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POLAR QUOTIENTS OF A PLANE CURVE AND THE NEWTON ALGORITHM

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Dedicated to Professor Arkadiusz Płoski

Abstract

Using the Newton algorithm we show how to compute all the polar quotients and their multiplicities of a plane curve $f = 0$, where f is a formal power series of two variables over an algebraically closed field $\mathbf{k}$ with characteristic zero. The curve is not necessarily reduced.

1 Introduction

Let $\mathbf{k}$ be an algebraically closed field with characteristic zero. We use standard notations; $\mathbf{k}[[X, Y]]$ is the ring of formal power series, $\operatorname{ord} f$ is the order of $f \in \mathbf{k}[[X, Y]]$ $(\operatorname{ord} 0 = +\infty)$. For elements $a_1, \dots, a_p$ of a given set we define the *system* $\mathscr{A} = \langle a_1, \dots, a_p\rangle$ as the sequence $a_1, \dots, a_p$ treated as unordered. Put $\deg \mathscr{A} = p$. Instead of $\langle \underbrace{a_1, \dots, a_1}_{m_1 \text{ times}}, \dots \underbrace{a_p, \dots, a_p}_{m_p \text{ times}}\rangle$ we write $\langle a_1 : m_1, \dots a_p : m_p\rangle$. For $\mathscr{A} = \langle a_1, \dots, a_p\rangle$ and $\mathscr{B} = \langle b_1, \dots, b_q\rangle$ we have a natural addition $\mathscr{A} \oplus \mathscr{B} = \langle a_1, \dots a_p, b_1, \dots b_q\rangle$ with the neutral element $\langle\ \rangle$. By convention $\langle a : 0\rangle = \langle\ \rangle$. (see [Wh], notion of *symetric power*).

Let $f(X, Y) \in \mathbf{k}[[X, Y]]$ be such a series that $p = \operatorname{ord} f(0, Y) > 1$. Recall the *polar curve* $\partial f/\partial Y = 0$ with its positive-order-roots $z_1(X), \dots, z_{p-1}(X)$ in the ring $\mathbf{k}[[X]]^* = \bigcup_{n\geq 1} \mathbf{k}[[X^{1/n}]]$ of the *Puiseux series*. We consider the system

$$\overline{\mathscr{Q}}(f, X) = \langle \operatorname{ord} f(X, z_1(X)), \dots, \operatorname{ord} f(X, z_{p-1}(X))\rangle \tag{1}$$

of *polar quotients* of f with respect to X. Every polar quotient is either a positive rational or $+\infty$. We omit the bar over $\mathscr{Q}$ to denote the system of finite quotients. We have $\overline{\mathscr{Q}}(f, X) = \mathscr{Q}(f, X)$ if and only if f is reduced.

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When X and f are transverse, the polar quotients are topological invariants called *polar invariants* ($\mathbf{k} = \mathbf{C}$). We have [T]:

(*Łojasiewicz exponent* of grad f near zero) = (maximal polar invariant) – 1.

The study of polar quotients and polar invariants extends over many authors ([T], [M], [D], [E], [Eph], [CA], [LMW1], [LMW2], [Ga], [GP1], [GP2], [LMP]). Let us mention Merle's [M] description of polar invariants for irreducible series (see also [GP1]); Delgado's developement of the case of two branches [D]; and the computation of polar invariants for multi-branched singularity using Egger's diagrams ([E], [Ga]). The authors of [GP2] give explicit formulae for the polar quotients in terms of characteristics and intersection multiplicities of branches. For a nondegenerate series f the polar invariants can be calculated using the Newton polygon of f [LP], [LMP].

This paper aims to compute the system of polar quotients using a version of the Newton algorithm ([W], [Can], [KP], [L2]). We generalize the approach from [LP], [LMP].

2 Main result

It is convenient to consider the ring $\mathbf{k}[[X^*, Y]] = \bigcup_{n\geq 1} \mathbf{k}[[X^{1/n}, Y]]$. Take $f = \sum f_{\alpha\beta} X^\alpha Y^\beta \in k[[X^*, Y]]$. As usual we define the *support* supp f as $\{(\alpha, \beta) : f_{\alpha\beta} \neq 0\}$, the *Newton diagram* $\Delta(f)$ as $\operatorname{conv}(\operatorname{supp} f + \mathbf{R}_+^2)$, and the *Newton polygon* $\mathcal{N}(f)$ as the set of compact faces of $\Delta(f)$. By $\delta(f)$ we denote the distance between $\Delta(f)$ and the horizontal axis.

For $S \in \mathcal{N}(f)$, by $|S|_1$ and $|S|_2$ we denote the lengths of projections of S onto the horizontal and vertical axies, respectively. We call the ratio $|S|_1/|S|_2$ the *inclination* of S. For $\theta > 0$ (or $\theta = -\infty$) it will be useful to consider the polygon $\mathcal{N}^\theta(f)$, which consists of all the faces $S \in \mathcal{N}(f)$ with an inclination strictly greater than θ. Let $\alpha(S)$ denote the abscissa of the point where the line determined by S intersects the horizontal axis. We define the *initial form* $\operatorname{in}(f, S) = \sum f_{\alpha\beta} X^\alpha Y^\beta$, where (α, β) runs over $S \cap \operatorname{supp} f$. By $t(f, S)$ we denote the number of different roots of the polynomial $\operatorname{in}(f, S)(1, Y) \in k[Y]$. The number $\varepsilon(S) \in \{-1, 0\}$ is defined as -1 when S touches the horizontal axis and as 0 otherwise. Put $d(f, S) = |S|_2 + \varepsilon(S) - t(f, S) + 1$. Note that $d(f, S) = 0$ if and only if every nonzero root of $\operatorname{in}(f, S)$ in $\mathbf{k}[[X]]^*$ is of multiplicity 1. Then we call the series f *nondegenerate* on S.

For any $\varphi \in \mathbf{k}[[X]]^*$, ord $\varphi > 0$ one can apply the substitution $f_\varphi(X, Y) = f(X, \varphi(X) + Y) \in \mathbf{k}[[X^*, Y]]$ ([Can], [GP1], [KP]). Clearly, $f_\varphi = f$ for $\varphi = 0$. Consider the ring $\mathbf{k}[X]^* = \bigcup_{n\geq 1} \mathbf{k}[X^{1/n}]$ of *Puiseux polynomials*. For $\varphi \in \mathbf{k}[X]^*$, $\deg \varphi < +\infty$. Put $\deg 0 = -\infty$. The set $T(f, X) \subset \mathbf{k}[X]^*$ of the *tracks* (of the Newton algorithm) for f is defined to be the minimal set satisfying two properties: (I) $0 \in T(f, X)$, (II) for every $\varphi(X) \in T(f, X)$, if there exists $S \in \mathcal{N}^{\deg\varphi}(f_\varphi)$, then for every nonzero root aX^θ of $\operatorname{in}(f_\varphi, S)$, $\varphi + aX^\theta \in T(f, X)$. We will write $\mathcal{N}_\varphi$ instead of $\mathcal{N}^{\deg\varphi}(f_\varphi)$ when f is fixed.

We call series $\psi \in \mathbf{k}[[X]]^*$ a *continuation* of $\varphi \in \mathbf{k}[X]^*$ if $\operatorname{ord}(\varphi - \psi) > \deg \varphi$. Then we write $\psi = \varphi + \cdots$. Let φ be a track of the Newton algorithm for f. Let $\bar{\mathcal{Q}}_\varphi(f, X) = \bigoplus \langle \operatorname{ord} f(X, z(X)) \rangle$, where $z(X)$ is a continuation of φ. By analogy we define $\mathcal{Q}_\varphi(f, X) \subset \mathcal{Q}(f, X)$ for $\operatorname{ord} f(X, z(X)) < +\infty$.

