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KODAI MATH. J.
24 (2001), 42–47

A REMARK ON EXPONENTIAL GROWTH AND THE SPECTRUM OF THE LAPLACIAN

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Abstract

In terms of the exponential growth of a non-compact Riemannian manifold, we give an upper bounds for the bottom of the essential spectrum of the Laplacian. This is an improvement of Brooks' result.

1. Introduction

Let M be a smooth, complete, non-compact Riemannian manifold, and Δ the Laplace-Beltrami operator on $L^2(M)$, where its sign is chosen so that it becomes a positive operator. We denote by λ_0 the bottom (that is, the greatest lower bound) of the spectrum of Δ and by λ_0^{ess} the bottom of the essential spectrum. It is easy to see that $\lambda_0 \le \lambda_0^{ess}$, and that $\lambda_0^{ess} = \lim_K \lambda_0(M-K)$, where K runs over an increasing set of compact subdomains of M such that $\cup K = M$ and $\lambda_0(M-K)$ stands for the bottom of the spectrum of Δ with the Dirichlet boundary condition on ∂K. For a compact manifold, the essential spectrum is empty, thus we put $\lambda_0^{ess} = \infty$.

There exist many works on the estimates for λ_0 or λ_0^{ess} (for instance, [1], [2], [3], [4], [5], [7], [8]). Among them, R. Brooks ([2], [3]) has given the upper bounds for λ_0^{ess} in terms of the *volume growth*: Pick a point $x_0 \in M$ and let $B(r)$ be the ball of radius r around x_0 and $V(r)$ the volume of this ball. It is shown that $\lambda_0^{ess} \le \bar{\mu}_v^2/4$ if the volume of M is infinite in [2] and that $\lambda_0^{ess} \le \bar{\mu}_f^2/4$ if that is finite in [3], where $\bar{\mu}_v$ and $\bar{\mu}_f$ are the exponential volume growth of M, respectively, defined as

$$(1.1)\quad \bar{\mu}_v = \limsup_{r\to\infty} \frac{1}{r} \log V(r) \quad \text{and} \quad \bar{\mu}_f = \limsup_{r\to\infty} \frac{-1}{r} \log(Vol(M) - V(r)).$$

The purpose of this note is to give an estimate for λ_0^{ess} using another kind of exponential growth than Brooks'. This estimate is not only a slight improve-

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1991 *Mathematics Subject Classification*: Primary 58G25, Secondary 35P15.

Key words and phrases: spectral geometry, growth function, essential spectrum.

[1] Partially supported by the Ministry of Education, Science, Sports and Culture of Japan under the Grant-in-Aid No. 09304022.

Received February 21, 2000; revised May 10, 2000.

ment of Brooks' results but a continuous analogue of the result for the discrete Laplacian on an infinite graph ([6]).

Let us state our result. For the ball $B(r) = B(r, x_0)$ of radius r whose origin is an arbitrary fixed point $x_0 \in M$ and any fixed positive number δ, we set

$$B_\delta(\partial B(r)) = \{x \in M - B(r) \mid \rho(x, \partial B(r)) \le \delta\}, \tag{1.2}$$

where $\rho(x, \partial B(r))$ is the distance between x and $\partial B(r)$, and denote by $S_\delta(r)$ the volume of $B_\delta(\partial B(r))$. Moreover, we set

$$\mu_0 = \lim_{\delta\to 0} \liminf_{r\to\infty} \mu_\delta(r) \quad \text{and} \quad \bar{\mu}_0 = \lim_{\delta\to 0} \limsup_{r\to\infty} \mu_\delta(r), \tag{1.3}$$

where $\mu_\delta(r) = (1/r) \log S_\delta(r)$. Our result is the following:

Theorem 1. *For a non-compact manifold M, we have $\lambda_0^{ess} \le \mu^2/4$, where $\mu = \max(\mu_0, 0)$ if the volume of M is infinite and $\mu = \bar{\mu}_0$ if that is finite.*

We have $0 \le \mu \le \bar{\mu}_v$ in the infinite case and $|\bar{\mu}_0| \le \bar{\mu}_f$ in the finite as is seen later. In this sense, Theorem 1 is somewhat better than Brooks'. The next corollary follows from Theorem 1 directly. We set

$$\mu_v = \liminf_{r\to\infty} \frac{1}{r} \log V(r) \quad \text{and} \quad \mu_f = \liminf_{r\to\infty} \frac{-1}{r} \log(Vol(M) - V(r)). \tag{1.4}$$

Corollary 2. *We have $\lambda_0^{ess} \le \mu_v^2/4$ if the volume of M is infinite, and $\lambda_0^{ess} \le \mu_f^2/4$ if that is finite.*

The author would like to express his sincere gratitude to Professors Tomoyuki Shirai and Yoichiro Takahashi for their comments.

