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KODAI MATH. J.
24 (2001), 36–41

SOME q-IDENTITIES ASSOCIATED WITH RAMANUJAN'S CONTINUED FRACTION

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Abstract

A continued fraction $C(-q,q)$ is defined as a special case of a general continued fraction $F(a,b,c,\lambda,q)$, which we have considered earlier in a separate paper. This continued fraction is also a special case of Ramanujan's continued fraction. In this paper we have found some very interesting q-identities and some identities analogous to identities given by Ramanujan involving $G(-q,q)$ and $H(-q,q)$ and one identity which gives the square of a continued fraction.

1. Introduction

In an earlier paper [4] we considered the continued fraction $C(-q,q)$, which is a special case of a continued fraction

$$F(a,b,c,\lambda,q) = 1 + \frac{(1-1/c)(aq+\lambda q)}{(1+aq/c)+} \, \frac{bq+\lambda q^2}{1+} \times \frac{(1-1/cq)(aq^2+\lambda q^3)}{(1+aq/c)+} \, \frac{bq^2+\lambda q^4}{1+\cdots\cdots} \tag{1.1}$$

$$= \frac{\sum_{n=0}^{\infty} (-\lambda/a)_n (c)_n (-aq/c)^n/(q)_n(-bq)_n}{\sum_{n=0}^{\infty} (-\lambda/a)_n (c)_n (-aq^2/c)^n/(q)_n(-bq)_n}$$

which is a generalization of the continued fraction of Ramanujan [1]. In the limit this continued fraction gives most of the classical results in continued fractions. For $c \to \infty$, it reduces to the *unusual* continued fraction of Ramanujan, which Andrews discovered in the "Lost" manuscript of Ramanujan [2]. By taking $a=0, b=0, \lambda=1$ and $c=-q$ in (1.1), we have the continued fraction $C(-q,q)$.

1991 *Mathematics Subject Classification*: 33D
Key words and phrases: continued fraction, basic hypergeometric series
Received January 24, 2000; revised July 25, 2000.

$$C(-q,q) = 1 + \frac{(1+1/q)q}{1+}\,\frac{q^2}{1+}\,\frac{(1+1/q^2)q^2}{1+}\,\frac{q^4}{1+\cdots\cdots}$$

$$= 1 + \frac{\sum_{n=0}^{\infty} q^{n(n-1)/2}(-q)_n/(q)_n}{\sum_{n=0}^{\infty} q^{n(n+1)/2}(-q)_n/(q)_n}$$

$$= 1 + \frac{(q^2;q^4)^2}{(q^3;q^4)_\infty (q;q^4)_\infty} \qquad \left(\begin{array}{l}\text{by using summation formula Slater} \\ \text{[3, eqn 8 and 13]} \\ \text{Also [4, p. 200, eqn. 2.2]}\end{array}\right)$$

2. Notation

$$(a;q^k)_n = \prod_{j=0}^{n-1}(1-aq^{kj}); \quad n \geq 1$$

$$(a;q^k)_0 = 1$$

$$(a;q^k)_\infty = \prod_{j\geq 0}(1-aq^{kj}),$$

when $k=1$, q^k shall be omitted from the various symbols, in case there is no chance of ambiguity.

3. An interesting q-identity

We shall prove the identity

$$(3.1) \qquad \left[\frac{(1+1/q)q}{1+}\,\frac{q^2}{1+}\,\frac{(1+1/q^2)q^3}{1+}\,\frac{q^4}{1+\cdots\cdots}\right]^2$$

$$= \frac{(q^2;q^4)_\infty^2}{(q^4;q^4)_\infty^2}\left[\sum_{n=0}^{\infty} q^{4n^2+2n}\frac{1+q^{4n+1}}{1-q^{4n+1}} - \sum_{n=0}^{\infty} q^{4n^2+6n+2}\frac{1+q^{4n+3}}{1-q^{4n+3}}\right]$$

