^x f_e^{\ a} + f_c v^a - v_c f^a = 0.$$

Thus Lemma 3.1 and this equation prove the lemma.

We next prove

LEMMA 3.3. *For an infinitesimal isometric invariant variation of an invariant submanifold, we have*

$$T_{cb} + T_{bc} = 0, \tag{3.10}$$

$$T_{cb} + T_{ed} f_c^{\ e} f_b^{\ d} = (\mathcal{L} f_c) f_b - (\mathcal{L} f_b) f_c \tag{3.11}$$

*and*

$$T^{cb} T_{cb} = 2T^{cb} \nabla_c v_b - 2(\mathcal{L} f^e)(\mathcal{L} f_e) - 4(\mathcal{L} f^e)(f_e^{\ d} v_d), \tag{3.12}$$

*where* $T^{cb} = T_{ed} g^{ec} g^{db}$.

*Proof.* For an isometric variation, we have from (2.12),

$$\nabla_b v_a + \nabla_a v_b = 2h_{ba}{}^x v_x, \tag{3.13}$$

from which, taking account of (1.19),

$$(\nabla_b v_a) f^b f^a = 0. \tag{3.14}$$

On the other hand, from (1.21), we have

$$h_{cb}{}^x = -h_{ed}{}^x f_c^{\ e} f_b^{\ d}. \tag{3.15}$$

Thus, from the definition (3.6) of $T_{cb}$, we see that $T_{cb} + T_{bc} = 0$.

On the other hand, using (3.14) and (3.15), we have, by a straightforward computation,

$$T_{ed} f_c^{\ e} f_b^{\ d} = (\nabla_e v_d) f_c^{\ e} f_b^{\ d} - \nabla_c v_b + (\nabla_c v_e) f^e f_b + (\nabla_e v_b) f^e f_c + 2h_{cb}{}^x v_x. \tag{3.16}$$

But

$$\mathcal{L} f_b = v^e f_{eb} + f_e \nabla_b v^e$$

and consequently (3.16) can be written as

$$\begin{aligned} T_{ed} f_c^{\ e} f_b^{\ d} = &-\{\nabla_c v_b - (\nabla_e v_d) f_c^{\ e} f_b^{\ d} - 2h_{cb}{}^x v_x + f_c (f_b^{\ e} v_e) - f_b (f_c^{\ e} v_e)\} \\ &- f_c \mathcal{L} f_b + f_b \mathcal{L} f_c. \end{aligned}$$

Thus, taking account of (3.6) and (3.14), we find (3.11). Finally from the definition (3.6) of $T_{cb}$ and (3.10) we have

$$T^{cb}T_{cb}=T^{cb}\{\nabla_c v_b-(\nabla_e v_d)f_c{}^e f_b{}^d+f_c(f_b{}^e v_e)-f_b(f_c{}^e v_e)\}.$$

Thus taking account of (3.11) and of

$$T_{eb}f^e=-\mathcal{L}f_b=-f^e T_{be},$$

we have

$$T^{cb}T_{cb}=2T^{cb}\nabla_c v_b-2(\mathcal{L}f^e)(\mathcal{L}f_e)-4(\mathcal{L}f^e)(f_e{}^d v_d)$$

and consequently the lemma is proved.

Now applying the operator $\nabla^c$ to (3.6) with (3.14) and using (1.18), (1.20) and (3.13), we see that

$$\nabla^c T_{cb}=\nabla^c\nabla_c v_b-\frac{1}{2}(\nabla_c\nabla_e v_d-\nabla_e\nabla_c v_d)f^{ce}f_b{}^d-2\nabla^c(h_{cb}{}^x v_x)$$

$$-v_b+(2n+1)(f^e v_e)f_b+2n(\nabla_e v_d)f^e f_b{}^d,$$

from which, using the Ricci identity and Lemma 2.5,

$$\nabla^c T_{cb}=2n\{(f^c v_c)f_b-v_b+(\nabla_e v_d)f^e f_b{}^d\}.$$

Hence taking account of

$$f_c\mathcal{L}f_b{}^c=(f^c v_c)f_b-v_b+(\nabla_e v_c)f^e f_b{}^c,$$

we obtain

$$\nabla^c T_{cb}=2n(f_c\mathcal{L}f_b{}^c)=-2n(f_b{}^c\mathcal{L}f_c). \tag{3.17}$$

