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MEAN GROWTH OF THE DERIVATIVE OF A BLASCHKE PRODUCT

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Abstract

If B is a Blaschke product with zeros $\{a_n\}$ and if $\sum_n (1-|a_n|)^\alpha$ is finite for some $\alpha \in (1/2, 1]$, then limits are found on the rate of growth of $\int_0^{2\pi} |B'(re^{it})|^p \, dt$ in agreement with a known result for $\alpha \in (0, 1/2)$. Also, a converse is established in the case of an interpolating Blaschke product, whenever $0 < \alpha < 1$.

1. Preliminaries

If $\{a_n\}$ is a sequence of complex numbers such that $0 < |a_n| < 1$ for all $n = 1, 2, \ldots$ and $\sum_n (1-|a_n|) < \infty$, the Blaschke product

$$B(z) = \prod_n \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z}$$

is an analytic function in the open unit disc U with zeros $\{a_n\}$. For each p with $0 < p < \infty$, the Hardy space H^p is the set of all functions analytic in U for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt$$

is finite. In [8], it is shown that if B is a Blaschke product with zeros $\{a_n\}$ such that

$$\sum_n (1-|a_n|)^\alpha < \infty \tag{1}$$

for some $\alpha \in (0, 1/2)$, then $B' \in H^{1-\alpha}$. In [7], M. Kutbi extended this result to the following Theorem.

Theorem A. *Let B be a Blaschke product with zeros $\{a_n\}$ such that condition* (1) *holds for some $\alpha \in (0, 1/2)$. Then for each $p > 1-\alpha$,*

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$$\int_0^{2\pi} |B'(re^{it})|^p \, dt = o\left(\frac{1}{(1-r)^{p+\alpha-1}}\right). \tag{2}$$

Kutbi also showed by means of examples that the exponent $p+\alpha-1$ is sharp.

If $1/2 < \alpha < 1$, the convergence of $\sum_n (1-|a_n|)^\alpha$ does not imply that $B' \in H^{1-\alpha}$; in fact, by a theorem of Frostman [3], there are Blaschke products B such that (1) holds for all $\alpha > 1/2$ and yet B' is not in the Nevanlinna class N. Nevertheless, we will show that Kutbi's theorem has an analogue in the case of $1/2 < \alpha < 1$. We will then restrict our attention to interpolating Blaschke products and investigate to what extent the condition (1) is necessary as well as sufficient for a condition like (2).

We say $g(r) = o(1/(1-r)^q)$ if $\lim_{r\to 1^-} g(r)(1-r)^q = 0$, and $g(r) = O(1/(1-r)^q)$ if $g(r)(1-r)^q$ is bounded for $r \in (0,1)$. Also, we will write $a_n \gtrsim b_n$ if there is a constant C such that $a_n \geq C b_n$ for all n. Finally, $a_n \asymp b_n$ if $a_n \gtrsim b_n$ and $b_n \gtrsim a_n$.

2. An analogue of Kutbi's theorem

To get our analogue of Theorem A, we will go through three steps.

Lemma 1. *Let B be a Blaschke product with zeros $\{a_n\}$ such that condition (1) holds for some $\alpha \in (1/2, 1]$. Then*

$$\sum_{n=1}^{\infty} \frac{(1-r_n)^\alpha}{(1-r_n r)^{2\alpha-1}} = o\left(\frac{1}{(1-r)^{2\alpha-1}}\right),$$

where $r_n = |a_n|$ for all n.

Proof. Following Kutbi, we note that

$$\frac{(1-r_n)^\alpha}{(1-r_n r)^{2\alpha-1}} < \frac{(1-r_n)^\alpha}{(1-r)^{2\alpha-1}} \quad \text{and} \quad \frac{(1-r_n)^\alpha}{(1-r_n r)^{2\alpha-1}} < (1-r_n)^{1-\alpha} \tag{3}$$

since $0 < r < 1$, $0 < r_n < 1$, and $\alpha > 1/2$. Let $\varepsilon > 0$. There exists an integer N such that $\sum_{n=N+1}^{\infty} (1-r_n)^\alpha < \varepsilon/2$. Then it follows from (3) that

$$\sum_{n=1}^{\infty} \frac{(1-r_n)^\alpha}{(1-r_n r)^{2\alpha-1}} < \sum_{n=1}^{N} (1-r_n)^{1-\alpha} + \frac{\varepsilon}{2(1-r)^{2\alpha-1}}.$$