The proof of this identity depends on Ramanujan's ${}_1\Psi_1$-summation [1, p. 101], namely

$$(3.2) \qquad {}_1\Psi_1[a;b;q,z] = \sum_{n=-\infty}^{\infty}(a)_n z^n/(b)_n$$

$$= \frac{(b/a)_\infty (az)_\infty (q/az)_\infty (q)_\infty}{(q/a)_\infty (b/az)_\infty (b)_\infty (z)_\infty}$$

We shall first prove a series of identity:

$$\sum_{n=0}^{\infty}\frac{q^{in}}{1-q^{4n+i}}=\sum_{n=0}^{\infty}q^{4n^2+2in}\frac{1+q^{4n+i}}{1-q^{4n+i}},\quad i=1,2,3. \tag{3.3}$$

Now

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{in}}{1-q^{4n+i}}&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{in+4nm+im}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^{i(n+m)+4(n+m)m+im}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{in+4n(m+n+1)+i(m+n+1)}\\&=\sum_{m=0}^{\infty}\frac{q^{4m^2+2im}}{1-q^{4m+i}}+\sum_{n=0}^{\infty}\frac{q^{4n^2+4n+2in+i}}{1-q^{4n+i}}\\&=\sum_{n=0}^{\infty}q^{4n^2+2in}\frac{1+q^{4n+i}}{1-q^{4n+i}}\end{aligned}$$

This proves (3.3).

By replacing q by q^4 and then setting $a=q^i$, $b=q^{4+i}$, $z=q^i$ in (3.2), we have for $i=1,3$.

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{q^{in}}{1-q^{4n+i}}&=\frac{1}{1-q^i}\,{}_1\Psi_1(q^i;q^{4+i};q^4,q^i)\\&=\frac{(q^4;q^4)_\infty^2(q^{2i};q^4)_\infty(q^{4-2i};q^4)_\infty}{(q^{4-i};q^4)_\infty^2(q^i;q^4)_\infty^2}\end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{4n^2+2n}\frac{1+q^{4n+1}}{1-q^{4n+1}}-\sum_{n=0}^{\infty}q^{4n^2+6n+2}\frac{1+q^{4n+3}}{1-q^{4n+3}}\\&=\sum_{n=0}^{\infty}\frac{q^n}{1-q^{4n+1}}-\sum_{n=0}^{\infty}\frac{q^{3n+2}}{1-q^{4n+3}},\quad\text{by (3.3)}\\&=\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+1}}\\&=\frac{(q^4;q^4)_\infty^2(q^2;q^4)_\infty^2}{(q^3;q^4)_\infty^2(q;q^4)_\infty^2},\quad\text{by (3.4)}\\&=\frac{(q^4;q^4)_\infty^2(q^2;q^4)_\infty^4}{(q^2;q^4)_\infty^2(q^3;q^4)_\infty^2(q;q^4)_\infty^2}\\&=\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}\left[\frac{(1+1/q)q}{1+}\,\frac{q^2}{1+}\,\frac{(1+1/q^2)q^3}{1+}\,\frac{q^4}{1+\cdots\cdots}\right]^2.\end{aligned}$$

This proves (3.1).

4. Some more identities

Let us define

$$G(-q,q)=\sum_{n=0}^{\infty}q^{n(n-1)/2}(-q)_n/(q)_n \tag{4.1}$$

$$=(-q,q)_\infty\left[\frac{1}{(q^2;q^4)_\infty}+\frac{(q^2;q^4)_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\right],$$

using summation formula Slater [3, eqn. 3] and

$$H(-q,q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}(-q)_n/(q)_n \tag{4.2}$$

$$=\frac{(-q;q)_\infty}{(q^2;q^4)_\infty}$$

using summation formula Slater [3, eqn. 8].

Obviously

$$C(-q,q)=\frac{G(-q,q)}{H(-q;q)} \tag{4.3}$$

and

$$G(-q,q)=H(-q,q)+\frac{(-q;q)_\infty(q^2;q^4)_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}. \tag{4.4}$$

Next we give a generalization of (3.4), namely

$$\sum_{n=-\infty}^{\infty}\frac{q^{in}}{1-q^{4n+j}}=\frac{(q^4;q^4)_\infty^2(q^{i+j};q^4)_\infty(q^{4-i-j};q^4)_\infty}{(q^j;q^4)_\infty(q^{4-j};q^4)_\infty(q^i;q^4)_\infty(q^{4-i};q^4)_\infty} \tag{4.5}$$

where $0<i\le 3$, $0<j\le 3$ and $i+j\ne 4$.