Thus, substituting (3.12) and (3.17) into the identity:

$$\nabla^c(T_{cb}v^b)=(\nabla^c T_{cb})v^b+T_{cb}\nabla^c v^b,$$

we have

$$\nabla^c(T_{cb}v^b)=\frac{1}{2}T^{cb}T_{cb}+(\mathcal{L}f^e)(\mathcal{L}f_e)+2(n+1)(\mathcal{L}f^e)(f_e{}^d v_d). \tag{3.18}$$

Now an infinitesimal variation which satisfies $\mathcal{L}f^b=\alpha f^b$, $\alpha$ being a certain function, is said to be *fiber-preserving*. Thus for a fiber-preserving variation, we have

$$\nabla^c(T_{cb}v^b)=\frac{1}{2}T^{cb}T_{cb}+\alpha^2,$$

from which, if the submanifold is compact orientable we have

$$\int\left[\frac{1}{2}T^{cb}T_{cb}+\alpha^2\right]dV=0,$$

$dV$ being the volume element of $M^{2n+1}$. Thus we have

PROPOSITION 3.4. *If an infinitesimal isometric invariant variation of a compact orientable invariant submanifold of a Sasakian manifold is fiber-preserving, then it is f-preserving.*

## § 4. Infinitesimal conformal variations.

An infinitesimal variation of a submanifold for which $\delta g_{cb}$ is proportional to $g_{cb}$ is said to be *conformal.* A necessary and sufficient condition for an infinitesimal variation (2.1) of a submanifold to be conformal is

$$\nabla_c v_b+\nabla_b v_c-2h_{cb}{}^x v_x=2\lambda g_{cb}, \tag{4.1}$$

where

$$\lambda=(1/(2n+1))(\nabla_e v^e-h_e{}^e{}_x v^x). \tag{4.2}$$

The purpose of the present section is to prove the following proposition as a generalization of Proposition 3.4.

PROPOSITION 4.1. *If an infinitesimal conformal invariant variation of a compact orientable invariant submanifold of a Sasakian manifold is fiber-preserving, then it is isometric and hence f-preserving.*

To prove this proposition we need following lemmas which will be proved in the same way as in the proofs of Lemmas 3.2 and 3.3.

LEMMA 4.2. *In order for an infinitesimal conformal invariant variation of an invariant submanifold to be f-preserving, it is necessary and sufficient that the tensor field $'T_{cb}$ defined by*

$$\begin{aligned}'T_{cb}=\nabla_c v_b-(\nabla_e v_d)f_c{}^e f_b{}^d-2h_{cb}{}^x v_x-f_c f_b(\nabla_e v_d)f^e f^d\\+f_c(f_b{}^e v_e)-f_b(f_c{}^e v_e)\end{aligned} \tag{4.3}$$

*vanishes identically.*

LEMMA 4.3. *For an infinitesimal conformal invariant variation of an invariant submanifold we have*

$$'T_{cb}+'T_{bc}=0,$$

$$'T_{cb}+'T_{ed}f_c{}^e f_b{}^d=(\mathcal{L}f_c)f_b-(\mathcal{L}f_b)f_c-2\lambda f_c f_b$$

*and*

$$'T^{cb}\,'T_{cb}=2'T^{cb}\nabla_c v_b-2(\mathcal{L}f^e)(\mathcal{L}f_e) -4(\mathcal{L}f^e)(f_e{}^d v_d)+2\lambda f_e(\mathcal{L}f^e), \tag{4.4}$$

*where* $'T^{cb}='T_{ed}g^{ec}g^{db}$.