Choose R such that

$$\sum_{n=1}^{N} (1-r_n)^{1-\alpha} < \frac{\varepsilon}{2(1-r)^{2\alpha-1}}$$

for all r with $R < r < 1$. Then $R < r < 1$ implies that

$$(1-r)^{2\alpha-1}\sum_{n=1}^{\infty}\frac{(1-r_n)^{\alpha}}{(1-r_n r)^{2\alpha-1}} < \varepsilon. \qquad \square$$

LEMMA 2. *Let B be a Blaschke product with zeros $\{a_n\}$. Suppose $1/2 < \alpha \leq 1$. If* (1) *holds, then*

$$\int_0^{2\pi} |B'(re^{it})|^{\alpha}\, dt = o\left(\frac{1}{(1-r)^{2\alpha-1}}\right).$$

Proof. Since $\alpha \leq 1$,

$$|B'(z)|^{\alpha} < \sum_{n=1}^{\infty}\left|\frac{1-|a_n|^2}{(1-\bar{a}_n z)^2}\right|^{\alpha} < 2^{\alpha}\sum_{n=1}^{\infty}\frac{(1-r_n)^{\alpha}}{|1-\bar{a}_n z|^{2\alpha}}.$$

Then since $2\alpha > 1$,

$$\begin{aligned}\int_0^{2\pi} |B'(re^{it})|^{\alpha}\, dt &< 2^{\alpha}\sum_{n=1}^{\infty}(1-r_n)^{\alpha}\int_0^{2\pi}\frac{1}{|1-\bar{a}_n re^{it}|^{2\alpha}}\, dt \\ &\leq C\sum_{n=1}^{\infty}\frac{(1-r_n)^{\alpha}}{(1-r_n r)^{2\alpha-1}}\end{aligned}$$

for some constant C independent of r. For the last inequality, one can see, for example, [2], pp. 65–66. Now the result follows from Lemma 1. $\square$

THEOREM 1. *Let B be a Blaschke product with zeros $\{a_n\}$. Suppose $1/2 < \alpha \leq 1$ and $p \geq \alpha$. If* (1) *holds, then*

$$\int_0^{2\pi} |B'(re^{it})|^{p}\, dt = o\left(\frac{1}{(1-r)^{p+\alpha-1}}\right).$$

Proof. Since $|B'(z)| \leq 1/(1-|z|)$,

$$\begin{aligned}\int_0^{2\pi} |B'(re^{it})|^{p}\, dt &= \int_0^{2\pi} |B'(re^{it})|^{p-\alpha}|B'(re^{it})|^{\alpha}\, dt \\ &\leq \left(\frac{1}{1-r}\right)^{p-\alpha}\int_0^{2\pi} |B'(re^{it})|^{\alpha}\, dt.\end{aligned}$$

Now apply Lemma 2. $\square$

If $\alpha = 1/2$, a result similar to Lemma 1 holds. We state without proof

THEOREM 2. *Let B be a Blaschke product with zeros $\{a_n\}$. If $\sum_n (1-|a_n|)^{1/2} < \infty$, then*

$$\int_0^{2\pi} |B'(re^{it})|^{1/2}\, dt = o\left(\log \frac{1}{1-r}\right).$$

3. Interpolating Blaschke products

A sequence $\{a_n\}$ of points in U is said to be uniformly separated if there is a constant $\delta > 0$ such that

$$\inf_n \prod_{m \neq n} \rho(a_m, a_n) \geq \delta > 0, \tag{4}$$

where ρ is the pseudohyperbolic metric in U and is given by

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|, \quad z, w \in U.$$

A Blaschke product whose zeros are uniformly separated is called an interpolating Blaschke product. In [1], W. Cohn proves that if B is an interpolating Blaschke product with zeros $\{a_n\}$ and $0 < \alpha < 1/2$, then (1) holds if and only if $B' \in H^{1-\alpha}$. It is natural then to investigate whether Theorem A and Theorem 1 have converses in the case of interpolating Blaschke products.

We will be using a lemma of Girela, Peláez and Vukotić that appears in [4]. It involves pseudohyperbolic discs $\Delta(a; r) = \{z \in U : \rho(z, a) < r\}$. Note that $\Delta(a; r)$ is a Euclidean disc with Euclidean center c and Euclidean radius R given by

$$c = \frac{1 - r^2}{1 - r^2|a|^2} a \quad \text{and} \quad R = \frac{1 - |a|^2}{1 - r^2|a|^2} r.$$

The lemma of Girela, Peláez and Vukotić says

Lemma A. *Let B be an interpolating Blaschke product with zeros $\{a_n\}$ and with constant δ as in* (4). *Then there exist two positive constants r_0 and γ depending only on δ such that the pseudohyperbolic discs $\{\Delta(a_n; r_0)\}_{n=1}^{\infty}$ are pairwise disjoint, and*

$$|B'(z)| \geq \gamma/(1 - |a_n|)$$

for all $z \in \Delta(a_n; r_0)$, $n = 1, 2, \ldots$.