Now, put $\mathscr{C}_\varphi = \langle \infty : \delta(f_\varphi) - 1 \rangle$ if $\delta(f_\varphi) > 1$ and $\mathscr{C}_\varphi = \langle\ \rangle$ if $\delta(f_\varphi) = 1$ or 0. For $S \in \mathcal{N}_\varphi$ we denote $\mathscr{A}_{\varphi, S} = \langle \alpha(S) : t(f_\varphi, S) - 1 \rangle$. We have the following

THEOREM 2.1 (main result).
(a) *For* $\varphi \in T(f, X)$ *we have* $\bar{\mathcal{Q}}_\varphi(f, X) = [\bigoplus_{S \in \mathcal{N}_\varphi} (\mathscr{A}_{\varphi, S} \oplus \mathscr{B}_{\varphi, S})] \oplus \mathscr{C}_\varphi$, *where* $\mathscr{B}_{\varphi, S}$ *is a system of quotients strictly greater than* $\alpha(S)$, $\deg \mathscr{B}_{\varphi, S} = d(f, S)$.
(b) $\mathscr{B}_{\varphi, S} = \bigoplus_{aX^\theta} \bar{\mathcal{Q}}_{\varphi + aX^\theta}(f, X)$, *where* aX^θ *runs over all multiple nonzero roots of* $\operatorname{in}(f_\varphi, S)$,
(c) $\mathcal{Q}(f, X) = \bigoplus_{\varphi \in T(f, X)} \bigoplus_{S \in \mathcal{N}_\varphi} \mathscr{A}_{\varphi, S}$

We prove the above theorem in the next section.

Remark 2.2. Clearly $\alpha(S)$ is a polar quotients if and only if $t(f_\varphi, S) > 1$. This condition is always satisfied for $S \in \mathcal{N}_\varphi$, which does not touch the horizontal axis. When $\alpha(S)$ is not a polar quotient then S touches the axis and $\operatorname{in}(f_\varphi, S) = bX^\zeta (Y - aX^\theta)^{|S|_2}$.

Remark 2.3. We can consider the system $\mathscr{A}_\varphi(f, X) = \bigoplus_{S \in \mathcal{N}^\varphi} \mathscr{A}_{\varphi, S}$ of finite quotients which are determined by the behaviour of f_φ on $\mathcal{N}_\varphi$. Clearly $\deg \mathscr{A}_\varphi(f, X) = \sum_{S \in \mathcal{N}^\varphi} [t(f_\varphi, S) - 1]$.

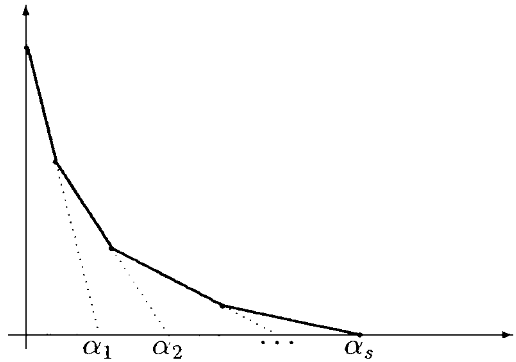
Remark 2.4. We state that the sum in (c) is in fact finite. We can replace $T(f, X)$ by the finite $T_{\min}(f, X)$, which contains all $\varphi \in T(f, X)$ such that there exists $S \in \mathcal{N}_\varphi$ with $t(f_\varphi, S) > 1$.

For $\varphi = 0$ we have the following two corollaries. We write $\mathscr{A}_S, \mathscr{B}_S$ instead of $\mathscr{A}_{0,S}, \mathscr{B}_{0,S}$ and $\mathscr{C}$ instead of $\mathscr{C}_0$.

COROLLARY 2.5. (a) $\bar{\mathcal{Q}}(f, X) = [\bigoplus_{S \in \mathcal{N}(f)} (\mathscr{A}_S \oplus \mathscr{B}_S)] \oplus \mathscr{C}$,
(b) $\mathscr{B}_S = \bigoplus_{aX^\theta} \bar{\mathcal{Q}}_{aX^\theta}(f, X)$, *where* aX^θ *runs over all possible multiple nonzero roots of* $\operatorname{in}(f, S)$.

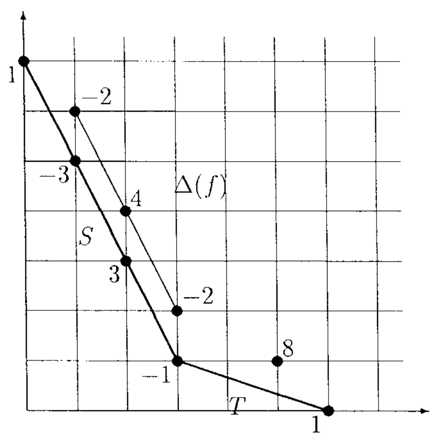
COROLLARY 2.6. *If* $\delta(f) \leq 1$ *and* $d(f, S) = 0$ *for every* $S \in \mathcal{N}(f)$ *then*

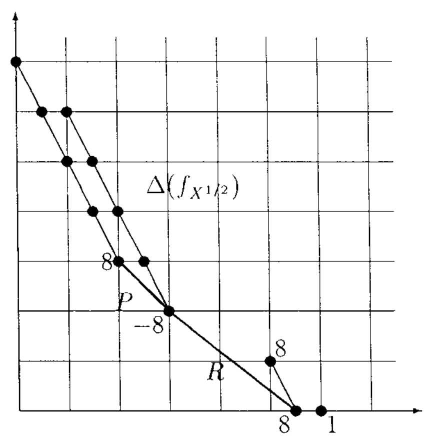
$$\bar{\mathcal{Q}}(f, X) = \mathcal{Q}(f, X) = \mathscr{A}_0(f, X) = \bigoplus_{S \in \mathcal{N}(f)} \langle \alpha(S) : |S|_2 + \varepsilon(S) \rangle.$$



When $S \in \mathcal{N}(f)$ touches the horizontal axis and $|S|_2 = 1$ then $\alpha(S)$ is not a polar quotient. We call such a face *exceptional* ([LP]).

Example 2.7. Consider the curve $f = Y(Y^2 - X)^3 - 2XY^2(Y^2 - X)^2 + 8X^5Y + X^6$. We have $\mathcal{N}(f) = \{S, T\}$, where S joins $(0,7)$ with $(3,1)$ and T joins $(3,1)$ with $(6,0)$. From Corollary 2.5 $\overline{\mathcal{Q}}(f, X) = \mathcal{A}_S \oplus \mathcal{B}_S \oplus \mathcal{A}_T \oplus \mathcal{B}_T \oplus \mathcal{C}$. Clearly $\mathrm{in}(f, S) = Y(Y^2 - X)^3 = Y(Y - X^{1/2})^3(Y + X^{1/2})^3$, $\alpha(S) = 7/2$, $t(f, S) = 3$, $d(f, S) = 4$; $\mathrm{in}(f, T) = -X^3Y + X^6 = -X^3(Y - X^3)$, $\alpha(T) = 6$, $t(f, T) = 1$, $d(f, T) = 0$ and $\delta(f) = 0$. Obviously $\mathcal{A}_T = \mathcal{B}_T = \mathcal{C} = \langle\ \rangle$.





Hence $\overline{\mathcal{Q}}(f, X) = \mathcal{A}_S \oplus \mathcal{B}_S = \langle 7/2, 7/2 \rangle \oplus \mathcal{B}_S$, where $\mathcal{B}_S$ contains four quotients strictly greater then $7/2$. Consider two multiple nonzero roots $X^{1/2}$ and $-X^{1/2}$ of $\mathrm{in}(f, S)$. We have $\mathcal{B}_S = \overline{\mathcal{Q}}_{X^{1/2}}(f, X) \oplus \overline{\mathcal{Q}}_{-X^{1/2}}(f, X)$. Taking $X^{1/2}$ as a track we obtain $f_{X^{1/2}} = f(X, X^{1/2} + Y) = (Y + X^{1/2})Y^3(Y + 2X^{1/2})^3 - 2X(Y + X^{1/2})^2Y^2(Y + 2X^{1/2})^2 + 8X^5(Y + X^{1/2}) + X^6$. The polygon $\mathcal{N}_{X^{1/2}}$ has two faces: P which joins $(2,3)$ with $(3,2)$ and R which joins $(3,2)$ with

