M. OZAWA
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A CHARACTERIZATION OF THE COSINE FUNCTION BY THE VALUE DISTRIBUTION

BY MITSURU OZAWA

1. Baker [1] has shown the following characterization of the exponential function.

If $f(z)$ is a transcendental entire function for which every value is linearly distributed, then with constants a, b, c,

$$f(z)=ae^{bz}+c.$$

Recently Kobayashi [3] has beautifully given a generalization of Baker's result. His main results may be stated in the following two theorems.

THEOREM A. *Let $f(z)$ be a transcendental entire function. Assume that there are three distinct finite complex numbers a_j and three distinct straight lines l_j in the complex plane on which all the solutions of $f(z)=a_j$ lie* ($j=1, 2, 3$). *Assume further that $f(z)$ has a finite deficient value other than a_1, a_2 and a_3. Then $f(z)=P$ (exp Az) with a quadratic polynomial P and a non-zero constant A.*

THEOREM B. *Let $f(z)$ be a transcendental entire function. Assume that there is an unbounded sequence $\{w_n\}$ so that each w_n is a linearly distributed value of $f(z)$. Then*

$$f(z)=P(\exp Az)$$

with a quadratic polynomial P and a non-zero constant A.

In this paper we shall give a characterization of the cosine function by the value distribution. Our form is of Baker's type. We need a deeper analysis for any extension of Kobayashi's type and this is just an open problem.

THEOREM 1. *Let $f(z)$ be a transcendental entire function, real for real z, of finite order. Assume that, for every real number w satisfying $w\geqq w_0$ or $w\leqq -w_0$, $f(z)=w$ has its roots on two straight lines l_{w1}, l_{w2} being parallel to the real axis and that, for every real number w satisfying $|w|<w_0$, $f(z)=w$ has its roots on two straight lines l_{w1}, l_{w2}. Then $f(z)=A\cos(Bz+C)+D$ with real constants A, B, C, D, $AB\neq 0$.*

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THEOREM 2. *Let $f(z)$ be a transcendental entire function, real for real z. Assume that $f(z)=w$ has either only non-real roots or only real roots for an arbitrary real number w. Then $f(z)=A\cos(Bz+C)+D$ with real constants A, B, C, D, $AB\neq 0$.*

In this theorem we cannot omit the reality of $f(z)$. However we have the following slightly extended theorem.

THEOREM 2′. *Let $f(z)$ be a transcendental entire function, having only real zeros and only real one-points. Assume that $f(z)=w$ has either only non-real roots or only real roots for an arbitrary real number w. Then $f(z)$ has one of the following three forms:*

1) $$f(z)=\frac{\sin(\xi z+\eta)\,e^{i(\xi z+\eta_1)}}{\sin(\eta-\eta_1)}, \qquad (\sin(\eta-\eta_1)\neq 0)$$

ξ, η, η_1: *real constants,*

2) $$f(z)=\frac{\sin p(\xi z+\eta)\,e^{i(p-1)(\xi z+\eta)}}{\sin(\xi z+\eta)}.$$

p: *an integer* ($\neq 0, 1$), $\qquad$ ξ, η: *real constants,*

3) $f(z)=A\cos(Bz+C)+D$,

A, B, C, D: *real*, $AB\neq 0$.

THEOREM 3. *Let $f(z)$ be a transcendental entire function, real for real z. Assume that $f(z)$ has only real zeros and only real one-points. Assume further that, for every real w, $f(z)=w$ has either only non-negative real roots or only non-real roots except only one negative root or only non-real roots. Then*

$$f(z)=A\cos B\sqrt{z}+D$$

with real A, B, D, $AB\neq 0$, $|D|\leqq|A|$, $|1-D|\leqq|A|$.

2. Lemmas. We need several known results.

LEMMA 1. [2]. *Let $f(z)$ be an entire function having only real zeros and real one-points. Then the order of $f(z)$ does not exceed one. If "real" is replaced by "positive", the order of $f(z)$ does not exceed 1/2.*

LEMMA 2. [2]. *Let $f(z)$ be a real entire function whose zeros and the one-points are all real. Then all the roots of $f(z)=h$ with real h in [0, 1] are real.*

LEMMA 3. [2]. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{w_\nu\}$ such that all the roots of the equations $f(z)=w_\nu$ ($\nu=1, 2, 3, \cdots$) be real. Then $f(z)$ is a quadratic polynomial.*

LEMMA 4. [2]. *Let $f(z)$ be an entire function which has only real zeros and real one-points. Assume that $f(z)$ is not real for real z. Then $f(z)$ is one of the following two forms:*

$$\text{i)}\quad f(z)=\frac{\sin(\xi z+\eta)\,e^{i(\xi z+\eta_1)}}{\sin(\eta-\eta_1)}\qquad (\sin(\eta-\eta_1)\neq 0),$$

where ξ, η, and η_1 are real constants,

$$\text{ii)}\quad f(z)=\frac{\sin p(\xi z+\eta)\,e^{i(p-1)(\xi z+\eta)}}{\sin(\xi z+\eta)},$$

where p ($\neq 0, 1$) is an integer and ξ, η are real constants.