*Proof of Proposition* 4.1. Differentiating (4.1) covariantly, we have

$$\nabla_c\nabla_b v_a+\nabla_c\nabla_a v_b=2\nabla_c(h_{ba}{}^x v_x+\lambda g_{ba}), \tag{4.5}$$

from which, using the Ricci identity,

$$\nabla_c\nabla_b v_a+\nabla_a\nabla_c v_b-K_{cabd}v^d=2\nabla_c(h_{ba}{}^x v_x+\lambda g_{ba}),$$

or, substituting (4.5),

$$\nabla_c\nabla_b v_a-\nabla_a\nabla_b v_c-K_{cabd}v^d=2\nabla_c(h_{ba}{}^x v_x+\lambda g_{ba})-2\nabla_a(h_{cb}{}^x v_x+\lambda g_{cb}).$$

Taking the skew-symmetric part of this equation with respect to $a$ and $b$ and making use of the Ricci identity, we find

$$\nabla_c\nabla_b v_a-\nabla_c\nabla_a v_b+K_{abcd}v^d+K_{acbd}v^d+K_{cbad}v^d$$
$$=-2\nabla_a(h_{cb}{}^x v_x+\lambda g_{cb})+2\nabla_b(h_{ca}{}^x v_x+\lambda g_{ca}),$$

or, using (4.5) and the first Bianchi identity,

$$\nabla_c\nabla_b v_a-K_{bacd}v^d=\nabla_c(h_{ba}{}^x v_x+\lambda g_{ba})+\nabla_b(h_{ca}{}^x v_x+\lambda g_{ca})$$
$$-\nabla_a(h_{cb}{}^x v_x+\lambda g_{cb})$$

Transvecting this with $g^{cb}$ and using $h_e{}^e{}_x=0$, we find

$$\nabla^c\nabla_c v_a+K_{ac}v^c-2\nabla^c(h_{ca}{}^x v_x)+(2n-1)\nabla_a\lambda=0. \tag{4.6}$$

Now applying the operator $\nabla^c$ to (4.3) and using the Ricci identity, we can verify that

$$\nabla^c\,'T_{cb}=\nabla^c\nabla_c v_b+\frac{1}{2}K_{ced}{}^a f^{ce}f_b{}^d v_a-2\nabla^c(h_{cb}{}^x v_x)-(\nabla^e\lambda)f_e f_b-v_b$$
$$+(2n+1)(f^e v_e)f_b+2n(\nabla_e v_d)f^e f_b{}^d$$

because of (1.18), (1.20), (4.1) and $(\nabla_e v_d)f^e f^d=\lambda$.

Substituting (1.27) into this equation and taking account of (4.6), we obtain

$$\nabla^c[\,'T_{cb}+2(n-1)\lambda g_{cb}+\lambda f_c f_b]=-2n f_b{}^c\mathcal{L}f_c.$$

Thus we have

$$\nabla^c[('T_{cb}+(2n-1)\lambda g_{cb}+\lambda f_c f_b)v^b]$$
$$=-2n(f_b{}^c v^b)\mathcal{L}f_c+['T_{cb}+(2n-1)\lambda g_{cb}+\lambda f_c f_b]\nabla^c v^b,$$

from which, substituting (4.4) and using (4.2) with $h_e{}^e{}_x=0$,

$$\nabla^c[('T_{cb}+(2n-1)\lambda g_{cb}+\lambda f_c f_b)v^b]$$
$$=\frac{1}{2}'T^{cb}\,'T_{cb}+4n^2\lambda^2+(\mathcal{L}f^e)(\mathcal{L}f_e)+2(n+1)(\mathcal{L}f^e)f_e{}^d v_d$$
$$-\lambda f_e(\mathcal{L}f^e).$$

The variation being conformal and fiber-preserving, from $\delta g_{cb}=2\lambda g_{cb}\varepsilon$ and $g_{cb}f^c f^b=1$, we see that $\delta f^a=-\lambda f^a\varepsilon$, that is, $\mathcal{L}f^a=-\lambda f^a$ by (3.5). Thus the above equation becomes

$$\nabla^c[('T_{cb}+(2n-1)\lambda g_{cb}+\lambda f_c f_b)v^b]=\frac{1}{2}'T^{cb}\,'T_{cb}+4n^2\lambda^2.$$

Thus if the submanifold is compact and orientable, we have

$$\int['T^{cb}\,'T_{cb}+8n^2\lambda^2]\,dV=0,$$

from which $\lambda=0$, and consequently the variation is isometric and hence $f$-preserving.

