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A MORSE INDEX THEOREM FOR GEODESICS ON A GLUED RIEMANNIAN SPACE

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Abstract

A glued Riemannian space is obtained from Riemannian manifolds M_1 and M_2 by identifying their isometric submanifolds B_1 and B_2. A curve on a glued Riemannian space which is a geodesic on each Riemannian manifold and satisfies certain passage law on the identified submanifold $B := B_1 \cong B_2$ is called a B-geodesic. Considering the variational problem with respect to arclength L of piecewise smooth curves through B, a critical point of L is a B-geodesic. A B-Jacobi field is a Jacobi field on each Riemannian manifold and satisfies certain passage condition on B. In this paper, we extend the Morse index theorem for geodesics in Riemannian manifolds to the case of a glued Riemannian space.

0. Introduction

In Riemannian manifolds, various results have been given on geodesics by many authors. Recently, N. Innami studied a geodesic reflecting at a boundary point of a Riemannian manifold with boundary in [5]. Let M be a Riemannian manifold with boundary B which is a union of smooth hypersurfaces. A curve on M is said to be a reflecting geodesic if it is a geodesic except at reflecting points and satisfies the reflection law. He dealt with the index form, conjugate points and so on, as in the case of a usual geodesic. Moreover, in [6], he generalized these to the case of a glued Riemannian manifold which is a space obtained from Riemannian manifolds with boundary by identifying their isometric boundary hypersurfaces. Some collapsing Riemannian manifolds are considered to be a kind of glued Riemannian manifolds. In [10] the author gave the definition of a glued Riemannian space which is obtained from Riemannian manifolds by identifying their isometric submanifolds B_1 and B_2 and is a generalization of a glued Riemannian manifold. A curve on a glued Riemannian space which is a geodesic on each Riemannian manifold and satisfies certain passage law on the identified submanifold $B := B_1 \cong B_2$ was called a B-geodesic. Considering the variational problem with respect to arclength L of piecewise smooth curves through B, a critical point of L is a B-geodesic. Also, the definitions of the index form of B-geodesics, B-Jacobi fields and B-conjugate

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points were given. A B-Jacobi field is a Jacobi field on each Riemannian manifold and satisfies certain passage condition on B. The purpose of this paper is to generalize the Morse index theorem for geodesics to the case of a glued Riemannian space. In Section 1, we review fundamental definitions, and results ([10]) on a glued Riemannian space. In Section 2, we give a precise statement of a Morse index theorem for B-geodesics, which relates the number of B-conjugate points on a B-geodesic γ, counted with their multiplicities, to the index of γ, and prove this theorem. Moreover, we make a comparison of the indices of B-geodesics in different glued Riemannian spaces, in Section 3.

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1. Preliminaries

Let N_μ and M_λ be manifolds (possibly with boundary) for $\mu = 1, \dots, k$ and $\lambda = 1, \dots, l$. We allow the case where $\dim N_\mu \neq \dim N_\nu$ and $\dim M_\kappa \neq \dim M_\lambda$ for $\mu \neq \nu$ and $\kappa \neq \lambda$. A map $\bar{\varphi} : \bar{N} \to \bar{M}$ from the topological direct sum $\bar{N} := N_1 \coprod \cdots \coprod N_k$ to $\bar{M} := M_1 \coprod \cdots \coprod M_l$ is *smooth* if $\bar{\varphi}|N_\mu$ is smooth. A *tangent bundle* $T\bar{M}$ *of* $\bar{M}$ is the direct sum $T\bar{M} = TM_1 \coprod \cdots \coprod TM_l$, where TM_λ denotes the tangent bundle of M_λ. We note that a tangent bundle $T\bar{M}$ on $\bar{M}$ is not constant rank vector bundle on $\bar{M}$. We put $T_p\bar{M} := T_pM_\lambda$ for $p \in M_\lambda$. We define a map $\pi_{\bar{M}} : T\bar{M} \to \bar{M}$ by

$$\pi_{\bar{M}}(v_p) := p \quad \text{for } v_p \in T_pM_\lambda.$$

A *vector field* $\bar{V}$ *on* $\bar{M}$ is a map $\bar{V} : \bar{M} \to T\bar{M}$ such that $\pi_{\bar{M}} \circ \bar{V} = \mathrm{id}_{\bar{M}}$, where $\mathrm{id}_{\bar{M}}$ is the identity map on $\bar{M}$. If $\bar{V}|M_\lambda : M_\lambda \to TM_\lambda$ is smooth vector field on each M_λ, then $\bar{V}$ is smooth. Let I_μ be a closed interval in $\boldsymbol{R}$ which is a manifold with boundary, for $\mu = 1, \dots, k$. A map $\bar{\alpha} : \bar{I} := I_1 \coprod \cdots \coprod I_k \to \bar{M}$ is called a *curve on* $\bar{M}$ if $\bar{\alpha}$ is smooth.

Let M_λ be a manifold (possibly with boundary) with a submanifold B_λ for $\lambda = 1, 2$ and ψ a diffeomorphism from B_1 to B_2. A *glued space* $M = M_1 \cup_\psi M_2$ is defined as follows: M is the quotient topological space obtained from the topological direct sum $\bar{M} = M_1 \coprod M_2$ of M_1 and M_2 by identifying $p \in B_1$ with $\psi(p) \in B_2$. We allow the case where $B_1 = B_2 = \emptyset$, $M_1 = \emptyset$ or $M_2 = \emptyset$, where ψ is the empty map. Let $\pi : \bar{M} \to M$ be the natural projection which is defined by $\pi(p) = [p]$, where $[p]$ is the equivalence class of p. Let N_λ be a manifold with a submanifold C_λ $(\lambda = 1, 2)$, $\tau : C_1 \to C_2$ a diffeomorphism and $N = N_1 \cup_\tau N_2$ a glued space. A *glued smooth map* $\varphi : \bar{N} \to M$ *on* $\bar{N}$ *derived from a smooth map* $\bar{\varphi} : \bar{N} \to \bar{M}$ or, simply, *a smooth map on* N is defined by $\varphi = \pi \circ \bar{\varphi}$. We note that a glued smooth map on $\bar{N}$ is considered as a map on N which, possibly, take two values at $[p]$ $(p \in C_\lambda)$. A glued smooth map φ is *continuous* if $\varphi(p) = \varphi(\tau(p))$ holds for any $p \in C_1$.