2. Proof of Theorem 1

For an arbitrary fixed $\delta > 0$, we set

$$\mu_\delta = \liminf_{r\to\infty} \mu_\delta(r) \quad \text{and} \quad \bar{\mu}_\delta = \limsup_{r\to\infty} \mu_\delta(r), \tag{2.1}$$

where $\mu_\delta(r) = (1/r) \log S_\delta(r)$. Theorem 1 follows from the following:

Theorem 3. 1) *If M has infinite volume, then $\lambda_0^{ess} \le \mu_\delta^2/4$ for any fixed $\delta > 0$. Moreover, $\lambda_0^{ess} = 0$ if $\mu_\delta < 0$.*
2) *If M has finite volume, then $\lambda_0^{ess} \le \bar{\mu}_\delta^2/4$ for any fixed $\delta > 0$.*

Proof of Theorem 1 *and Corollary* 2 *from Theorem* 3. We first assume that M has infinite volume. It is obvious that $\mu_{\delta_1} \le \mu_{\delta_2}$ if $\delta_1 < \delta_2$. Then there exists $\mu_0 = \lim_{\delta\to 0} \mu_\delta$, and we clearly have

$$\mu_0 \le \mu_\delta \le \mu_v \le \bar{\mu}_v. \tag{2.2}$$

Let $\mu_0 < 0$. Then, from the above, there exists $\delta > 0$ such that $\mu_\delta < 0$. Thus $\lambda_0^{ess} = 0$.

Next, we assume that M has finite volume. It holds also here that $\bar{\mu}_{\delta_1} \le \bar{\mu}_{\delta_2}$ if $\delta_1 < \delta_2$. Therefore there exist $\bar{\mu}_0 = \lim_{\delta \to 0} \bar{\mu}_\delta$ and $\bar{\mu}_\infty = \lim_{\delta \to \infty} \bar{\mu}_\delta$; we have

$$\bar{\mu}_0 \le \bar{\mu}_\delta \le \bar{\mu}_\infty \le \limsup_{r \to \infty} \frac{1}{r} \log(Vol(M) - V(r)) = -\mu_f \le 0. \tag{2.3}$$

Let $\bar{\mu}_\delta < 0$. In this case, for any $\varepsilon > 0$ such that $\bar{\mu}_\delta + \varepsilon < 0$, there exists r_0 such that, for any $r \ge r_0$, $(1/r) \log S_\delta(r) < \bar{\mu}_\delta + \varepsilon$. Then we get, for any $r \ge r_0$,

$$Vol(M) - V(r) = \sum_{k=0}^{\infty} S_\delta(r + k\delta) \le \sum_{k=0}^{\infty} \exp((\bar{\mu}_\delta + \varepsilon)(r + k\delta)).$$

Thus, we have

$$\frac{1}{r} \log(Vol(M) - V(r)) \le \bar{\mu}_\delta + \varepsilon - \frac{1}{r} \log(1 - \exp((\bar{\mu}_\delta + \varepsilon)\delta))$$

and $-\mu_f \le \bar{\mu}_\delta + \varepsilon$. Since we can select arbitrary small $\varepsilon > 0$, $-\mu_f \le \bar{\mu}_\delta$. Therefore, by (2.3), $-\mu_f = \bar{\mu}_\delta$; we have also $-\mu_f = \bar{\mu}_\delta$ for any $\delta > 0$. Moreover, we easily get $\bar{\mu}_\delta = 0$ for any δ and $\bar{\mu}_\delta = -\mu_f = 0$ if $\bar{\mu}_\delta = 0$ for some δ. Consequently, we have $\bar{\mu}_0 = \bar{\mu}_\infty = -\mu_f$ and $|\bar{\mu}_0| = \mu_f \le \bar{\mu}_f$. Hence the proof is completed. □