$(11/2, 0)$. We have $\overline{\mathscr{Q}}_{X^{1/2}}(f, X) = \mathscr{A}_{X^{1/2}, P} \oplus \mathscr{B}_{X^{1/2}, P} \oplus \mathscr{A}_{X^{1/2}, R} \oplus \mathscr{B}_{X^{1/2}, R} \oplus \mathscr{C}_{X^{1/2}}$. Note that $\text{in}(f_{X^{1/2}}, P) = 8X^2Y^3 - 8X^3Y^2 = 8X^2Y^2(Y - X)$, $\alpha(P) = 5$, $t(f_{X^{1/2}}, P) = 2$, $d(f_{X^{1/2}}, P) = 0$; $\text{in}(f_{X^{1/2}}, R) = -8X^3Y^2 + 8X^{11/2} = -8X^3 \cdot (Y - X^{5/4})(Y + X^{5/4})$, $\alpha(R) = 11/2$, $t(f_{X^{1/2}}, R) = 2$, $d(f_{X^{1/2}}, R) = 0$ and $\delta(f_{X^{1/2}}) = 0$. Since $\mathscr{B}_{X^{1/2}, P} = \mathscr{B}_{X^{1/2}, R} = \mathscr{C}_{X^{1/2}} = \langle\ \rangle$ we finish with $\overline{\mathscr{Q}}_{X^{1/2}}(f, X) = \mathscr{A}_{X^{1/2}, P} \oplus \mathscr{A}_{X^{1/2}, R} = \langle 5, 11/2 \rangle$. Analogously, $\overline{\mathscr{Q}}_{-X^{1/2}}(f, X) = \langle 5, 11/2 \rangle$ and finally $\overline{\mathscr{Q}}(f, X) = \mathscr{A}_0(f, X) \oplus \overline{\mathscr{Q}}_{X^{1/2}}(f, X) \oplus \overline{\mathscr{Q}}_{-X^{1/2}}(f, X) = \langle 7/2, 7/2, 5, 5, 11/2, 11/2 \rangle$. As a result we see that f is reduced. We can compute the Milnor number from Teissier's formula $\mu_0(f) = \sum \mathscr{Q}(f, X) - \text{ord}\, f(0, Y) + 1 = 22$. Note that $T_{\min}(f, X) = \{0, X^{1/2}, -X^{1/2}\}$.

3 Proof of the main result

The proof of the main result will be performed in several steps. The first step is the classical Newton-Puiseux theorem, which provides a description of the Puiseux roots of a series in terms of its Newton polygon (Theorem 3.1). The second and fundamental step is the description of the Newton polygon of the derivative. In the final steps of the proof we applicate these methods to the Newton algorithm.

The Newton-Puiseux theorem

Let $f \in \mathbf{k}[[X^*, Y]]$ be a series such that $p = \text{ord}\, f(0, Y) > 1$. Let $\text{Zer}\, f = \langle y_1(X), \dots, y_p(X) \rangle$ be the system of all the positive-order-roots of $f = 0$ in $\mathbf{k}[[X]]^*$. Consider $S \in \mathscr{N}(f)$ and define the form $\text{in}(f, S)^\circ$ by the equation $\text{in}(f, S) = X^{a_S} Y^{b_S}\, \text{in}(f, S)^\circ$ where a_S and b_S are the maximal possible powers. Let us note that $\deg \text{Zer}\, \text{in}(f, S)^\circ = |S|_2$. For any $\theta > 0$ (or $\theta = -\infty$) we define $\text{Zer}^\theta f$, which contains those roots of $\text{Zer}\, f$ which satisfy $\text{ord}\, y_i(X) > \theta$. By the *height* of $\mathscr{N}^\theta(f)$ we mean $\sum_{S \in \mathscr{N}^\theta(f)} |S|_2$ and we denote it by $|\mathscr{N}^\theta(f)|$. We need the Newton-Puiseux theorem in the following form (we use convention $\text{in}\, 0 = 0$).

THEOREM 3.1. *Let* $\text{Zer}^\theta f = \langle y_1(X), \dots, y_s(X) \rangle$. *Then*
(i) $\langle \text{ord}\, y_1(X), \dots, \text{ord}\, y_s(X) \rangle = \bigoplus_{S \in \mathscr{N}^\theta(f)} \langle |S|_1 / |S|_2 : |S|_2 \rangle \oplus \langle +\infty : \delta(f) \rangle$,
(ii) $\langle \text{in}\, y_1(X), \dots, \text{in}\, y_s(X) \rangle = \bigoplus_{S \in \mathscr{N}^\theta(f)} \text{Zer}\, \text{in}(f, S)^\circ \oplus \langle 0 : \delta(f) \rangle$.
(iii) $s = |\mathscr{N}^\theta(f)| + \delta(f)$.

The following property geometrically expresses the orders from (1). For $\theta > 0$ and any closed nonempty subset $Z \subset \mathbf{R}^2_+$ we define the number

$$\alpha(\theta, Z) = \min\{\alpha + \beta\theta : (\alpha, \beta) \in Z\}.$$

Obviously $\alpha(\theta, \Delta_f) = \alpha(\theta, \text{supp}\, f)$. This number is the abscissa of the point where the line of inclination θ, supporting $\Delta(f)$, intersects the horizontal axis. We have the following simple

PROPERTY 3.2. *Let* $z(X) \in \mathbf{k}[[X]]^*$, $0 < \operatorname{ord} z(X) < +\infty$. *Then*
(a) $\operatorname{ord} f(X, z(X)) \geq \alpha(\operatorname{ord} z, \Delta_f)$,
(b) *the inequality is sharp if and only if there exists a face* $S \in \mathcal{N}(f)$ *such that* $|S|_1/|S|_2 = \operatorname{ord} z(X)$ *and* $\operatorname{in} z(X)$ *is a root of* $\operatorname{in}(f, S)$.

Polygon of the derivative

In this next step we describe the polygon $\mathcal{N}(\partial f/\partial Y)$. By eliminating the points from $\operatorname{supp} f$ that lie on the horizontal axis and by moving all remaining points one unit down, we obtain the support of $\partial f/\partial Y$. In effect, if $S \in \mathcal{N}(f)$ does not touch the horizontal axis, then $T = S - (0, 1)$ is a face of the polygon $\mathcal{N}(\partial f/\partial Y)$ and

$$\operatorname{in}\left(\frac{\partial f}{\partial Y}, T\right) = \frac{\partial}{\partial Y} \operatorname{in}(f, S). \tag{2}$$

If S touches the horizontal axis then there exists the corresponding family of faces $T \in \mathcal{N}(\partial f/\partial Y)$ such that $|T|_1/|T|_2 \geq |S|_1/|S|_2$ and $\sum |T|_2 + \delta(\partial f/\partial Y) = |S|_2 - 1$. If there exists an element T, of the family which is parallel to S, then (2) is also satisfied. As a result we have

COROLLARY 3.3 (see [L1], Corollary 5.4). *Let* $T \in \mathcal{N}(\partial f/\partial Y)$.
(a) *If* T *is parallel to face* $S \in \mathcal{N}(f)$, *then* (2) *is satisfied.*
(b) *If* T *is not parallel to any face of* $\mathcal{N}(f)$, *then*
· $|T|_1/|T|_2 > |S|_1/|S|_2$ *for every* $S \in \mathcal{N}(f)$,
·· *the polygon* $\mathcal{N}(f)$ *touches the horizontal axis.*

We need a more detailed analysis of the relations between roots of $\operatorname{in}(f, S)$ and $\partial/\partial Y \operatorname{in}(f, S)$. Consider the factorization

$$\operatorname{in}(f, S) = bX^{a_S} Y^{b_S} L_1^{r_1} \cdots L_k^{r_k}, \tag{3}$$

where $Y, L_1, \ldots, L_k$ are different monic linear factors of $\operatorname{in}(f, S)$ in $\mathbf{k}[X]^*[Y]$ $(b \neq 0, k > 0, r_i > 0)$. Every factor has the form $Y - aX^\theta$, where $\theta = |S|_1/|S|_2$. We have the following

LEMMA 3.4 (see [LMP], Lemma 4.1).
(i) *If* S *does not touch the horizonatal axis* $(b_S > 0)$, *then*

$$\frac{\partial}{\partial Y} \operatorname{in}(f, S) = b' X^{a_S} Y^{b_S - 1} L_1^{r_1 - 1} \cdots L_k^{r_k - 1} L_1' \cdots L_k',$$

where $L_1', \ldots, L_k'$ *are monic linear factors different from* $Y, L_1, \ldots, L_k$.
(ii) *If* S *touches the horizontal axis* $(b_S = 0)$ *and* $k > 1$ *then*

$$\frac{\partial}{\partial Y} \operatorname{in}(f, S) = b' X^{a_S} L_1^{r_1 - 1} \cdots L_k^{r_k - 1} L_1' \cdots L_{k-1}',$$

where $L_1', \ldots, L_{k-1}'$ *are monic linear factors different from* $L_1, \ldots, L_k$.

Proof. If $b_S > 0$ then Y-differentiation of $\mathrm{in}(f,S) = X^{a_S} Y^{b_S}\, \mathrm{in}(f,S)^\circ$ moves the support of $\mathrm{in}(f,S)$ by the vector $(0,-1)$. We obtain

$$\frac{\partial}{\partial Y}\,\mathrm{in}(f,S) = X^{a_S} Y^{b_S-1} h(X,Y), \tag{4}$$

where the powers a_S and $b_S - 1$ are the maximal possible. Let us consider a factor of $\mathrm{in}(f,S)$ of the form $L^r = (Y - aX^\theta)^r$. By substitution $aX^\theta + Y$ in place of Y, we obtain Y^r in place of L^r. We use (4) to show that L^{r-1} divides $\partial/\partial Y\,\mathrm{in}(f,S)$ and $r-1$ is the maximal possible power.