LEMMA 5. [3]. *Let $f(z)$ be an entire function of finite lower order. If all the zeros of $f(z)$ ($f(z)-1$) lie in the strip $|\Im z|\leqq h$, the order of $f(z)$ is at most one.*

3. *Proof of Theorem* 2. Let E be the set of real numbers w for which $f(z)=w$ has only real roots and F the one for which $f(z)=w$ has no real root. By Lemma 3 E is bounded and hence $F\neq\phi$. Evidently $f(0)$ and $f(\epsilon)$ are real for a real number ϵ. Hence $f(z)=f(0)$ and $f(z)=f(\epsilon)$ have only real roots by our assumption. Hence Lemma 1 implies that $f(z)$ is of order at most one. By Lemma 2 E is a closed interval $[w_*, w^*]$, $w_*<w^*$. Let $\{x_j\}$ be the set of (real) roots of $f(z)=w^*$. Let x move the open interval $(x_j-\epsilon, x_j+\epsilon)$ along the real axis. Then $f(x)$ moves from $f(x_j-\epsilon)$ to $f(x_j)=w^*$ and then turns back from w^* to $f(x_j+\epsilon)$. $f(x)$ cannot traverse w^*. Hence x_j is a multiple root of $f(z)=w^*$ of even order. The same holds for w_*. Hence

$$f(z)-w^*=g(z)^2,$$

$$f(z)-w_*=h(z)^2.$$

Thus

$$(g(z)-h(z))(g(z)+h(z))=w_*-w^*\neq 0,$$

which shows that

$$h(z)-g(z)=\sqrt{w^*-w_*}\,e^{i(bz+c)}$$

$$h(z)+g(z)=\sqrt{w^*-w_*}\,e^{-i(bz+c)}$$

Hence

$$f(z)=\frac{w^*+w_*}{2}+\frac{w^*-w_*}{2}\cos 2(bz+c)$$

with real b, c. This is the desired result.

If the reality of $f(z)$ is omitted in this theorem, the result is not true. Let us consider the function i) in Lemma 4. For this function $f(z)$ and for a real number w

$$f(z)=w$$

can be solved and with $u=\xi z+\eta$, $x=\eta_1-\eta$

$$2u=-\imath\frac{1}{2}\log(1-4w\sin^2 x+4w^2\sin^2 x)+\theta+2p\pi+(\eta-\eta_1),$$

$$\cos x+\imath(1-2w)\sin x$$

$$=(1-4w\sin^2 x+4w^2\sin^2 x)^{1/2}e^{i\theta}$$

Hence for $w\neq 0, 1$ $f(z)=w$ has only non-real roots and for $w=0$ or $w=1$ $f(z)=w$ has only real roots.

Similarly we have the same fact for the function ii) in Lemma 4.

4. *Proof of Theorem* 3. Let us put $F(z)=f(z^2)$. Then $F(z)$ satisfies the assumption in Theorem 2. Hence $F(z)=A\cos(Bz+C)+D$. Further $F(z)$ is real for purely imaginary z. Hence $C=0$. Evidently $|D|\leqq|A|$, $|1-D|\leqq|A|$, since $f(z)$ has only real zeros and real ones.

5. *Proof of Theorem* 1. Firstly by Lemma 5 the order of f is at most one. Again by Lemma 3 for $w\geqq w_1$ and $w\leqq -w_1$ $l_{w1}\neq l_{w2}$. By the reality of $f(z)$

$$f(z)-w=Ce^{Az}\prod\left(1-\frac{z}{\bar{b}_j}\right)e^{z/\bar{b}_j}\prod\left(1-\frac{z}{b_j}\right)e^{z/b_j}$$

with real C and A. Further $\Im\bar{b}_j=\alpha_{w1}=-\alpha_{w2}=\Im b_j$. Hence for real y

$$\log|f(\imath y)-w|$$

$$=\log|C|+\frac{1}{2}\sum\log\left(1+\frac{2(\Re b_j)^2-2\alpha^2{}_{w1}}{|b_j|^4}y^2+\frac{y^4}{|b_j|^4}\right)$$

$$\geqq\log|C|+\frac{1}{2}\sum_{|j|\geqq n_0}\log\left(1+\frac{y^4}{|b_j|^4}\right)-\frac{K}{2}\log\frac{y^4}{|\alpha_{w1}|^4}$$

Here for $|j|\geqq n_0$ $2(\Re b_j)^2-2\alpha_{w1}^2\geqq 0$. The above inequality holds for sufficiently large y^4. Now consider

$$L(z)=\prod_{|j|\geqq n_0}\left(1+\frac{z}{|b_j|^4}\right),\quad \sum\frac{1}{|b_j|^2}<\infty$$

For this function the maximum modulus $M(r, L)=L(r)$. Further $\log M(r, L)/\log r\to\infty$ as $r\to\infty$. Therefore $\log|f(iy)-w|\to\infty$ as $y\to\pm\infty$ and so $|f(iy)|\to\infty$ as $y\to\pm\infty$. Now the Lindelöf-Iversen-Gross theorem [4] implies

$$|f(z)|\to\infty$$

uniformly as $|z|\to\infty$ in $|\arg z-\pi/2|\leqq\pi/2-\delta$ or in $|\arg z-3\pi/2|\leqq\pi/2-\delta$ for every $\delta>0$.