A *glued tangent bundle* TM *of* M is the glued space $TM_1 \cup_{\psi_*} TM_2$, where $\psi_* : TB_1 \to TB_2$ is the differential map of ψ. Let $\hat{\pi} : T\bar{M} \to TM$ be the natural projection which is defined by $\hat{\pi}(v) = [v]$, where $[v]$ is the equivalence class of v.

For $p \in \overline{M}$, we set $T_pM := \{\hat{\pi}(T_p\overline{M}) = [v] \in TM | v \in T_p\overline{M}\}$. We define a map $\pi_M : TM \to M$ by

$$\pi_M([v_p]) := [p] \quad \text{for } v_p \in T_p\overline{M}.$$

We note that $\pi \circ \pi_{\overline{M}} = \pi_M \circ \hat{\pi}$ holds. A *glued vector field* $V : \overline{M} \to TM$ *on* $\overline{M}$ *derived from a vector field* $\overline{V}$ *on* $\overline{M}$ or, simply, *a vector field on* M is defined by $V = \hat{\pi} \circ \overline{V}$. A glued vector field V is called a *smooth glued vector field* provide V is glued smooth. If a glued vector field V on $\overline{M}$ is continuous, then we can regard it as a cross section of TM over M; that is $\pi_M \circ V = \mathrm{id}_M$. Similarly, we can define a *glued vector field* (or *vector field*) *along a curve* $\bar{\alpha} : \bar{I} := I_1 \coprod I_2 \to \overline{M}$.

Let $T_p^*\overline{M}$ be the dual vector space of $T_p\overline{M}$. We put $T^*\overline{M} = T^*M_1 \coprod T^*M_2$, where T^*M_λ is the cotangent bundle of M_λ. For $\bar{\theta}_p$ $(\in T_p^*\overline{M})$, $\bar{\omega}_q$ $(\in T_q^*\overline{M}) \in T^*\overline{M}$, we define an equivalence relation $\sim$ as follows: $\bar{\theta}_p \sim \bar{\omega}_q$ if and only if $\bar{\theta}_p = \bar{\omega}_q$ $(p = q)$ or $\bar{\theta}_p|_{T_pB_1} = \psi^*(\bar{\omega}_q)$ $(p \in B_1, q = \psi(p))$ or $\bar{\omega}_q|_{T_qB_1} = \psi^*(\bar{\theta}_p)$ $(q \in B_1, p = \psi(q))$, where ψ^* is the dual map of ψ_*. The quotient space obtained from $T^*\overline{M}$ by this equivalence relation is denoted by T^*M. Let $\hat{\pi} : T^*\overline{M} \to T^*M$ be the natural projection, that is, $\hat{\pi}(\bar{\theta}) := [\bar{\theta}]$, where $[\bar{\theta}]$ is the equivalence class of $\bar{\theta}$. For $p \in \overline{M}$, we set $T_p^*M := \hat{\pi}(T_p^*\overline{M})$ and define a map $[\bar{\theta}] : T_pM \to \boldsymbol{R}$ by $[\bar{\theta}]([\bar{v}]) := \bar{\theta}(\bar{v})$ for $\bar{\theta} \in T_p^*\overline{M}$ and $\bar{v} \in T_p\overline{M}$. Then we can regard T_p^*M as the dual of T_pM. We put $T^{r,s}(\overline{M}) := T^{r,s}(M_1) \coprod T^{r,s}(M_2)$, where $T^{r,s}(M_\lambda)$ is the (r,s)-tensor bundle of M_λ. An (r,s)-*tensor field on* $\overline{M}$ is a cross section of $T^{r,s}(\overline{M})$. The definition of the *smoothness* of a tensor field on $\overline{M}$ is similar to that of a vector field on $\overline{M}$. Similarly, we can define the equivalence relation on $T^{r,s}(\overline{M})$ induced from those on $T\overline{M}$ and $T^*\overline{M}$, and denote the quotient space by $T^{r,s}(M)$. Let $\hat{\pi} : T^{r,s}(\overline{M}) \to T^{r,s}(M)$ be the natural projection. A *glued tensor field* T *derived from a tensor field* $\overline{T}$ *on* $\overline{M}$ is defined by $T = \hat{\pi} \circ \overline{T}$. A glued tensor field T derived from a tensor field $\overline{T}$ on $\overline{M}$ is (*glued*) *smooth* if $\overline{T}$ is smooth.

Definition 1.1. Let (M_λ, g_λ) be a Riemannian manifold with a Riemannian submanifold B_λ for $\lambda = 1, 2$ and ψ an isometry from B_1 to B_2. Let $\bar{g}$ be the *metric on* $\overline{M}$ which is defined to be $\bar{g}_p = (g_\lambda)_p$ for $p \in M_\lambda$. A *glued Riemannian space* $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$ is a pair of a glued space $M = M_1 \cup_\psi M_2$ and a *glued metric* g on M derived from $\bar{g}$ which is a glued tensor field derived from the $(0,2)$-tensor field $\bar{g}$.

We note that, for any glued smooth vector fields V and W on $\overline{M}$ derived from smooth vector fields $\overline{V}$ and $\overline{W}$ on $\overline{M}$, respectively, a map $g(V, W) : \overline{M} \to \boldsymbol{R}$ defined by

$$g(V, W)(p) := \bar{g}(\overline{V}_p, \overline{W}_p)$$

is glued smooth on $\overline{M}$ derived from a smooth map $\bar{g}(\overline{V}, \overline{W}) : \overline{M} \to \boldsymbol{R}$.