## § 5. An integral formula.

We put

$$S_b{}^a=(\nabla_b v^e)f_e{}^a-f_b{}^e(\nabla_e v^a)-2h_b{}^e{}_x v^x f_e{}^a+(f_b v^a-v_b f^a).$$

Then, as is shown in Lemma 3.1, an invariant variation of an invariant submanifold is $f$-preserving if and only if $S_b{}^a=0$. We put $\|S_{cb}\|^2=S_{cb}S^{cb}$ where $S_{cb}=S_c{}^e g_{eb}$ and $S^{cb}=g^{ce}S_e{}^b$. Then, using (1.19) and (1.21), we obtain by a straightforward computation,

$$\|S_{cb}\|^2=2\|\nabla_c v_b\|^2-8(h_{ed}{}^x v_x)\nabla^e v^d+4\|h_{cb}{}^x v_x\|^2 \tag{5.1}$$
$$-\|(\nabla_b v_e)f^e\|^2-\|(\nabla_e v_b)f^e\|^2-2f^{cb}(\nabla_c v^e)(\nabla_b v_a)f_e{}^a$$
$$+2(f^e\nabla_e v^b)f_b{}^a v_a+2(f^e\nabla_b v_e)f_a{}^b v^a+2v_a v^a-2(f_a v^a)^2.$$

On the other hand, putting

$$w^b=(\nabla^b v^a)v_a-f^{ba}(\nabla_a v_c)f_e{}^c v^e+(f_a{}^b v^a)(f^e v_e),$$

we have

$$\nabla_b w^b=(\nabla^b\nabla_b v_a)v^a+\|\nabla_b v_a\|^2-f^{cb}(\nabla_c\nabla_b v_a)v^e f_e{}^a-f^{cb}(\nabla_c v^e)(\nabla_b v_a)f_e{}^a$$
$$+2n(f^e\nabla_e v_d)f_c{}^d v^c+2f^e(\nabla_d v_e)f_c{}^d v^c+(2n+1)(f_e v^e)^2-(v_e v^e),$$

from which, using (1.27) and the Ricci identity,

$$\nabla_b w^b=v^a(\nabla^b\nabla_b v_a+K_{ba}v^b)+\|\nabla_b v_a\|^2-f^{cb}(\nabla_c v^e)(\nabla_b v_a)f_e{}^a$$
$$+2n(f^e\nabla_e v_d)f_c{}^d v^c+2f^e(\nabla_d v_e)f_c{}^d v^c-2n(v_e v^e)+2n(f_e v^e)^2. \tag{5.2}$$

Comparing (5.1) with (5.2), we have

$$\|S_{cb}\|^2=2\nabla^b w_b-2v^a(\nabla^b\nabla_b v_a+K_{ba}v^b)-8(h_{ba}{}^x v_x)\nabla^b v^a+4\|h_{cb}{}^x v_x\|^2$$
$$+(4n+2)\{v_e v^e-(f_e v^e)^2+(f^e\nabla_e v^d)f_d{}^c v_c\}-2f^e(\nabla_d v_e)f_c{}^d v^c$$
$$-\|(\nabla_b v_e)f^e\|^2-\|(\nabla_e v_b)f^e\|^2,$$

or equivalently

$$\|S_{cb}\|^2=2\nabla^b(w_b-2h_{ba}{}^x v_x v^a)-2v^a[\nabla^b\nabla_b v_a+K_{ba}v^b-2\nabla^b(h_{ba}{}^x v_x)]$$
$$-2(h^{cb}{}_y v^y)(\nabla_c v_b+\nabla_b v_c-2h_{cb}{}^x v_x)$$
$$-\|\mathcal{L}f_a\|^2-\|\mathcal{L}f^a\|^2-4(n+1)(\mathcal{L}f^e)f_e{}^d v_d.$$