For any $p > 0$ and $\alpha > -1$, A^p_α is the space of all functions f analytic in U for which

$$\iint_U |f(re^{it})|^p (1 - r)^\alpha \, dA < \infty.$$

In [4], Girela, Peláez and Vukotić give a proof that if B is an interpolating Blaschke product with zeros $\{a_n\}$ and $1 < p < 2$, then B' is in the Bergman space

A_0^p if and only if $\sum_n (1-|a_n|)^{2-p} < \infty$. We will extend their proof to arbitrary $\alpha > -1$.

THEOREM 3. *Let B be an interpolating Blaschke product with zeros $\{a_n\}$. Then for $\alpha > -1$ and $\alpha + 1 < p < \alpha + 2$, $B' \in A_\alpha^p$ if and only if*

$$\sum_n (1-|a_n|)^{2-p+\alpha} < \infty.$$

Proof. If $\sum_n (1-|a_n|)^{2-p+\alpha} < \infty$, then $B' \in A_\alpha^p$ by a theorem of H. O. Kim [6]. For the converse, assume that $B' \in A_\alpha^p$. By Lemma A, there exist positive constants r_0 and γ such that the discs $\{\Delta(a_n; r_0)\}_{n=1}^{\infty}$ are pairwise disjoint and $|B'(z)| \geq \gamma/(1-|a_n|)$ for all $z \in \Delta(a_n; r_0)$, $n = 1, 2, \ldots$. Let $\Delta_n = \Delta(a_n; r_0)$. If c_n and R_n denote the Euclidean center and Euclidean radius of Δ_n for $n = 1, 2, \ldots$, then simple computations show that $R_n \asymp 1 - |a_n|$ and that $1 - |z| \geq 1 - |c_n| - |R_n| \gtrsim 1 - |a_n|$ for all $z \in \Delta_n$. Thus,

$$\begin{aligned}\iint_U |B'(re^{it})|^p (1-r)^\alpha \, dA &\geq \sum_{n=1}^{\infty} \iint_{\Delta_n} |B'(re^{it})|^p (1-r)^\alpha \, dA \\ &\gtrsim \sum_{n=1}^{\infty} \frac{1}{(1-|a_n|)^p} (1-|a_n|)^\alpha (1-|a_n|)^2 \\ &= \sum_{n=1}^{\infty} (1-|a_n|)^{2-p+\alpha}. \qquad \square\end{aligned}$$

Theorem 3 was proved for $p \geq 1$ by A. Gluchoff in [5]. We are now ready to give a partial converse to Theorem 1 and Theorem A, which will also show that the exponent $p + \alpha - 1$ in Theorem 1 (and Theorem A) is sharp.

THEOREM 4. *Let B be an interpolating Blaschke product and suppose $0 < \alpha < 1$. Suppose there exists a positive number p such that*

$$\int_0^{2\pi} |B'(re^{it})|^p \, dt = O\left(\frac{1}{(1-r)^{p+\alpha-1}}\right),$$

where $p > 1 - \alpha$ if $0 < \alpha \leq 1/2$ and $p \geq \alpha$ if $1/2 < \alpha < 1$. Then, $\sum_n (1-|a_n|)^{\alpha'} < \infty$ for all $\alpha' > \alpha$.

Proof. For all $q < 1$,

$$\int_0^{2\pi} |B'(re^{it})|^p (1-r)^{p+\alpha-1-q} \, dt = O((1-r)^{-q}).$$

Then,

$$\iint_U |B'(re^{it})|^p (1-r)^{p+\alpha-1-q} \, dA < \infty,$$

and so, $B' \in A^p_{p+\alpha-1-q}$. Note that if we restrict q so that $\alpha < q < 1$, then $p+\alpha-1-q > -1$ and $(p+\alpha-1-q)+1 < p < (p+\alpha-1-q)+2$. So by Theorem 3,

$$\sum_n (1-|a_n|)^{2-p+(p+\alpha-1-q)} < \infty.$$

In other words, $\sum_n (1-|a_n|)^{\alpha+1-q} < \infty$, which is what was required. $\square$

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