From now on, identifying B_1 with B_2 by ψ, we put $B := B_1 \cong B_2$ and $T_pB := T_pB_1 \cong T_pB_2$ for $p \in B$ and omit the symbol $[\cdot]$ of the equivalence class. In particular, $[M_\lambda] := \pi(M_\lambda)$ will be denoted by M_λ. We call a map $\alpha : [a, t_0] \coprod [t_0, b] \to M$ a *glued curve derived from a curve* $\bar{\alpha} : [a, t_0] \coprod [t_0, b] \to \overline{M}$ or, simply, *a curve on M* if $\alpha : [a, t_0] \coprod [t_0, b] \to M$ is a continuous glued smooth map derived from $\bar{\alpha}$. Let $\alpha : [a, t_0] \coprod [t_0, b] \to M$ be a glued curve derived from a curve $\bar{\alpha} : [a, t_0] \coprod [t_0, b] \to \overline{M}$. The (*glued*) *velocity vector field of* α is $\alpha' := \hat{\pi} \circ \bar{\alpha}'$. We put $\alpha'(t_0 - 0) := \hat{\pi} \circ \bar{\alpha}_1'(t_0)$ and $\alpha'(t_0 + 0) := \hat{\pi} \circ \bar{\alpha}_2'(t_0)$, where $\bar{\alpha}_1 := \bar{\alpha} \,|\, [a, t_0] : [a, t_0] \to \overline{M}$ and $\bar{\alpha}_2 := \bar{\alpha} \,|\, [t_0, b] : [t_0, b] \to \overline{M}$. We note that a glued velocity vector field is considered as a glued vector field along $\bar{\alpha}$ and not generally continuous. We call $\alpha : [a, b] \to M$ a *piecewise smooth curve on M* provided there is a partition $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ of $[a, b]$ such that $\alpha \,|\, [a_{i-1}, a_{i+1}] : [a_{i-1}, a_i] \coprod [a_i, a_{i+1}] \to M$ is a glued curve. We call a_j $(j = 1, \ldots, k)$ the *break*. A function $\lambda : [a, t_0] \coprod [t_0, b] \to \{1, 2\}$ is defined by

$$\lambda(t) := \begin{cases} 1 & \text{on } [a, t_0] \\ 2 & \text{on } [t_0, b] \end{cases}.$$

For simplicity, we put $\lambda := \lambda(t)$.

If M is a glued Riemannian space such that $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$, then, for $t_0 \in (a, b)$, let $\Omega_{t_0}(M_1, M_2; B) =: \Omega_{t_0}$ be the set of all piecewise smooth curves $\alpha : [a, b] \to M$ such that $\alpha(t_0) \in B$, $\alpha([a, t_0]) \subset M_1$ and $\alpha([t_0, b]) \subset M_2$. Moreover, if p and q are points of M_1 and M_2, respectively. Then let $\Omega_{t_0}(p, q) \subset \Omega_{t_0}$ be the set of all piecewise smooth curves $\alpha \in \Omega_{t_0}$ such that $\alpha(a) = p$ and $\alpha(b) = q$. The projection from T_pM_λ to T_pB is denoted by tan. Let D^λ be Levi-Civita connection of Riemannian manifold M_λ for $\lambda = 1, 2$. A curve $\gamma \in \Omega_{t_0}$ is a *B-geodesic* if γ satisfies the following conditions:

$$D^\lambda_{\gamma'}\gamma' = 0 \quad \text{on } M_\lambda, \tag{1.1}$$

that is, $\gamma \,|\, [a, t_0]$ and $\gamma \,|\, [t_0, b]$ are geodesics on M_1 and M_2, respectively,

$$\tan \gamma'(t_0 - 0) = \tan \gamma'(t_0 + 0), \tag{1.2}$$

$$g_1(\gamma'(t_0 - 0), \gamma'(t_0 - 0)) = g_2(\gamma'(t_0 + 0), \gamma'(t_0 + 0)). \tag{1.3}$$

We assume that geodesics and B-geodesics are parametrized by arclength.

Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. We define a linear map $Q_{u,v} : T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\} \to T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$ as

$$Q_{u,v}(w) = \left\{ w - \frac{g_1(w, \mathrm{nor}_1\, u)}{g_1(u, \mathrm{nor}_1\, u)}\, \mathrm{nor}_1\, u \right\} + \frac{g_1(w, \mathrm{nor}_1\, u)}{g_1(u, \mathrm{nor}_1\, u)}\, \mathrm{nor}_2\, v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\}$, where $\mathrm{nor}_\lambda : T_qM_\lambda \to T_qB^\perp$ is the projection. The following hold:

$$Q_{u,v}(x) = x \quad \text{for any } x \in T_qB.$$

$$Q_{u,v}(\mathrm{nor}_1\, u) = \mathrm{nor}_2\, v.$$

$$g_2(Q_{u,v}(w), x) = g_1(w, x)$$

for any $x \in T_qB$ and $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\ u\}$.

$$g_2(Q_{u,v}(w), Q_{u,v}(w)) = g_1(w, w)$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\ u\}$. Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. Then we have

$$Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(\gamma'(t_0-0)) = \gamma'(t_0+0).$$

Remark. Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. If we define a linear map $Q_{v,u} : T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\ v\} \to T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\ u\}$ as

$$Q_{v,u}(z) = \left\{ z - \frac{g_2(z, \mathrm{nor}_2\ v)}{g_2(v, \mathrm{nor}_2\ v)}\ \mathrm{nor}_2\ v \right\} + \frac{g_2(z, \mathrm{nor}_2\ v)}{g_2(v, \mathrm{nor}_2\ v)}\ \mathrm{nor}_1\ u$$

for any $z \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\ v\}$. The following hold:

$$Q_{u,v} \circ Q_{v,u} = \mathrm{id}, \quad Q_{v,u} \circ Q_{u,v} = \mathrm{id},$$

$$g_2(Q_{u,v}(w), z) = g_1(w, Q_{v,u}(z))$$

for $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\ u\}$ and $z \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\ v\}$.