Remark 4. It is obvious that $\lim_{\delta \to 0} S_\delta(r)/\delta = S(r)$, where $S(r) = S(r, x_0)$ is the surface area of the distance sphere of radius r at x_0. In addition, it is also obvious that $\liminf_{r \to \infty} (1/r) \log(S_\delta(r)/\delta) = \mu_\delta$ and $\limsup_{r \to \infty} (1/r) \log(S_\delta(r)/\delta) = \bar{\mu}_\delta$ for any fixed δ. Note that, for any r and any fixed $\delta > 0$, there exists $q_{r,\delta} \in (r, r + \delta)$ such that

$$S(q_{r,\delta}) = (B(r + \delta) - B(r))/\delta = S_\delta(r)/\delta.$$

Then, setting $\mu_s = \liminf_{r \to \infty} (1/r) \log S(r)$ and $\bar{\mu}_s = \limsup_{r \to \infty} (1/r) \log S(r)$, we have $\mu_s \le \mu_0 \le \bar{\mu}_0 \le \bar{\mu}_s$. When $\mu_s = \mu_0$ or $\bar{\mu}_s = \bar{\mu}_0$, we can substitute in Theorem 1 μ_s or $\bar{\mu}_s$ for μ_0 or $\bar{\mu}_0$, respectively.

Now let us prove Theorem 3 following an idea in [6] and Brooks' one in [2].

Proof of Theorem 3. Let $\lambda_0(M - K)$ be the bottom of the spectrum of Δ on $L^2(M - K)$ with the Dirichlet boundary condition on ∂K. It is well-known that

$$\lambda_0(M - K) = \inf \frac{\int_M \|\mathrm{grad}\, f\|^2}{\int_M f^2}, \tag{2.4}$$

where f runs over uniformly Lipshitz functions with compact support on $M - K$. Then we only have to show the following: for any fixed $\delta > 0$, for any compact subset K and for any sufficiently small $\varepsilon, \varepsilon_1 > 0$, there exists a function f supported in $M - K$ such that

$$\frac{\int_M \|\text{grad } f\|^2}{\int_M f^2} < \alpha^2(\varepsilon) + \varepsilon_1, \tag{2.5}$$

where $\alpha(\varepsilon) \to \mu_\delta/2$ as $\varepsilon \to 0$.

Consider a test function $f(x) = \exp(h(x)) \cdot \chi(x)$, where $\chi(x)$ has compact support in $M - K$. Then

$$\begin{aligned}\int_{M-K} \|\text{grad } f\|^2 &= \int_{M-K} e^{2h(x)}(\|\text{grad } h \cdot \chi + \text{grad } \chi\|^2) \\ &\leq \int_{M-K} e^{2h(x)}(2\chi \cdot \langle \text{grad } h, \text{grad } \chi\rangle + \|\text{grad } \chi\|^2) \\ &\quad + \int_{M-K} f^2 \|\text{grad } h\|^2.\end{aligned} \tag{2.6}$$

For $x \in M$, let $\rho(x) = \rho(x, x_0)$ denote the distance from a fixed point $x_0 \in M$. For r sufficiently large so that $K \subset B(r - \delta)$, we set χ as follows:

$$\chi(x) = \chi_r(x) = \begin{cases} 0, & \text{if } x \in K \text{ or } \rho(x) > r + \delta, \\ \rho(x, K)/\delta, & \text{if } 0 < \rho(x, K) \leq \delta, \\ 1 - \rho(x, B(r))/\delta, & \text{if } r \leq \rho(x) \leq r + \delta, \\ 1, & \text{otherwise.} \end{cases} \tag{2.7}$$

Then grad χ is supported in $B_\delta(\partial B(r))$ and a neibourhood $B_\delta(\partial K)$ of radius δ about ∂K; moreover, $\|\text{grad } \chi\| \leq 1/\delta$. In addition, we put, for a fixed number $\alpha \geq 0$ and for a positive integer j,

$$h_j(x) = \begin{cases} \alpha\rho(x), & \text{if } \rho(x) \leq j, \\ 2\alpha j - \alpha\rho(x), & \text{if } \rho(x) > j. \end{cases} \tag{2.8}$$

Note that, for every j, $\|\text{grad } h_j\|^2 \leq \alpha^2$, and that h_j increases pointwise to $h = \alpha\rho$. Thus, for r and j sufficiently large and $r > j$, we have

$$\int_{M-K} \|\text{grad } f\|^2 \leq \alpha^2 \int_{M-K} f^2 + (2\alpha/\delta + 1/\delta^2)\left(\int_{B_\delta(\partial K)} e^{2h_j} + \int_{B_\delta(\partial B(r))} e^{2h_j}\right) \tag{2.9}$$

and there exists a finite constant C independent of r and j such that

$$(2\alpha/\delta + 1/\delta^2)\int_{B_\delta(\partial K)} e^{2h_j} \leq C. \tag{2.10}$$

From here, we divide our proof into two cases: the case of M with infinite volume and that of M with finite volume.