Therefore, assuming that the submanifold is compact orientable, we apply Green's theorem and obtain

$$\int\Big[\frac{1}{2}\|S_{cb}\|^2+v^a\{\nabla^b\nabla_b v_a+K_{ba}v^b-2\nabla^b(h_{ba}{}^x v_x)\}$$
$$+(h^{cb}{}_y v^y)(\nabla_c v_b+\nabla_b v_c-2h_{cb}{}^x v_x)$$
$$+\frac{1}{2}(\|\mathcal{L}f_a\|^2+\|\mathcal{L}f^a\|^2)+2(n+1)(\mathcal{L}f^e)f_e{}^d v_d\Big]dV=0. \tag{5.3}$$

From (2.12) and (2.13) we see that the infinitesimal variation of $dV$ is given by

$$\delta dV=(\nabla_a v^a-h_a{}^a{}_x v^x)dV\varepsilon. \tag{5.4}$$

On the other hand it is known [2] that the integral formula

$$\int\Big[v^a(\nabla^b\nabla_b v_a+K_{ba}v^b)+\frac{1}{2}\|\nabla_b v_a+\nabla_a v_b\|^2-(\nabla_e v^e)^2\Big]dV=0$$

is valid for any vector field $v^a$ in a compact orientable Riemannian manifold. From this we have

$$\begin{aligned}(5.5)\qquad &\int\Big[v^a\{\nabla^b\nabla_b v_a+K_{ba}v^b-2\nabla^b(h_{ba}{}^x v_x)+\nabla_a(h_e{}^e{}_x v^x)\}\\ &+\frac{1}{2}\|\nabla_b v_a+\nabla_a v_b-2h_{ba}{}^x v_x\|^2-(\nabla_e v^e-h_e{}^e{}_x v^x)(\nabla_b v^b)\\ &+(h^{cb}{}_y v^y)(\nabla_c v_b+\nabla_b v_c-2h_{cb}{}^x v_x)\Big]dV=0.\end{aligned}$$

Comparing (5.3) with (5.5) and taking account of $h_e{}^e{}_x=0$, we have

$$\begin{aligned}&\int\Big[\|S_{cb}\|^2-\|\nabla_c v_b+\nabla_b v_c-2h_{cb}{}^x v_x\|^2+2(\nabla_e v^e)^2\\ &+\|\mathcal{L}f_a\|^2+\|\mathcal{L}f^a\|^2+4(n+1)(\mathcal{L}f^e)f_e{}^d v_d\Big]dV=0,\end{aligned}$$

or using $S_e{}^e=g^{ed}S_{ed}=0$,

$$\begin{aligned}(5.6)\qquad &\int\Big[\|\nabla_c v_b+\nabla_b v_c-2h_{cb}{}^x v_x\|^2-\|S_{cb}+\sqrt{2/(2n+1)}(\nabla_e v^e)g_{cb}\|^2\\ &+\|\mathcal{L}f_a\|^2+\|\mathcal{L}f^a\|^2+4(n+1)(\mathcal{L}f^e)f_e{}^d v_d\Big]dV=0.\end{aligned}$$

Now, when we have $\delta f^a=(\mathcal{L}f^a)\varepsilon=0$, we say that the variation is *strictly fiber-preserving.* From (5.6) we have

PROPOSITION 5.1. *In order for an infinitesimal strictly fiber-preserving invariant variation of a compact orientable invariant submanifold of a Sasakian manifold to be isometric, it is necessary and sufficient that the variation is volume-preserving and f-preserving.*

Now if an infinitesimal variation is affine we have from (2.19) $\nabla_c(\nabla_e v^e)=0$, since $h_e{}^e{}_x=0$. Thus $\nabla_e v^e=$const. and consequently if the submanifold is compact we have $\nabla_e v^e=0$. Thus from Proposition 5.1, we have

PROPOSITION 5.2. *A strictly fiber-preserving invariant variation of a compact orientable submanifold of a Sasakian manifold is isometric if and only if the variation is affine and f-preserving.*

*Remark.* From (5.6) we see that if a fiber-preserving variation of a submanifold is conformal, then $\lambda=0$ and $S_{cb}=0$. This gives another proof of Proposition 4.1.

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