If $\gamma \in \Omega_{t_0}(p,q)$ is a B-geodesic with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$, the set $T_\gamma\Omega_{t_0}$ consists of all vector fields Y along γ which satisfy the following condition:

$$Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y(t_0-0)) = Y(t_0+0). \tag{1.4}$$

A subspace $T_\gamma\Omega_{t_0}(p,q)$ in $T_\gamma\Omega_{t_0}$ is defined by

$$T_\gamma\Omega_{t_0}(p,q) := \{Y \in T_\gamma\Omega_{t_0} \mid Y(a) = 0,\ Y(b) = 0\}.$$

For $\lambda = 1, 2$, let R^λ be the Riemannian curvature tensor of a Riemannian manifold M_λ defined as

$$R^\lambda(X, Y)W := D^\lambda_X D^\lambda_Y W - D^\lambda_Y D^\lambda_X W - D^\lambda_{[X,Y]} W,$$

for any vector field X, Y and W on M_λ, and S^λ_Z the shape operator of $B \subset M_\lambda$ defined as

$$S^\lambda_Z(V) := -\tan D^\lambda_V Z,$$

for any vector field V tangent to B and Z normal to B. Especially, if $B = \{p\}$, we have that $S^\lambda_Z = 0$ for $Z \in T_pM_\lambda$. A vector field Y along a piecewise smooth curve $\alpha \in \Omega_{t_0}$ is a *tangent to* α if $Y = f\alpha'$ for some function f on $[a,b]$ and *perpendicular to* α if $g_\lambda(Y, \alpha') = 0$. If $\|\alpha'\|_\lambda \neq 0$, then each tangent space $T_{\alpha(t)}M_\lambda$ has a direct sum decomposition $\mathrm{Span}\{\alpha'(t)\} + \{\alpha'(t)\}^\perp$. Hence each vector field Y along α has a unique expression $Y = Y^T + Y^\perp$, where Y^T is tangent to α and $Y^\perp$ is perpendicular to α, that is,

$$Y^\perp = Y - \frac{g_\lambda(Y, \alpha')}{g_\lambda(\alpha', \alpha')}\alpha'.$$

If α is a B-geodesic, then $(Y^T)' = (Y')^T$ and $(Y^\perp)' = (Y')^\perp$.

Let $q \in B$ and $v \in T_qM_\lambda$ $(\lambda = 1, 2)$ is not tangent to B. A linear operator $P_\lambda^v : T_qB \oplus \mathrm{Span}\{\mathrm{nor}_\lambda\, v\} \to T_qB$ is defined by

$$P_\lambda^v(w) := w - \frac{g_\lambda(w, \mathrm{nor}_\lambda\, v)}{g_\lambda(v, \mathrm{nor}_\lambda\, v)} v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_\lambda\, v\}$ $(\subset T_qM_\lambda)$. We note that P_λ^v is surjective and $P_\lambda^v(v) = 0$.

Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. We define a symmetric linear map $A_{u,v} : T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\} \to T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$ as

$$A_{u,v}(w) = (S^1_{\mathrm{nor}_1 u} - S^2_{\mathrm{nor}_2 v})(P_2^v(w)) - \frac{g_2((S^1_{\mathrm{nor}_1 u} - S^2_{\mathrm{nor}_2 v})(P_2^v(w)), v)}{g_2(v, \mathrm{nor}_2\, v)}\, \mathrm{nor}_2\, v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$. We call this map $A_{u,v}$ a *passage endomorphism*. The following hold:

$$A_{u,v}(w) \perp v \quad \text{and} \quad A_{u,v}(v) = 0.$$

The *index form* $I_\gamma : T_\gamma\Omega_{t_0} \times T_\gamma\Omega_{t_0} \to \boldsymbol{R}$ of a B-geodesic $\gamma \in \Omega_{t_0}$ with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$ is the symmetric bilinear form defined as

$$\begin{aligned} I_\gamma(Y, W) = &\int_a^{t_0} \{g_1(Y^{\perp\prime}, W^{\perp\prime}) - g_1(R^1(Y, \gamma')\gamma', W)\}\, dt \\ &+ \int_{t_0}^{b} \{g_2(Y^{\perp\prime}, W^{\perp\prime}) - g_2(R^2(Y, \gamma')\gamma', W)\}\, dt \\ &+ g_2(A_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0 + 0)), W(t_0 + 0)), \end{aligned}$$

for all $Y, W \in T_\gamma\Omega_{t_0}$. It follows that

$$I_\gamma(Y, W) = I_\gamma(Y^\perp, W^\perp) \quad \text{for all } Y, W \in T_\gamma\Omega_{t_0}.$$

Thus there is no loss of information in restricting the index form I_γ to

$$T_\gamma^\perp\Omega_{t_0} := \{Y \in T_\gamma\Omega_{t_0} \mid Y \perp \gamma'\}.$$

We write $I_\gamma^\perp$ for this restriction. For $\gamma \in \Omega_{t_0}(p, q)$, we put

$$T_\gamma^\perp\Omega_{t_0}(p, q) := \{Y \in T_\gamma\Omega_{t_0}(p, q) \mid Y \perp \gamma'\}$$

and write $I_\gamma^{0,\perp}$ for the restriction of the index form I_γ to this.

Let $\mathrm{pr}_1 : T_{\gamma(t_0)}M_1 \to T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_1\, \gamma'(t_0 - 0)\}$ and $\mathrm{pr}_2 : T_{\gamma(t_0)}M_2 \to T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_2\, \gamma'(t_0 + 0)\}$ be orthogonal projections. For proofs of Lemmas without the proof in this section we refer the reader to [10]. The following holds:

Lemma 1.2. *Let $\gamma \in \Omega_{t_0}(p, q)$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. If Y and $W \in T_\gamma\Omega_{t_0}(p, q)$ have breaks $a_1 < \cdots < t_0 = a_j < \cdots < a_k$, then we have that*

$$
\begin{aligned}
&I_\gamma(Y, W) \\
&\quad = -\left\{\int_a^{t_0} g_1(Y^{\perp''} + R^1(Y,\gamma')\gamma', W^\perp)\,dt + \int_{t_0}^b g_2(Y^{\perp''} + R^2(Y,\gamma')\gamma', W^\perp)\,dt\right\} \\
&\qquad + g_2(A_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y(t_0+0)), W(t_0+0)) \\
&\qquad + g_1(\mathrm{pr}_1(Y^{\perp'}(t_0-0)), W^\perp(t_0-0)) - g_2(\mathrm{pr}_2(Y^{\perp'}(t_0+0)), W^\perp(t_0+0)) \\
&\qquad + \sum_{i=1}^{j-1} g_1(Y^{\perp'}(a_i-0) - Y^{\perp'}(a_i+0), W^\perp(a_i)) \\
&\qquad + \sum_{i=j+1}^{k} g_2(Y^{\perp'}(a_i-0) - Y^{\perp'}(a_i+0), W^\perp(a_i)) \\
&\qquad + g_2(Y^{\perp'}(b), W^\perp(b)) - g_1(Y^{\perp'}(a), W^\perp(a)).
\end{aligned}
$$

Let $\gamma \in \Omega_{t_0}$ be a B-geodesic. If it holds $a \le t_1 < t_2 \le t_0$, we set $T_{\gamma|[t_1,t_2]}\Omega = \{Y \mid \text{vector fields along } \gamma|[t_1,t_2]\}$. Then we define the map $\tilde{I}_{\gamma|[t_1,t_2]} : T_{\gamma|[t_1,t_2]}\Omega \times T_{\gamma|[t_1,t_2]}\Omega \to \boldsymbol{R}$ by

$$\tilde{I}_{\gamma|[t_1,t_2]}(Y, W) = \int_{t_1}^{t_2} \{g_1(Y^{\perp'}, W^{\perp'}) - g_1(R^1(Y,\gamma')\gamma', W)\}\,dt,$$

for all $Y, W \in T_{\gamma|[t_1,t_2]}\Omega$. If it holds $t_0 < t_1 < t_2 \le b$, we set $T_{\gamma|[t_1,t_2]}\Omega = \{Y \mid \text{vector fields along } \gamma|[t_1,t_2]\}$. Then we define the map $\tilde{I}_{\gamma|[t_1,t_2]} : T_{\gamma|[t_1,t_2]}\Omega \times T_{\gamma|[t_1,t_2]}\Omega \to \boldsymbol{R}$ by

$$\tilde{I}_{\gamma|[t_1,t_2]}(Y, W) = \int_{t_1}^{t_2} \{g_2(Y^{\perp'}, W^{\perp'}) - g_2(R^2(Y,\gamma')\gamma', W)\}\,dt,$$

for all $Y, W \in T_{\gamma|[t_1,t_2]}\Omega$.

Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. If $Y \in T_\gamma\Omega_{t_0}$ satisfies

$$Y'' + R^\lambda(Y,\gamma')\gamma' = 0 \quad \text{on } M_\lambda \ (\lambda = 1,2), \tag{1.5}$$

$$
\begin{aligned}
&-A_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y(t_0+0)) \\
&\qquad = Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(\mathrm{pr}_1(Y'(t_0-0))) - \mathrm{pr}_2(Y'(t_0+0)),
\end{aligned} \tag{1.6}
$$

and

$$g_1(Y'(t_0-0), \gamma'(t_0-0)) = g_2(Y'(t_0+0), \gamma'(t_0+0)), \tag{1.7}$$

then Y is called a *B-Jacobi field along γ*. Let $\mathscr{J}_\gamma$ be the set of all B-Jacobi fields along γ. A B-Jacobi field Y along γ is *perpendicular* if Y is perpendicular to γ. Let $\mathscr{J}_\gamma^\perp$ be the set of all perpendicular B-Jacobi fields along γ. Let $\mathscr{J}_\gamma^0$ be the set of all B-Jacobi field $Y \in \mathscr{J}_\gamma$ such that $Y(a) = 0$.

If Y is a B-Jacobi field along γ, then we have that

$$I_\gamma(Y, Y) = g_2(Y^{\perp'}(b), Y^\perp(b)) - g_1(Y^{\perp'}(a), Y^\perp(a)). \tag{1.8}$$

Lemma 1.3. *Let $\gamma \in \Omega_{t_0}(p,q)$ be a B-geodesic with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. Then $Y \in T_\gamma^\perp \Omega_{t_0}(p,q)$ is an element of the nullspace of $I_\gamma^{0,\perp}$ if and only if Y is a B-Jacobi field along γ.*

Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. We say that $\gamma(t_2)$ ($t_2 \in (a,b]$) is a *B-conjugate point to $\gamma(t_1)$ ($t_1 \in [a,b), t_1 < t_2$) along γ* if there exists a B-Jacobi field Y along γ such that $Y(t_1) = 0$, $Y(t_2) = 0$ and $Y \mid [t_1, t_2]$ is nontrivial.

B-conjugate points in M_1 are always usual ones but the converse is not true in general. We give an example which shows this:

Example 1. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following M_λ and B a submanifold of M_λ ($\lambda = 1,2$):

$$M_1 = S^2(1) = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}, \quad M_2 = \boldsymbol{E}^3, \quad B = \{(0,-1,0)\},$$

and g_1 is a Riemannian metric induced from the natural Euclidean metric of $\boldsymbol{E}^3$ and g_2 is the natural Euclidean metric of $\boldsymbol{E}^3$. We defined a B-geodesic $\gamma : [-\pi/2, +\infty) \to M$ by

$$\gamma(t) = \begin{cases} (0, \cos t, \sin t) & \text{on } [-\pi/2, \pi] \\ (0, -t + \pi - 1, 0) & \text{on } [\pi, +\infty) \end{cases}.$$

Then, $T_\gamma \Omega_{t_0}$ is the set of all vector fields Y along γ such that $Y \mid [a, t_0]$ and $Y \mid [t_0, b]$ are piecewise smooth vector fields on M_1 and M_2, respectively, and, $Y(t_0 - 0) = d\gamma'(t_0 - 0)$ and $Y(t_0 + 0) = d\gamma'(t_0 + 0)$ for some $d \in \boldsymbol{R}$. Hence, $\gamma(\pi/2)$ is a conjugate point to $\gamma(-\pi/2)$ but not a B-conjugate point.