Remark 3.5. If S touches the horizontal axis ($b_S = 0$) and $k = 1$ in (3), then $\mathrm{in}(f,S) = bX^{a_S}L^r$ and $\partial/\partial Y(\mathrm{in}(f,S)) = b'X^{a_S}L^{r-1}$.

From the above lemma and the previous statements we have

PROPOSITION 3.6. *Let* $S \in \mathcal{N}(f)$.
(a) *If* $t = t(f,S) > 1$ *then there exists* $t-1$ *solutions* $\psi_1'(X),\dots,\psi_{t-1}'(X)$ *of* $\partial f/\partial Y = 0$ *such that* $\mathrm{ord}\, f(X,\psi_i'(X)) = \alpha(S)$.
(b) *If* $d = d(f,S) > 0$ *then there exists* d *solutions* $\psi_1(X),\dots,\psi_d(X)$ *of* $\partial f/\partial Y = 0$ *such that* $\mathrm{ord}\, f(X,\psi_i(X)) > \alpha(S)$.

Proof. (a) If S does not touch the horizontal axis, then by using notations of Lemma 3.4, $k = t(f,S) - 1$. Since $T = S - (0,1)$ is a face of $\mathcal{N}(\partial f/\partial Y)$, from Theorem 3.1 there exist solutions $\psi_1'(X),\dots,\psi_k'(X)$ of $\partial f/\partial Y = 0$ that correspond to the factor $L_1'\cdots L_k'$ of $\partial/\partial Y(\mathrm{in}(f,S)) = \mathrm{in}(\partial f/\partial Y, T)$. Clearly $\mathrm{ord}\,\psi_i'(X) = |T|_1/|T|_2 = |S|_1/|S|_2$. Since $L_1',\dots,L_k'$ are different from $Y, L_1,\dots, L_k$, then in ψ_i' is not a root of $\mathrm{in}(f,S)$. By Property 3.2 we obtain

$$\mathrm{ord}\, f(X,\psi_i'(X)) = \alpha(|S|_1/|S|_2, \Delta_f) = \alpha(S). \tag{5}$$

If S touches the horizontal axis, then $k = t(f,S)$. Let s be the number of appearances of Y in the sequence $L_1',\dots,L_{k-1}'$. We can assume that $L_1',\dots, L_{k-s-1}'$ are different from Y. Analogously, as before, we construct solutions $\psi_1'(X),\dots,\psi_{k-s-1}'(X)$ that satisfy (5). For $s > 0$, by the Newton-Puiseux theorem, there exist s solutions $\psi_{k-s}'(X),\dots,\psi_{k-1}'(X)$ of $\partial f/\partial Y = 0$ that correspond to the family of faces $T \in \mathcal{N}(\partial f/\partial Y)$, that lie below the line $\beta = s$, and to the distance $\delta(\partial f/\partial Y)$. We have $\sum |T|_2 + \delta(\partial f/\partial Y) = s$. If $\psi_i'(X)$ comes from a face T of the family, then by Corollary 3.3 we have $\mathrm{ord}\,\psi_1(X) = |T|_1/|T|_2 > |S|_1/|S|_2$. Since S touches the horizontal axis, it is the lower possible face of $\mathcal{N}(f)$. Therefore, the line supporting Δ_f, which is parallel to T, meets Δ_f at the vertex lying on the horizontal axis. The vertex has the abscissa $\mathrm{ord}\, f(X,0)$. From Property 3.2 we obtain $\mathrm{ord}\, f(X,\psi_i'(X)) = \alpha(|T|_1/|T|_2, \Delta_f) = \mathrm{ord}\, f(X,0)$. If $\psi_i(X) = 0$, then similarly $\mathrm{ord}\, f(X,\psi_i'(X)) = \mathrm{ord}\, f(X,0)$. Since $\mathrm{ord}\, f(X,0) = \alpha(S)$ we finish with (5) in both cases.

(b) Note that $d = d(f,S) = (r_1 - 1) + \cdots + (r_k - 1)$. If $d > 0$ then there exists a face $T \in \mathcal{N}(\partial f/\partial Y)$ which is parallel to S. By Theorem 3.1 and by

Lemma 3.4 there exist solutions $\psi_1(X), \dots, \psi_d(X)$ of $\partial f/\partial Y = 0$ corresponding to the factor $L_1^{r_1-1} \cdots L_k^{r_k-1}$ of $\partial/\partial Y(\mathrm{in}(f, S)) = \mathrm{in}(\partial f/\partial Y, T)$. Clearly $\mathrm{ord}\, \psi_i(X) = |T|_1/|T|_2 = |S|_1/|S|_2$ and $\mathrm{in}\, \psi_i(X)$ is a root of $\mathrm{in}(f, S)$. From Property 3.2 it follows that $\mathrm{ord}\, f(X, \psi_i(X)) > \alpha(|S|_1/|S|_2, \Delta_f) = \alpha(S)$, which completes the proof.

The Newton polygon relative to a Puiseux polynomial

Let $\varphi \in \mathbf{k}[X]^*$ be an arbitrary Puiseux polynomial of a positive order. We begin by describing the polygon $\mathcal{N}_\varphi = \mathcal{N}^{\deg \varphi}(f_\varphi)$. Applying the Weierstrass preparation theorem we can write

$$f(X, Y) = U(X, Y)(Y - y_1(X)) \cdots (Y - y_p(X)), \tag{6}$$

where $U(X, Y) \in \mathbf{k}[[X, Y]]$ is a unit and, as before, $\mathrm{Zer}\, f = \langle y_1, \dots, y_p \rangle$. Hence

$$f_\varphi(X, Y) = f(X, \varphi(X) + Y) = U'(X, Y)[Y - (y_1 - \varphi)] \cdots [Y - (y_p - \varphi)],$$

where $U'(X, Y) \in \mathbf{k}[[X^*, Y]]$ is also a unit. Therefore

$$\mathrm{Zer}\, f_\varphi = \langle y_1 - \varphi, \dots, y_p - \varphi \rangle.$$

Using Theorem 3.1 (a) with $\theta = -\infty$ we obtain

$$\langle \mathrm{ord}(y_1 - \varphi), \dots, \mathrm{ord}(y_p - \varphi) \rangle = \bigoplus_{S \in \mathcal{N}(f_\varphi)} \langle |S|_1/|S|_2 : |S|_2 \rangle \oplus \langle +\infty : \delta(f_\varphi) \rangle.$$

Let $\mathrm{Zer}_\varphi\, f = \langle y_1, \dots, y_s \rangle$ denote the system of such solutions from $\mathrm{Zer}\, f$ that $\mathrm{ord}(y_i - \varphi) > \deg \varphi$ (i.e. $y_1(X), \dots, y_s(X)$ are continuations of φ). As a consequence of Theorem 3.1 applied with $\theta = \deg \varphi$ we have

Corollary 3.7.
(a) $\langle \mathrm{ord}(y_1 - \varphi), \dots, \mathrm{ord}(y_s - \varphi) \rangle = \bigoplus_{S \in \mathcal{N}_\varphi} \langle |S|_1/|S|_2 : |S|_2 \rangle \oplus \langle +\infty : \delta(f_\varphi) \rangle$
(b) $\langle \mathrm{in}(y_1 - \varphi), \dots, \mathrm{in}(y_p - \varphi) \rangle = \bigoplus_{S \in \mathcal{N}_\varphi} \mathrm{Zer}\, \mathrm{in}(f_\varphi, S)^\circ \oplus \langle 0 : \delta(f_\varphi) \rangle$
(c) $s = |\mathcal{N}_\varphi| + \delta(f_\varphi)$.