The proof is similar to that of (3.4).

With the help of (4.1), (4.2), (4.4) and (4.5) we have the following identities, which are analogous to the identities of Ramanujan [2, pp. 197–198]

$$\frac{(q^4;q^4)_\infty^2}{(-q;q)_\infty^2}[H(-q,q)]^2=\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+2}},\quad (i=1,j=2 \text{ in } (4.5)) \tag{4.6}$$

$$\frac{(q^4;q^4)_\infty^2}{(-q;q)_\infty^2}[G(-q;q)-H(-q;q)]^2=\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+1}},\quad (i=1,j=1 \text{ in } (4.5)) \tag{4.7}$$

(4.8) $$\frac{(q^4;q^4)_\infty^2}{(-q;q)_\infty^2}[H(-q,q)]^2 = \sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{4n+1}}, \quad (i=2, j=1 \text{ in } (4.5))$$

(4.9) $$\frac{(q^4;q^4)_\infty^2}{(-q;q)_\infty^2}[G(-q,q)-(H(-q,q)]^2 = \sum_{n=-\infty}^{\infty} q^{2n(2n+1)}\frac{1+q^{4n+1}}{1-q^{4n+1}},$$

(Using (4.7) and and putting in for $i = 1$)

5. Another continued fraction of the rogers-ramanujan type

By making $q \to q^2$ and then putting $a = 0$, $b = 0$, $c = q^2$ and $\lambda = (1 + 1/q)$ in (1.1), we have

(5.1) $$\frac{(1-1/q^2)(q+q^2)}{1+}\,\frac{q^3+q^4}{1+}\,\frac{(1-1/q^4)(q^5+q^6)}{1+\cdots\cdots}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n(1+1/q)^n q^{n^2+n}}{\sum_{n=0}^{\infty}(-1)^n(1+1/q)^n q^{n^2-n}}$$

Let

(5.2) $$\frac{\sum_{n=0}^{\infty}(-1)^n(1+1/q)^n q^{n^2+n}}{\sum_{n=0}^{\infty}(-1)^n(1+1/q)^n q^{n^2-n}} = \sum_{n=0}^{\infty} a_n q^n, \quad \text{(say)}$$

Thus

$$1-(1+1/q)q^2+(1+1/q)^2q^6-(1+1/q)^3q^{12}+\cdots\cdots\cdots\cdots\cdots$$
$$= (a_0+a_1q+a_2q^2+\cdots\cdots)$$
$$\times[1-(1+1/q)+(1+1/q)^2q^2-(1+1/q)^3q^6\cdots\cdots\cdots]$$

or

$$q-q^2-q^3+q^5+2q^6+q^7-q^{10}-3q^{11}-3q^{12}-q^{13}\cdots\cdots\cdots\cdots$$
$$= (a_0+a_1q+a_2q^2+\cdots\cdots)$$
$$\times(-1+q+2q^2+q^3-q^4-3q^5-3q^6-q^7+\cdots\cdots\cdots)$$

Comparing the coefficients of the various powers of q, we get

$$\sum_{n=1}^{\infty} a_n q^n = -q-q^3-2q^4-4q^5-8q^6-\cdots\cdots\cdots\cdots$$

Hence

$$\frac{(1-1/q^2)(q+q^2)}{1+} \frac{q^3+q^4}{1+} \frac{(1-1/q^4)(q^5+q^6)}{1+\cdots\cdots}$$
$$= -q - q^3 - 2q^4 - 4q^5 - 8q^6 - \cdots\cdots\cdots\cdots$$

This continued fraction $C(-q, q)$ is also a special case of Ramanujan's work (Entry 9 and 13 in chapter 16 of Ramanujan's Second Note Book Memoirs of the AMS 53 (1985) No. 315).

The present paper was motivated by Andrews' treatment of the Rogers-Ramanujan continued fraction [1, 2] and the technique employed in the proof is a straight forward modification of his technique.

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