We define the function $\rho_K : [a,b] \to \boldsymbol{R}$ and $f_K : [a,b] \to \boldsymbol{R}$ by

$$\rho_K(t) = \begin{cases} t & \text{if } K = 0 \\ \dfrac{1}{\sqrt{K}} \tan \sqrt{K}t & \text{if } K > 0 \\ \dfrac{1}{\sqrt{-K}} \tanh \sqrt{-K}t & \text{if } K < 0 \end{cases}$$

and

$$f_K(t) = \begin{cases} t & \text{if } K = 0 \\ \dfrac{1}{\sqrt{K}} \sin \sqrt{K}t & \text{if } K > 0 \\ \dfrac{1}{\sqrt{-K}} \sinh \sqrt{-K}t & \text{if } K < 0 \end{cases},$$

respectively.

Lemma 1.4. *Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. Then there are $\tilde{a}$ and $\tilde{b}$ ($a \le \tilde{a} < t_0 < \tilde{b} \le b$) such that $\gamma(t)$ is not a conjugate point to $\gamma(\tilde{a})$ for any $t \in (\tilde{a}, t_0]$ and $\gamma(t)$ is not a B-conjugate point to $\gamma(\tilde{a})$ for any $t \in (t_0, \tilde{b}]$.*

To show this lemma it is necessary to use the following proposition:

Proposition ([11]). *Let* $\gamma \in \Omega_{t_0}$ *be a* B*-geodesic with* $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. *Let* K_1 *be any real number such that* $f_{K_1}(t-a) > 0$ *for any* $t \in (a, t_0]$. *Let* δ *be any real number. We assume that* $K_2 := K_1$ *if* $\delta = 0$ *and* K_2 *is any real number if* $\delta \neq 0$. *Let* $b_1(> t_0)$ *be the smallest value which satisfies*

$$\delta = \frac{-1}{\rho_{K_1}(t_0 - a)} + \frac{-1}{\rho_{K_2}(t - t_0)},$$

and $b_2(> t_0)$ *the smallest value which satisfies* $f_{K_2}(t - t_0) = 0$, *where* $b_i := \infty$ $(i = 1, 2)$ *if there are no such* b_i. *Moreover, we put* $\tilde{b} := \min\{b, b_1, b_2\}$. *Assume that* $\dim B > 0$,

$$(\text{the maximal eigenvalue of } R_t^{\lambda}) \leq K_{\lambda} \quad \textit{for any } t \in [a, b]$$

and

$$(\text{the minimal eigenvalue of } A) \geq \delta.$$

Then there are no conjugate points along $\gamma \,|\, [a, t_0]$ *and no* B*-conjugate points along* $\gamma \,|\, [a, \tilde{b})$ *to* $\gamma(a)$.

Proof of Lemma 1.4. In case where $\dim B = 0$, the assertion is trivial. We assume that $\dim B > 0$. Choose a real number K and δ such that

$$(\text{the maximal eigenvalue of } R_t^{\lambda}) \leq K \quad \text{for any } t \in [a, b]$$

and

$$(\text{the minimal eigenvalue of } A) \geq \delta.$$

Moreover, choose $\tilde{a}$ $(a \leq \tilde{a} < t_0)$ such that

$$f_K(t - \tilde{a}) > 0 \quad \text{for any } t \in (\tilde{a}, t_0].$$

Let $b_1(> t_0)$ be the smallest value which satisfies

$$\delta = \frac{-1}{\rho_K(t_0 - \tilde{a})} + \frac{-1}{\rho_K(t - t_0)},$$

and $b_2(> t_0)$ the smallest value which satisfies $f_K(t - t_0) = 0$, where $b_i := \infty$ $(i = 1, 2)$ if there are no such b_i. Moreover, we put $b_0 := \min\{b, b_1, b_2\}$. Then, by taking $\tilde{b}$ as $t_0 < \tilde{b} < b_0$ the assertion holds from the above proposition. □

Lemma 1.5. *Let* $\gamma \in \Omega_{t_0}$ *be a* B*-geodesic with* $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. *We assume that* $\gamma(t_0)$ *and* $\gamma(b)$ *are not* B*-conjugate points to* $\gamma(a)$. *Then, for any* $v_1 \in T_{\gamma(a)}M_1$ *and* $v_2 \in T_{\gamma(b)}M_2$, *there is a unique* $Y \in \mathcal{J}_\gamma$ *with* $Y(a) = v_1$ *and* $Y(b) = v_2$.

Lemma 1.6. *Let* $\gamma \in \Omega_{t_0}$ *be a* B*-geodesic with* $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. *If* $\gamma(t)$ *is not a conjugate point to* $\gamma(a)$ *for any* $t \in (a, t_0]$ *and* $\gamma(t)$ *is not a* B*-conjugate point* *to* $\gamma(a)$ *for any* $t \in (t_0, b]$, *then, for any* $Y \in T_\gamma \Omega_{t_0}$ *with* $Y(a) = 0$, *there exist a unique B-Jacobi field* $J \in \mathscr{J}_\gamma^0$ *such that* $J(b) = Y(b)$ *and*

$$I_\gamma(J, J) \le I_\gamma(Y, Y).$$

In particular, the equality holds if and only if $J^\perp = Y^\perp$.

Lemma 1.7. *Let* $\gamma \in \Omega_{t_0}$ *be a B-geodesic with* $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. *If* $\gamma(t)$ *is not a conjugate point to* $\gamma(a)$ *for any* $t \in (a, t_0]$ *and* $\gamma(t)$ *is not a B-conjugate point to* $\gamma(a)$ *for any* $t \in (t_0, b]$, *then, for any* $Y \in T_\gamma \Omega_{t_0}$, *there exist a unique B-Jacobi field* $J \in \mathscr{J}_\gamma$ *such that* $J(a) = Y(a)$, $J(b) = Y(b)$ *and*

$$I_\gamma(J, J) \le I_\gamma(Y, Y).$$

In particular, the equality holds if and only if $J^\perp = Y^\perp$.