First, we assume that M has infinite volume. Then it is obvious that

$$\int_{M-K} f^2 = \int_{M-K} e^{2h_j}\chi_r^2 \to \infty \quad \text{as } r, j \to \infty. \tag{2.11}$$

Let $\mu_\delta \geq 0$. It follows form the definition of μ_δ that, for any $\varepsilon > 0$, there exists a sequence $\{r_n\}$ such that

$$\mu_\delta(r_n) = \frac{1}{r_n} \log S_\delta(r_n) \leq \mu_\delta + \varepsilon \tag{2.12}$$

and $r_n > 2j(2 + \mu_\delta/\varepsilon)$ for every n. Therefore, setting $\alpha = \alpha(\varepsilon) = (\mu_\delta + 2\varepsilon)/2$, we have

$$\int_{B_\delta(\partial B(r_n))} e^{2h_j} \leq \exp((2j - r_n)(\mu_\delta + 2\varepsilon)) \cdot \exp((\mu_\delta + \varepsilon)r_n) \leq 1. \tag{2.13}$$

By (2.9), (2.10), (2.11) and (2.13), we can select n and j such that

$$\frac{\int_M \|\text{grad } f_n\|^2}{\int_M f_n^2} \leq \alpha^2(\varepsilon) + \varepsilon_1 \tag{2.14}$$

for any $\varepsilon_1 > 0$, where $f_n = e^{h_j}\chi_{r_n}$.

If $\mu_\delta < 0$, then, for any $\varepsilon > 0$ satisfying $\mu_\delta + \varepsilon < 0$, there exists a sequence $\{r_n\}$ such that $\mu_\delta(r_n) \leq \mu_\delta + \varepsilon$. Setting $\alpha = 0$, that is, $\exp(h_j(x)) = \mathbf{1}$, we have

$$\int_{B_\delta(\partial B(r_n))} e^{2h_j} = S_\delta(r_n) \leq \exp((\mu_\delta + \varepsilon)r_n) < 1. \tag{2.15}$$

Thus, for any $\varepsilon_1 > 0$, we can select n such that $\int_M \|\text{grad } f_n\|^2 / \int_M f_n^2 \leq \varepsilon_1$, where $f_n = \mathbf{1}\chi_{r_n}$. We finish the proof in the case of infinite volume.

Next, let M have finite volume. Then, we clearly have $-\infty \leq \bar{\mu}_\delta \leq 0$; we may assume $-\infty < \bar{\mu}_\delta \leq 0$. It follows form the definition of $\bar{\mu}_\delta$ that, for any sufficiently small $\varepsilon > 0$, there exists a sequence $\{r_n\}$ such that

$$\bar{\mu}_\delta - \varepsilon \leq \mu_\delta(r_n) = \frac{1}{r_n} \log S_\delta(r_n) \leq \bar{\mu}_\delta + \varepsilon \tag{2.16}$$

and $r_n > 2j(2\varepsilon - \bar{\mu}_\delta)/(\varepsilon - 2\bar{\mu}_\delta)$ for every n. Here we can assume this sequence $\{r_n\}$ satisfies $r_{n+1} - r_n \geq \delta$ for any n. Setting $\alpha = \alpha(\varepsilon) = -(\bar{\mu}_\delta - 2\varepsilon)/2$ and $g(x) = e^{\alpha\rho(x)}$, we have

$$\int_{B(r)} g^2 \geq \sum_{A(r)} \int_{B(r_n+\delta) - B(r_n)} g^2 \geq \sum_{A(r)} \exp(2\alpha r_n) \cdot S_\delta(r_n) \geq \sum_{A(r)} \exp(\varepsilon r_n) \to \infty \tag{2.17}$$

as $r \to \infty$, where $A(r) = \{n | r_n + \delta \leq r\}$. Then we have

$$\int_{M-K} f^2 = \int_{M-K} e^{2h_j}\chi_r^2 \geq \int_{B(j)} g^2 - \int_K g^2 \to \infty \quad \text{as } r, j \to \infty \tag{2.18}$$

and

$$\int_{B_\delta(\partial B(r_n))} e^{2h_j} \leq \exp((2j - r_n)(2\varepsilon - \bar{\mu}_\delta)) \cdot \exp((\bar{\mu}_\delta + \varepsilon)r_n) \leq 1. \tag{2.19}$$

In the same way as in the case of infinite volume, selecting sufficiently large n and j, one obtain the desired estimate. $\square$

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