We will need the following simple

Property 3.8. *Let $g \in \mathbf{k}[[X^*, Y]]$ be a nonzero series. Fix $\theta > 0$.*
(i) *There exists a unique representation*

$$g = g_0 + g_1 + g_2 + \cdots$$

such that every g_i is a quasi-homogeneous form of the weights $(\theta, 1)$.
(ii) *If r is the maximal power such that Y^r divides g_0, then*

$$|\mathcal{N}^\theta(g)| + \delta(g) = r.$$

(iii) *If there exists $S \in \mathcal{N}(g)$ such that $|S|_1/|S|_2 = \theta$, then $g_0 = \mathrm{in}(f, S)$.*

Now, we can prove the following

Proposition 3.9. *Assume that* $\varphi \in \mathbf{k}[X]^*$ *is such a polynomial that* $\mathcal{N}_\varphi$ *is nonempty. Let* $S \in \mathcal{N}_\varphi$ *and let* aX^θ *be a nonzero root of* $\mathrm{in}(f, S)$. *Then*

$$|\mathcal{N}_{\varphi+aX^\theta}| + \delta(f_{\varphi+aX^\theta}) = \textit{mutiplicity of } aX^\theta \textit{ as a root of } \mathrm{in}(f, S).$$

Proof. For $\theta = |S|_1/|S|_2$ let us consider the representation

$$f_\varphi = g_0 + g_1 + g_2 + \cdots$$

according to Property 3.8. We have $g_0 = \mathrm{in}(f_\varphi, S)$. Let us write $\mathrm{in}(f_\varphi, S) = (Y - aX^\theta)^r h(X, Y)$, where r is the maximal possible power. We have $r > 0$. Let us note that $f_{\varphi+aX^\theta}(X, Y) = f_\varphi(X, aX^\theta + Y)$. Hence

$$f_{\varphi+aX^\theta} = \tilde{g}_0 + \tilde{g}_1 + \tilde{g}_2 + \cdots, \tag{7}$$

where $\tilde{g}_i(X, Y) = g_i(X, aX^\theta + Y)$. Since $aX^\theta + Y$ is the homogeneous form of the weights $(\theta, 1)$, (7) is the unique representation guaranteed by Property 3.8. We have $\tilde{g}_0(X, Y) = \mathrm{in}(f_\varphi, S)(X, aX^\theta + Y) = Y^r \tilde{h}(X, Y)$. From Property 3.8(b) follows that

$$r = |\mathcal{N}_{\varphi+aX^\theta}| + \delta(f_{\varphi+aX^\theta}),$$

which concludes the proof.

Tracks of the Newton algorithm

We give here two different characterizations of the set $T(f, X)$, of the tracks of the Newton algorithm for f, and prove their equivalency with the definition from Section 2. Recall

Definition 3.10. $T(f, X) \subset \mathbf{k}[X]^*$ is the minimal subset satisfying two properties: (I) $0 \in T(f, X)$, (II) for every $\varphi(X) \in T(f, X)$, if there exists $S \in \mathcal{N}_\varphi$, then for every nonzero root aX^θ of $\mathrm{in}(f_\varphi, S)$, $\varphi + aX^\theta \in T(f, X)$.

Let us define

$$T_1(f, X) = \{\varphi \in \mathbf{k}[X]^* : \exists y(X) \in \mathrm{Zer}\, f \text{ such that } \mathrm{ord}(y(X) - \varphi) > \deg \varphi\}$$

and

$$T_2(f, X) = \{\varphi \in \mathbf{k}[X]^* : |\mathcal{N}_\varphi| + \delta(f_\varphi) > 0\}.$$

Proposition 3.11. *If* $\mathrm{Zer}\, f$ *is nonempty then* $T(f, X) = T_1(f, X) = T_2(f, X)$.

Proof. The equality $T_1(f, X) = T_2(f, X)$ follows directly from Corollary 3.7 (c). Since $T(f, X)$ is the minimal set with properties (I) and (II), it suffices to show that $T_2(f, X)$ satisfies both properties in order to verify that $T(f, X) \subset T_2(f, X)$. Because $\mathrm{Zer}\, f$ is nonempty, the first condition $0 \in T_2(f, X)$ is clear. In order to check the second condition, let us consider $\varphi \in T_2(f, X)$. Let us assume that there exists $S \in \mathcal{N}_\varphi$ and let aX^θ be an arbitrary nonzero root of

$\mathrm{in}(f_\varphi, S)$. We must show that $\varphi + aX^\theta \in T_2(f, X)$, which follows immediately from Proposition 3.9.

To finish the proof it suffices to verify that $T_1(f, X) \subset T(f, X)$. Let $\varphi = a_1 X^{\theta_1} + \cdots + a_n X^{\theta_n}$ be a non zero element of $T_1(f, X)$ i.e. there exists $y(X) \in \mathrm{Zer}\, f$ such that $\mathrm{ord}(y(X) - \varphi) > \deg \varphi$. Let us put $\varphi_0 = 0$ and $\varphi_k = a_1 X^{\theta_1} + \cdots + a_k X^{\theta_k}$ for $k = 1, \dots, n-1$. We will show the implication $\varphi_k \in T(f, X) \Rightarrow \varphi_{k+1} \in T(f, X)$. Assume that $\varphi_k \in T(f, X)$. We have $\mathrm{in}(y(X) - \varphi_k) = a_{k+1} X^{\theta_{k+1}}$. By Corollary 3.7 there exists $S \in \mathcal{N}_{\varphi_k}$ such that $a_{k+1} X^{\theta_{k+1}}$ is a root of $\mathrm{in}(f_{\varphi_k}, S)$. From property (II) we obtain $\varphi_{k+1} = \varphi_k + a_{k+1} X^{\theta_{k+1}} \in T(f, X)$. Since $0 \in T(f, X)$, by induction we show that $\varphi = \varphi_n \in T(f, X)$, completing the proof.

End of the proof

We are in a good position to finish the proof of the main result (Theorem 2.1).

Proof of (a). Let $\varphi \in T(f, X)$. By Proposition 3.11 $s = |\mathcal{N}_\varphi| + \delta(f_\varphi) > 0$. By Theorem 3.1 there exist solutions $\psi_1(X), \dots, \psi_s(X)$ of $f_\varphi = 0$ such that $\deg \psi_i > \deg \varphi$. By using previous notation we can write $\mathrm{Zer}^{\deg \varphi}(f_\varphi) = \langle \psi_1, \dots, \psi_s \rangle$. On the other hand, we have the system $\mathrm{Zer}_\varphi f = \langle y_1, \dots, y_s \rangle$ of roots of $f = 0$ which are continuations of φ. Corollary 3.7 states the one-to-one correspondence between $\mathrm{Zer}^{\deg \varphi}(f_\varphi)$ and $\mathrm{Zer}_\varphi f$ by $\psi \mapsto y = \varphi + \psi$ (the inverse: $y \mapsto \psi = y - \varphi$). Since $(\partial f / \partial Y)_\varphi = \partial f_\varphi / \partial Y$, this construction can be directly applied to $\mathrm{Zer}_\varphi(\partial f / \partial Y)$. If $s = |\mathcal{N}_\varphi| + \delta(f_\varphi) > 1$, then by Y-differentiation and by Theorem 3.1, there exist $s - 1$ solutions $\psi_1, \dots, \psi_{s-1}$ of $\partial f_\varphi / \partial Y = 0$ such that $\mathrm{ord}\, \psi_i > \deg \varphi$. Clearly

$$\mathrm{Zer}_\varphi(\partial f / \partial Y) = \langle \phi + \psi_1, \dots, \phi + \psi_{s-1} \rangle. \tag{8}$$

Consider $S \in \mathcal{N}_\varphi$. If $t = t(f_\varphi, S) > 1$, then by Proposition 3.6 (a) there exist $t - 1$ solutions $\psi'_1, \dots, \psi'_{t-1}$ of $\partial f_\varphi / \partial Y = 0$ such that $\mathrm{ord}\, f_\varphi(X, \psi'_i(X)) = \alpha(S)$. Then for $z'_i(X) = \varphi(X) + \psi'_i(X) \in \mathrm{Zer}_\varphi(\partial f / \partial Y)$ we have

$$\mathrm{ord}\, f(X, z'_i(X)) = \mathrm{ord}\, f(X, \varphi(X) + \psi'_i(X)) = \mathrm{ord}\, f_\varphi(X, \psi'_i(X)) = \alpha(S).$$

If $d = d(f, S) > 0$ then by Proposition 3.6 (b) we analogously construct solutions $z_1(X), \dots, z_d(X) \in \mathrm{Zer}_\varphi(\partial f / \partial Y)$ such that $\mathrm{ord}\, f(X, z_i(X)) > \alpha(S)$. We finish with the observation that if $\delta(f_\varphi) > 1$, then $\varphi(X)$ is a common root of $f = 0$ and $\partial f / \partial Y = 0$ with multiplicity $\delta(f_\varphi) - 1$ and $\mathrm{ord}\, f(X, \varphi(X)) = \mathrm{ord}\, f_\varphi(X, 0) = +\infty$.