Proof. By Lemma 1.6, we obtain that

$$0 \le I_\gamma(J - Y, J - Y) = I_\gamma(J, J) - 2I_\gamma(J, Y) + I_\gamma(Y, Y). \tag{1.9}$$

Moreover, from (1.8), we get

$$\begin{aligned} I_\gamma(J, Y) &= g_2({J^\perp}'(b), Y^\perp(b)) - g_1({J^\perp}'(a), Y^\perp(a)) \\ &= g_2({J^\perp}'(b), J^\perp(b)) - g_1({J^\perp}'(a), J^\perp(a)) = I_\gamma(J, J). \end{aligned}$$

It follows that $I_\gamma(J, J) \le I_\gamma(Y, Y)$, and the equality of (1.9) holds if and only if $J^\perp - Y^\perp = (J - Y)^\perp = 0$. $\square$

2. Index theorem

Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. Given a B-conjugate point $\gamma(c)$, $a < c \le b$, to $\gamma(a)$, its *multiplicity* (or *order*) $\tilde{\mu}$ is defined to be the dimension of the space of all B-Jacobi fields along γ which vanish at a and c. We note that if $\gamma(c)$ is not B-conjugate point to $\gamma(a)$, the multiplicity of $\gamma(c)$ is zero. Moreover, we note that, for B-conjugate point $\gamma(c)$ $(a < c < t_0)$ to $\gamma(a)$, (the multiplicity of $\gamma(c)$) $\le$ (the multiplicity of $\gamma(c)$ as a conjugate point), since B-conjugate points in M_1 are always usual ones but the converse is not true. We assume that $\gamma(t_0)$ is not conjugate point to $\gamma(a)$, then $\tilde{\mu} \le m_2 - 1$ since $\dim \mathscr{J}_\gamma^{0,\perp} = m_2 - 1$ where $\mathscr{J}_\gamma^{0,\perp} := \mathscr{J}_\gamma^0 \cap \mathscr{J}_\gamma^\perp$ and $m_2 = \dim M_2$ (see [10]).

In general, given a symmetric bilinear form I on a vector space V, the *index* $i(I)$, the *augmented index* $a(I)$ and the *nullity* $n(I)$ of I are defined by

$i(I) :=$ the maximum dimension of those subspaces of V on which I is negative definite;

$a(I) :=$ the maximum dimension of those subspaces of V on which I is negative semi-definite;

$n(I) := \dim\{v \in V \mid I(v, w) = 0 \text{ for all } w \in V\}$.

Lemma 2.1 ([7]). *If I is a symmetric bilinear form on a finite-dimensional vector space V, then $a(I) = i(I) + n(I)$.*

For a B-geodesic $\gamma \in \Omega_{t_0}(p,q)$ with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$, we put

$$L := \{Y \in T_\gamma^\perp \Omega_{t_0}(p,q) \mid I_\gamma^\perp(Y,W) = 0 \text{ for all } W \in T_\gamma^\perp \Omega_{t_0}(p,q)\}.$$

We consider the index, the augmented index and the nullity of the index form $I_\gamma^{0,\perp}$ restricted I_γ to $T_\gamma^\perp \Omega_{t_0}(p,q)$. The purpose of this section is to give a proof of the index theorem:

Theorem 2.2 (Index theorem). *Let $\gamma \in \Omega_{t_0}(p,q)$ be a B-geodesic such that $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$ and $\gamma(t_0)$ is not conjugate point to $\gamma(a)$. Then there are only finitely many points $\gamma(t_1),\dots,\gamma(t_m)$ $(a < t_1 < \cdots < t_m < t_0)$ which are conjugate to $\gamma(a)$ along $\gamma \mid [a,t_0]$ and finitely many points $\gamma(t_{m+1}),\dots,\gamma(t_l)$ $(t_0 < t_{m+1} < \cdots < t_l < b)$ other than $\gamma(b)$ which are B-conjugate to $\gamma(a)$ along γ. Let μ_i be the multiplicity of $\gamma(t_i)$ $(i = 1,\dots,m)$ as a conjugate point to $\gamma(a)$ and $\tilde{\mu}_i$ $(i = 1,\dots,l)$ the multiplicity of $\gamma(t_i)$. Then it holds that*

$$i(I_\gamma^{0,\perp}) = \mu_1 + \cdots + \mu_m + \tilde{\mu}_{m+1} + \cdots + \tilde{\mu}_l \geq \tilde{\mu}_1 + \cdots + \tilde{\mu}_l.$$

We give an example where $\mu_1 + \cdots + \mu_m + \tilde{\mu}_{m+1} + \cdots + \tilde{\mu}_l \neq \tilde{\mu}_1 + \cdots + \tilde{\mu}_l$ holds.

Example 2. In example 1, $\gamma(\pi/2)$ is a conjugate point to $\gamma(-\pi/2)$ but not a B-conjugate point. Let μ_1 be the multiplicity of $\gamma(\pi/2)$ as a conjugate point to $\gamma(-\pi/2)$ and $\tilde{\mu}_1$ the multiplicity of $\gamma(\pi/2)$. Then it holds that

$$i(I_\gamma^{0,\perp}) = \mu_1 = 1 > \tilde{\mu}_1 = 0.$$

Theorem 2.3. *Let $\gamma \in \Omega_{t_0}(p,q)$ be a B-geodesic with $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$. Then*

(1) *$n(I_\gamma^{0,\perp}) = 0$ if $\gamma(b)$ is not B-conjugate point to $\gamma(a)$,*

(2) *$n(I_\gamma^{0,\perp}) =$ the multiplicity of $\gamma(b)$ if $\gamma(b)$ is B-conjugate point to $\gamma(a)$.*

Proof. By Lemma 1.3, we have

$$n(I_\gamma^{0,\perp}) = \dim L = \dim\{Y \in T_\gamma^\perp \Omega_{t_0}(p,q) \mid Y \in \mathscr{J}_\gamma\}.$$