For every real w there occur five possibilities on l_{w1}, l_{w2}: 1) $l_{w1}\equiv l_{w2}$ and l_{w1} coincides with the real axis, 2) $l_{w1}\not\equiv l_{w2}$ but l_{w1}, l_{w2} are parallel to the real axis, 3) l_{w1} coincides with the real axis and l_{w2} is perpendicular to the real axis, 4) both of l_{w1}, l_{w2} are perpendicular to the real axis, and 5) l_{w1}, l_{w2} intesect at a real point and have inclinations m, $-m$ respectively. Here $0<m<\infty$.

Suppose that 4) occurs. Then on l_{w1}, l_{w2} there are only finitely many w-points of $f(z)$. Hence $f(z)=w+Pe^{az}$, where P is a real polynomial and a is a real non-zero constant. In this case it is easy to show that there are infinitely many roots of $f(z)=\tilde{w}$ for $\tilde{w}\neq w$ in the direction of the imaginary axis. This is absurd. Hence 4) does not occur. Similarly 5) does not occur either. If 3) occurs, there are only finitely many w-points of $f(z)$ on l_{w2}. Of course in this case there are at least two non-real w-points of $f(z)$ on l_{w2}, since, if not, 1) appears instead of 3).

Let E be the set of real numbers for which 1) occurs, F the one for which 2) occurs, and D the one for which 3) occurs. For w satisfying $|w|>w_0$ either 1) or 2) occurs. However by Lemma 3 the case 1) does not occur for any unbounded sequence $\{w_n\}$. Therefore F covers two unbounded parts of the real axis. Since $f(z)=f(0)$ has at least one real root 0, $E\cup D\neq\phi$ and is a bounded closed set. Let $\{w_*, w^*\}$ be the connected component of $E\cup D$ containing $f(0)$. Firstly $E\cup D=[w_*, w^*]$. Indeed $\{f(x)\}$ for real x is connected and $\{f(x)\}\subset E\cup D$ and further for every $w\in E\cup D$ there is a real number x such that $f(x)=w$. Next we shall prove that $w_*, w^*\in E$. Suppose that $w^*\in D$. Evidently w^* is an end-point of F, that is, $w\in F$ if $w>w^*$. Let $\{x_j\}$ and $\{z_s\}$ be the sets of real roots and of non-real roots of $f(z)=w^*$, respectively. As in the proof of Theorem 2 x_j is a multiple root of $f(z)=w^*$ of even order. Thus for every $w_1\in F$, $w^*<w_1<w^*-\delta$ for a sufficiently small $\delta>0$, $f(z)=w_1$ has two complex roots around all the x_j. Hence we have two l_{w_11}, l_{w_12} which are parallel to the real axis. If $\delta\to 0$, l_{w_11}, l_{w_12} tend to the real axis. $f(z)=w_1$ has no other root which does not lie on l_{w_11}, l_{w_12}. However every small neighborhood of z_s corresponds to some neighborhood of w^*. Hence there must exist a point z_s' such that $f(z_s')=w_1$. z_s' does not lie on l_{w_11}, l_{w_12}. This is a contradiction. This shows that $w^*\in E$. The same holds for w_*. Further $f(z)=w^*$

and $f(z)=w_*$ have only real roots of even order. Thus we have the desired result as in the proof of Theorem 2.

Theorem 2 does not hold without the reality of $f(z)$. Let us start from the second function in Lemma 4 with $p=3$. Let us put $u=\xi z+\eta$ and $x=\exp(2iu)$. For $f(z)=w$ we have $x^3-wx+w-1=0$. $x=1$ does not give any root of the original equation. Therefore

$$x=\frac{-1\pm\sqrt{4w-3}}{2}.$$

This gives the roots

$$u=-\frac{i}{2}\log\frac{\sqrt{4w-3-1}}{2}+q\pi$$

and

$$u=-\frac{i}{2}\log\frac{\sqrt{4w-3+1}}{2}+\frac{\pi}{2}+q\pi$$

with integers $q=0, \pm 1, \cdots$. For $w=1$, the first members are disappeared. Except for $w=0$ and $w=1$, and $f(z)=w$ has roots lying on two distinct straight lines which are parallel to the real axis.

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Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku,
Tokyo, Japan