Proof of (b). Consider roots $z_1(X), \dots, z_d(X) \in \mathrm{Zer}_\varphi(\partial f / \partial Y)$ such that

$$\mathcal{B}_{\varphi, S} = \langle \mathrm{ord}\, f(X, z_1(X)), \dots, \mathrm{ord}\, f(X, z_d(X)) \rangle.$$

Fix $i \in \{1, \dots, d\}$. We must show the fact that z_i has the form $\varphi + aX^\theta + \cdots$, where aX^θ is a multiple root of $\mathrm{in}(f_\varphi, S)$. Let $\psi_i = z_i - \varphi$. Since $\mathrm{in}\, \psi_i = aX^\theta$, this fact follows immediately from the construction described in Proposition 3.6.

Proof of (c). We begin with an observation that $T_{\min}(f,X)$ is finite (Remark 2.4). It is enough to consider the tracks generated by solutions of $f=0$ with infinite number of termes. Let us fix such a solution $y(X)=a_1X^{\theta_1}+a_2X^{\theta_2}+\cdots\in \mathrm{Zer}\, f$. Let us define the sequence of tracks $\varphi_0=0,\ldots,$ $\varphi_j=a_1X^{\theta_1}+\cdots+a_jX^{\theta_j},\ldots$ and the sequences of series $f^{(j)}:=f_{\varphi_{j-1}}$ and polygons $\mathcal{N}^{(j)}:=\mathcal{N}_{\varphi_{j-1}}$. For any $j=1,2,\ldots,$ according to Corollary 3.7, there exists a face $S^{(j)}\in\mathcal{N}^{(j)}$ with the inclination θ_j, such that $a_jX^{\theta_j}$ is a root of $\mathrm{in}(f^{(j)},S^{(j)})$ with multiplicity $r_j>0$. We write $l_j=\deg_Y \mathrm{in}(f^{(j)},S^{(j)})$. Of course, $l_j\geq r_j$. From Proposition 3.9 we have $r_j=|\mathcal{N}^{(j+1)}|+\delta(f^{(j+1)})\geq l_{j+1}$. As the result we have the infinite sequence $l_1\geq r_1\geq l_2\geq r_2\geq\cdots$ of positive integers. Therefore, there exists j such that $r_j=l_{j+1}=r_{j+1}=\cdots$. The first equality means that $S^{(j+1)}$ is the highest face of $\mathcal{N}^{(j)}$. The second equality means that $\mathrm{in}(f^{(j+1)},S^{(j+1)})$ has a unique nonzero root, hence $t(f^{(j+1)},S^{(j+1)})=1$. Since $S^{(j+1)}$ touches the horizontal axis (Remark 3.5), $\mathcal{N}^{(j+1)}$ has only one face.

To conclude the proof we need to show that for every solution $z(X)\in \mathrm{Zer}(\partial f/\partial Y)$, such that

$$\mathrm{ord}\, f(X,z(X))<+\infty \tag{9}$$

there exists a track $\varphi\in T(f,X)$ and a face $S\in\mathcal{N}_\varphi$ such that $\mathrm{ord}\, f(X,z(X))=\alpha(S)$. According to (6) and (9) we have $\mathrm{ord}(z(X)-y(X))<+\infty$ for every $y(X)\in\mathrm{Zer}\, f$. Let us choose a solution $y(X)$ with the longest common track $\varphi(X)$ of both series $z(X)$ and $y(X)$. We have $z(X)=\varphi(X)+\psi_z(X)$ and $y(X)=\varphi(X)+\psi_y(X)$, where $\mathrm{ord}\,\psi_z>\deg\varphi$, $\mathrm{ord}\,\psi_y>\deg\varphi$ and $\mathrm{in}\,\psi_y\neq\mathrm{in}\,\psi_z$. Since $0=(\partial f/\partial Y)(X,z(X))=(\partial f_\varphi/\partial Y)(X,\psi_z(X))$, therefore $\mathrm{in}\,\psi_z$ can be described by $\mathcal{N}'_\varphi:=\mathcal{N}^{\deg\varphi}(\partial f_\varphi/\partial Y)$ by virtue of Corollary 3.7.

If $\mathrm{ord}\,\psi_z=+\infty$, then $z(X)=\varphi(X)$ and according to (9) we have $\mathrm{ord}\, f(X,z(X))=\mathrm{ord}\, f(X,\varphi(X))=\mathrm{ord}\, f_\varphi(X,0)<+\infty$. Hence, there exists $S\in\mathcal{N}_\varphi$ which touches the horizontal axis and $\mathrm{ord}\, f_\varphi(X,0)=\alpha(S)$. If $\mathrm{ord}\,\psi_z(X)$ is finite, but does not appear as an inclination of any face of $\mathcal{N}_\varphi$, then according Corollary 3.3 (b) the line supporting $\Delta(f_\varphi)$, of inclination $\mathrm{ord}\,\psi_z$, intersects the horizontal axis at the point $(\mathrm{ord}\, f_\varphi(X,0),0)$. Hence, by Property 3.2 we have $\mathrm{ord}\, f(X,z(X))=\mathrm{ord}\, f_\varphi(X,\psi_z(X))=\mathrm{ord}\, f_\varphi(X,0)$. Therefore, in both cases $\mathrm{ord}\, f(X,z(X))=\alpha(S)$.

Let us assume that $\mathrm{ord}\,\psi_z=|S|_1/|S|_2$ for a certain $S\in\mathcal{N}_\varphi$ but that $\mathrm{in}\,\psi_z$ is not a root of $\mathrm{in}(f_\varphi,S)$. By Property 3.2 we have $\mathrm{ord}\, f(X,z(X))=\alpha(S)$ as before. If $\mathrm{in}\,\psi_z$ is a root of $\mathrm{in}(f_\varphi,S)$, then by Theorem 3.1 there exists $y(X)\in\mathrm{Zer}\, f$ of the form $y(X)=\varphi(X)+\psi_z(X)+\cdots$, which contradicts the definition of $y(X)$. Because of this contradiction, the last possibility cannot happen and we conclude the proof of the main result.

4 Eggers' example

The following example shows that curves which are not equisingular can have the same polar invariants, counting their multiplicities. Let us consider the following four Puiseux series

$$y_1 = X + X^{3/2} + X^{15/4}, \qquad y_3 = X + X^{5/2} + X^{11/4},$$
$$y_2 = 2X + X^{5/2} + X^{13/4}, \quad y_4 = 2X + X^{3/2} + X^{17/4}.$$

Let $f_1, f_2, f_3, f_4 \in \mathbf{k}[[X, Y]]$ be the minimal series respectively for y_1, y_2, y_3, y_4. We put $f = f_1 f_2$ and $f' = f_3 f_4$. Let us begin with f. We have the following cycle for y_1

$$y_1^{(i)} = \varepsilon^{4i} X^{4/4} + \varepsilon^{6i} X^{6/4} + \varepsilon^{15i} X^{15/4}$$

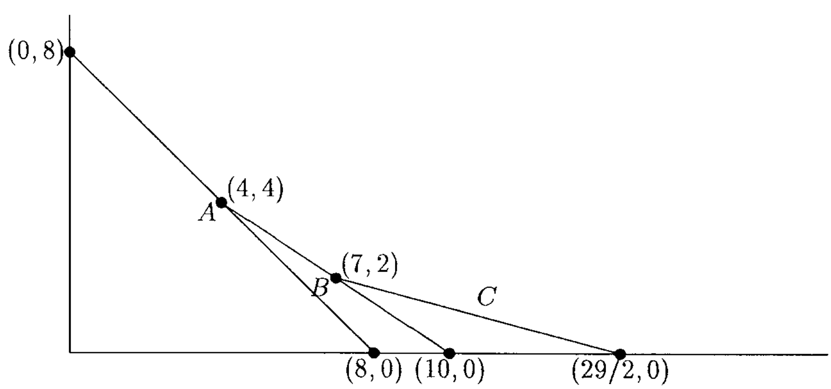
as well as for y_2

$$y_2^{(i)} = 2\varepsilon^{4i} X^{4/4} + \varepsilon^{10i} X^{10/4} + \varepsilon^{13i} X^{13/4},$$

where $i = 0, 1, 2, 3$ and where ε is a primitive root of unity of degree four. On the basis of the table provided below we can analyze tracks $0, X, X + X^{3/2}$ and $X - X^{3/2}$

$f(X, Y)$	$f(X, X+Y)$	$f(X, X+X^{3/2}+Y)$	$f(X, X-X^{3/2}+Y)$
$y_1^{(0)} = X + X^{3/2} + X^{15/4}$	$X^{3/2}$	$X^{15/4}$	$2X^{3/2}$
$y_1^{(1)} = X - X^{3/2} - \varepsilon X^{15/4}$	$-X^{3/2}$	$-2X^{3/2}$	$-\varepsilon X^{15/4}$
$y_1^{(2)} = X + X^{3/2} - X^{15/4}$	$X^{3/2}$	$-X^{15/4}$	$2X^{3/2}$
$y_1^{(3)} = X - X^{3/2} + \varepsilon X^{15/4}$	$-X^{3/2}$	$-2X^{3/2}$	$\varepsilon X^{15/4}$
$y_2^{(0)} = 2X + X^{5/2} + X^{13/4}$	X	X	X
$y_2^{(1)} = 2X - X^{5/2} + \varepsilon X^{13/4}$	X	X	X
$y_2^{(2)} = 2X + X^{5/2} - X^{13/4}$	X	X	X
$y_2^{(3)} = 2X - X^{5/2} - \varepsilon X^{13/4}$	X	X	X

The first column presents all the roots of $f = 0$; the second—the initial forms of solutions of $f(X, X + Y) = 0$; the third—$f(X, X + X^{3/2} + Y) = 0$ and the fourth—$f(X, X - X^{3/2} + Y) = 0$. This information allows us to reconstruct the relative polygons as well as their initial forms.

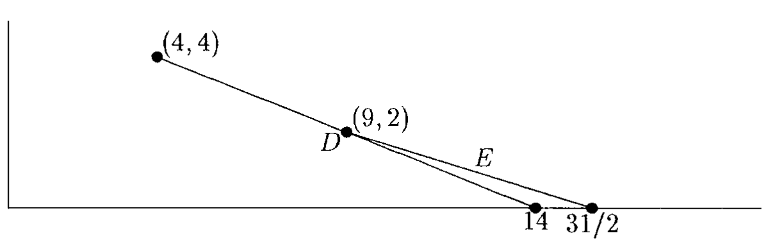


Polygon $\mathcal{N}(f)$ has one face A, which joins points $(0,8)$ and $(8,0)$, $\mathrm{in}(f,A) = (Y-X)^4(Y-2X)^4$, thus $t = t(f,A) = 2$; from this we obtain invariant $\alpha(A) = 8$ with multiplicity $t-1=1$. Polygon $\mathcal{N}_X$ has one face B, which joins points $(4,4)$ and $(10,0)$, $\mathrm{in}(f_X,B) = X^4(Y-X^{3/2})^2(Y+X^{3/2})^2$, thus $t = t(f_X,B) = 2$; we obtain invariant $\alpha(B) = 10$ with multiplicity $t-1=1$. Finally polygon $\mathcal{N}_{X+X^{3/2}}$ has a single face C, which joins $(7,2)$ and $(29/2,0)$, $\mathrm{in}(f_{X+X^{3/2}},C) = 4X^7(Y-X^{15/4})(Y+X^{15/4})$, thus $t = t(f_{X+X^{3/2}},C) = 2$; therefore we obtain invariant $\alpha(C) = 29/2$ with multiplicity $t-1=1$. Taking into account track $X-X^{3/2}$ we again obtain $\alpha(C) = 29/2$ with multiplicity 1.

With the help of the second table we analyze tracks $2X, 2X+X^{5/2}$ as well as $2X-X^{5/2}$.

$f(X,Y)$	$f(X,2X+Y)$	$f(X,2X+X^{5/2}+Y)$	$f(X,2X-X^{5/2}+Y)$
$y_1^{(0)} = X+X^{3/2}+X^{15/4}$	$-X$	$-X$	$-X$
$y_1^{(1)} = X-X^{3/2}-\varepsilon X^{15/4}$	$-X$	$-X$	$-X$
$y_1^{(2)} = X+X^{3/2}-X^{15/4}$	$-X$	$-X$	$-X$
$y_1^{(3)} = X-X^{3/2}+\varepsilon X^{15/4}$	$-X$	$-X$	$-X$
$y_2^{(0)} = 2X+X^{5/2}+X^{13/4}$	$X^{5/2}$	$X^{13/4}$	$2X^{5/2}$
$y_2^{(1)} = 2X-X^{5/2}+\varepsilon X^{13/4}$	$-X^{5/2}$	$-2X^{5/2}$	$\varepsilon X^{13/4}$
$y_2^{(2)} = 2X+X^{5/2}-X^{13/4}$	$X^{5/2}$	$-X^{13/4}$	$2X^{5/2}$
$y_2^{(3)} = 2X-X^{5/2}-\varepsilon X^{13/4}$	$-X^{5/2}$	$-2X^{5/2}$	$-\varepsilon X^{13/4}$

Polygon $\mathcal{N}_{2X}$ has one segment D, which joins $(4,4)$ and $(14,0)$, $\mathrm{in}(f_{2X},D) = X^4(Y-X^{5/2})^2(Y+X^{5/2})^2$, thus $t = t(f_{2X},D) = 2$; we obtain invariant $\alpha(D) = 14$ with multiplicity $t-1=1$.

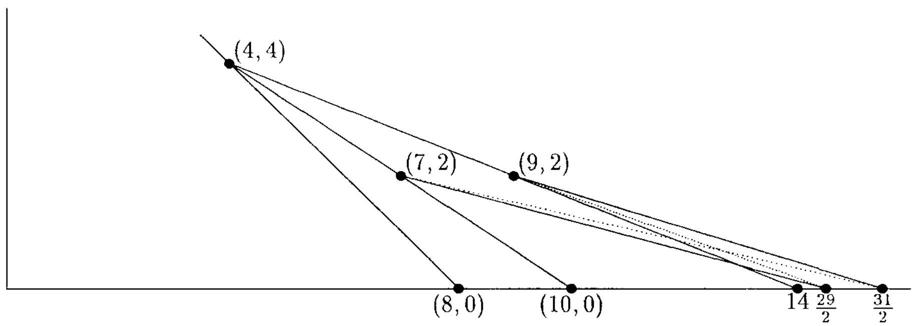


Polygon $\mathcal{N}_{2X+X^{5/2}}$ contains one face E, which joins $(9,2)$ and $(31/2,0)$, $\mathrm{in}(f_{2X+X^{5/2}},E) = 4X^9(Y-X^{13/4})(Y+X^{13/4})$, thus $t = t(f_{2X+X^{5/2}},E) = 2$; we obtain invariant $\alpha(E) = 31/2$ with multiplicity $t-1=1$. Considering track $2X-X^{5/2}$ we again obtain $\alpha(E) = 31/2$ with multiplicity 1.

The remaining tracks do not lead to new solutions. The complete system of invariants is as follows

$$\langle 8, 10, 14, 29/2, 29/2, 31/2, 31/2 \rangle.$$

An analogous analysis of series f' results in the same system. The idea behind Eggers' example becomes visible when we simultaneously mark faces A, B, C, D, E. Dotted lines signify new faces, which take the place of C and E for f'.



5 The case of one branch

Let us consider the case when $f \in \mathbf{k}[[X, Y]]$ is an irreducible series Y-regular of order p $(p > 1)$. Let us fix Puiseux solution

$$y(X) = a_1 X^{v_1/p} + a_2 X^{v_2/p} + \cdots \tag{10}$$

of equation $f = 0$ ($a_j \neq 0$, $0 < v_1 < v_2 <$ integers, $\mathrm{GCD}(p, v_1, v_2, \ldots) = 1$). This solution generates other solutions in the form of a cycle

$$y_i(X) = a_1 \varepsilon^{v_1 i} X^{v_1/p} + a_2 \varepsilon^{v_2 i} X^{v_2/p} + \cdots, \quad i = 0, \ldots, p-1, \tag{11}$$

where ε is a primitive root of unity of degree p. Let $\theta_j = v_j/p$. Let us consider a sequence of tracks of the Newton algorithm for f constructed from solution (10): $\varphi_0(X) = 0, \ldots, \varphi_j(X) = a_1 X^{\theta_1} + \cdots + a_j X^{\theta_j}, \ldots$. Let us put $f^{(j)} := f_{\varphi_{j-1}}$ and $\mathcal{N}^{(j)} := \mathcal{N}_{\varphi_{j-1}}$. It is convenient to denote $v_0 = p$, $\theta_0 = 0$. We have

Property 5.1.
(i) *Polygon* $\mathcal{N}^{(j)}$ *consists of one face* $S^{(j)}$, *with inclination* v_j/p, *which touches the horizontal axis.*
(ii) $\deg_Y \mathrm{in}(f^{(j)}, S^{(j)}) = \mathrm{GCD}(v_0, \ldots, v_{j-1})$ *for* $j = 1, 2, \ldots$.
(iii) *Every root of* $\mathrm{in}(f^{(j)}, S^{(j)})$ *has the multiplicity* $\mathrm{GCD}(v_0, \ldots, v_j)$.
(iv) $t(f^{(j)}, S^{(j)}) = \dfrac{\mathrm{GCD}(v_0, \ldots, v_{j-1})}{\mathrm{GCD}(v_0, \ldots, v_j)}$,

(v) $\alpha(S^{(j)}) = \sum_{j'=1}^{j}(\mathrm{GCD}(v_0,\dots,v_{j'-1}) - \mathrm{GCD}(v_0,\dots,v_{j'}))\dfrac{v_{j'}}{p} +$
$\mathrm{GCD}(v_0,\dots,v_j)\dfrac{v_j}{p}$.

Recall the definition of a generalized characteristic $(b_0,\dots,b_h)$:

$$b_0 = v_0,\dots,b_k = \min\{v_j > b_{k-1} \mid \mathrm{GCD}(v_0,\dots,v_{j-1}) > \mathrm{GCD}(v_0,\dots,v_j)\},$$
$$\dots,b_h = \min\{v_j \mid \mathrm{GCD}(v_0,\dots,v_j) = 1\}.$$

If $p = v_0 \le v_1$, then the generalized characteristic coincides with the topological characteristic ($\mathbf{k} = \mathbf{C}$). We call $b_1/p,\dots,b_h/p$ characteristic exponents. From Property 5.1 we obtain

Corollary 5.2. *$t(f^{(j)}, S^{(j)}) > 1$ if and only if v_j/p is a characteristic exponent.*

By $j_1,\dots,j_h$ let us denote the values of index j for characteristic exponents and let $\alpha_k = \alpha(S^{(j_k)})$, $k = 1,\dots,h$. Furthermore let

$$e_k = \mathrm{GCD}(v_0,\dots,v_{j_k}) = \mathrm{GCD}(b_0,\dots,b_k).$$

We have

Corollary 5.3. *All polar quotients obtained on the tracks of solution* (10) *have the form*

$$\alpha_k = \sum_{k'=1}^{k}(e_{k'-1} - e_{k'})\frac{b_{k'}}{p} + e_k\frac{b_k}{p}, \quad k = 1,\dots,h,$$

and appear with multiplicity $(e_{k-1}/e_k) - 1$.

Let $n_k = e_{k-1}/e_k$. To take into consideration the multiplicity deriving from the remaining tracks, let us write cycle (11) in the form

$$y_i(X) = \psi_0(X) + \psi_1(\varepsilon_1^{i_1}X^{1/n_1}) + \psi_2(\varepsilon_2^{i_1+n_1 i_2}X^{1/(n_1 n_2)})$$
$$+\cdots+\psi_h(\varepsilon_h^{i_1+n_1 i_2+\cdots+n_1 n_2\cdots n_{h-1}i_h}X^{1/(n_1 n_2\cdots n_h)}),$$

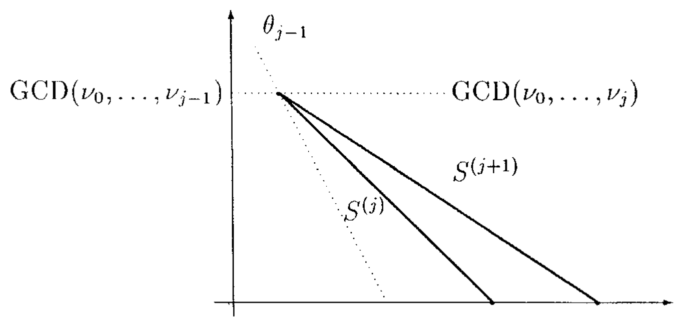
where $\psi_0,\dots,\psi_{h-1}\in\mathbf{k}[X]$, $\psi_h\in\mathbf{k}[[X]]$, $\varepsilon_h = \varepsilon$, $\varepsilon_k = \varepsilon^{n_{k+1}\cdots n_h}$ $(k = 1,\dots,h-1)$ and $i = i_1 + n_1 i_2 + \cdots + n_1 n_2\cdots n_{h-1}i_h$ is the unique decomposition of $i\in\{0,\dots,p-1\}$ such that $0\le i_1\le n_1 - 1,\dots,0\le i_h\le n_h - 1$. Therefore the track

$$\varphi_{j_k}(X) = \psi_0(X) + \psi_1(X^{1/n_1}) + \cdots + \psi_{k-1}(X^{1/(n_1\cdots n_{k-1})})$$

generates the cycle composed of $n_1 \cdots n_{k-1}$ elements. Hence the multiplicity of polar quotient α_k is

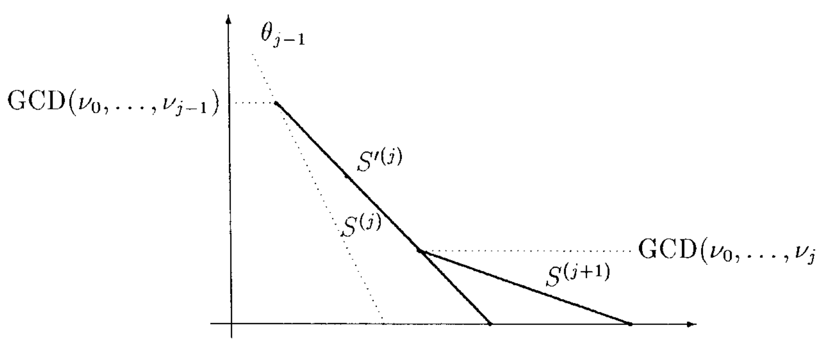
$$n_1 \cdots n_{k-1}(n_k - 1) = \frac{e_0}{e_1} \cdots \frac{e_{k-2}}{e_{k-1}} \left(\frac{e_{k-1}}{e_k} - 1 \right) = \frac{e_0}{e_k} - \frac{e_0}{e_{k-1}}.$$

Commentary. From Property 5.1 result the common relations between faces $S^{(j)}$ and $S^{(j+1)}$ depending whether or not j stands in the characteristic position. If $j \notin \{j_1, \ldots, j_h\}$, then $\mathrm{GCD}(v_0, \ldots, v_{j-1}) = \mathrm{GCD}(v_0, \ldots, v_j)$ and faces $S^{(j)}, S^{(j+1)}$ have a common upper end.



This means that polygon $\mathcal{N}^{(j+1)}$ does not contain a segment parallel to $S^{(j)}$.

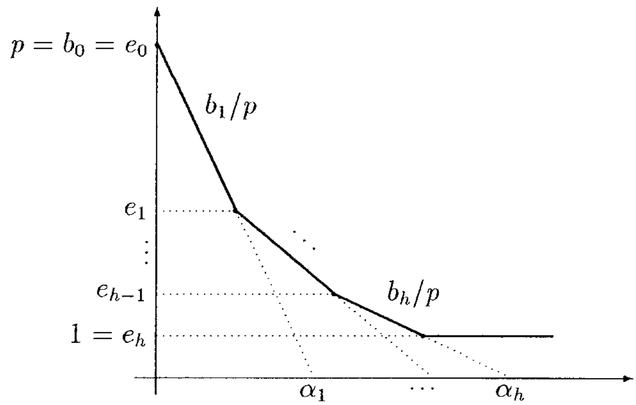
If $j = j_k$, then $\mathrm{GCD}(v_0, \ldots, v_{j-1}) = e_{k-1} > e_k = \mathrm{GCD}(v_0, \ldots, v_j)$.



Let us divide face $S^{(j)}$ into $n_k = e_{k-1}/e_k$ equal parts. Face $S^{(j+1)}$ appears in place of the lowest part. Polygon $\mathcal{N}^{(j+1)}$ contains a face $S'^{(j)}$ that is parallel to $S^{(j)}$. The length of the projection of $S'^{(j)}$ onto the vertical axis is

$$\mathrm{GCD}(v_0, \ldots, v_{j-1}) - \mathrm{GCD}(v_0, \ldots, v_j) = e_{k-1} - e_k.$$

The sequence of polygons $\mathcal{N}^{(1)}, \mathcal{N}^{(2)}, \ldots$ approaches the polygon of series $f(X, y(X) + Y)$, which is composed of segments $S'^{(j_1)}, \ldots, S'^{(j_h)}$.



The numbers $\alpha_1, \dots, \alpha_h$ represent the abscissae of the points where the lines determined by the faces intersect the horizontal axis (see: [GP1]).

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