This proves (1) and (2). □

Theorem 2.4. *Let $\gamma \in \Omega_{t_0}(p,q)$ be a B-geodesic such that $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$ and $\gamma(t_0)$ is not conjugate point to $\gamma(a)$. Then*

$$a(I_\gamma^{0,\perp}) = i(I_\gamma^{0,\perp}) + n(I_\gamma^{0,\perp}).$$

Proof. We will construct a finite-dimensional subspace L_1 of $T_\gamma^\perp \Omega_{t_0}(p,q)$ such that $i(I_\gamma^{0,\perp}) = i(I_\gamma|L_1)$, $a(I_\gamma^{0,\perp}) = a(I_\gamma|L_1)$ and $n(I_\gamma^{0,\perp}) = n(I_\gamma|L_1)$. By

Lemma 1.4, we can take a subdivision $a = a_0 < a_1 < \cdots < a_j = t_0 < a_{j+1} < \cdots < a_k < a_{k+1} = b$ of the interval $[a, b]$ such that $\gamma(t)$ is not a conjugate point to $\gamma(a_i)$ for any $t \in (a_i, a_{i+1}]$ $(i = 0, 1, \ldots, j-2, j+1, \ldots, k)$, $\gamma(t)$ is not a conjugate point to $\gamma(a_{j-1})$ for any $t \in (a_{j-1}, t_0]$ and $\gamma(t)$ is not a B-conjugate point to $\gamma(a_{j-1})$ for any $t \in (t_0, a_{j+1}]$. We set

$$\begin{aligned} L_1 &:= L(a_0, \ldots, a_{k+1}) \\ &:= \{ Y \in T_\gamma^\perp \Omega_{t_0}(p, q) \mid Y \text{ is a Jacobi field along } \gamma \,|\, [a_i, a_{i+1}] \text{ for} \\ &\qquad i = 0, \ldots, j-2,\ j+1, \ldots, k \text{ and a } B\text{-Jacobi field along } \gamma \,|\, [a_{j-1}, a_{j+1}] \}. \end{aligned}$$

Let $N(a_i)$ be the normal space to γ at $\gamma(a_i)$, that is,

$$N(a_i) = \{\gamma'(a_i)\}^\perp := \{ v \in T_{\gamma(t_0)} M_\lambda \mid g_\lambda(v, \gamma'(a_i)) = 0 \},$$

and define a linear map

$$\mathcal{N} : L_1 \to N := N(a_1) \times \cdots \times N(a_{j-1}) \times N(a_{j+1}) \times \cdots \times N(a_k)$$

by

$$\mathcal{N}(Y) := (Y(a_1), \ldots, Y(a_{j-1}), Y(a_{j+1}), \ldots, Y(a_k)).$$

Lemma 2.5. (1) *$\mathcal{N}$ is a linear isomorphism of L_1 onto N;*
(2) *Define a map $\rho : T_\gamma^\perp \Omega_{t_0}(p, q) \to L_1$ by setting*

$$\rho(Y) := \mathcal{N}^{-1}(Y(a_1), \ldots, Y(a_{j-1}), Y(a_{j+1}), \ldots, Y(a_k))$$

for $Y \in T_\gamma^\perp \Omega_{t_0}(p, q)$. Then

$$I_\gamma(Y, Y) \geq I_\gamma(\rho(Y), \rho(Y)) \quad \textit{for} \;\; Y \in T_\gamma^\perp \Omega_{t_0}(p, q),$$

and the equality holds if and only if $Y \in L_1$.
(3) *$i(I_\gamma^{0,\perp}) = i(I_\gamma | L_1)$, $a(I_\gamma^{0,\perp}) = a(I_\gamma | L_1)$ and $n(I_\gamma^{0,\perp}) = n(I_\gamma | L_1)$.*

Lemma 2.1 and Lemma 2.5(3) imply Theorem 2.4. □

Proof of Lemma 2.5. (1) Suppose $Y \in L_1$ and $\mathcal{N}(Y) = 0$ so that $Y(a_i) = 0$ for $i = 1, \ldots, j-1, j+1, \ldots, k$. By our choice of a_i, $Y = 0$, proving that $\mathcal{N}$ is injective. To show that $\mathcal{N}$ is surjective, it suffices to prove that, given vectors v_i at $\gamma(a_i)$ and v_{i+1} at $\gamma(a_{i+1})$, there is a Jacobi field Y along $\gamma \,|\, [a_i, a_{i+1}]$ which extends v_i and v_{i+1} for $i = 1, \ldots, j-2, j+1, \ldots, k-1$, and given vectors v_{j-1} at $\gamma(a_{j-1})$ and v_{j+1} at $\gamma(a_{j+1})$, there is a B-Jacobi field Y along $\gamma \,|\, [a_{j-1}, a_{j+1}]$ which extends v_{j-1} and v_{j+1}. Since $\gamma(a_{i+1})$ is not conjugate point to $\gamma(a_i)$, $Y \mapsto (v_i, v_{i+1})$ defines a linear isomorphism of the space of Jacobi fields along $\gamma \,|\, [a_i, a_{i+1}]$ into the direct sum of the tangent spaces at $\gamma(a_i)$ and $\gamma(a_{i+1})$ for $i = 1, \ldots, j-2$, $j+1, \ldots, k-1$. Moreover since $\gamma(t_0)$ and $\gamma(a_{j+1})$ are not conjugate points to $\gamma(a_{j-1})$, $Y \mapsto (v_{j-1}, v_{j+1})$ defines a linear isomorphism of the space of B-Jacobi fields along $\gamma \,|\, [a_{j-1}, a_{j+1}]$ into the direct sum of the tangent spaces at $\gamma(a_{j-1})$ and $\gamma(a_{j+1})$. Since they are linear isomorphisms of a vector space into a vector space of the same dimension (cf. Lemma 1.5), it must be surjective. This completes the proof of (1).

(2) With the notations in the Section 1, we have

$$I_\gamma^\perp(Y,Y)=\sum_{i=0}^{j-2}\tilde I_{\gamma\,|\,[a_i,a_{i+1}]}(Y,Y)+I_{\gamma\,|\,[a_{j-1},a_{j+1}]}(Y,Y)+\sum_{i=j+1}^{k}\tilde I_{\gamma\,|\,[a_i,a_{i+1}]}(Y,Y)$$

and

$$I_\gamma^\perp(\rho(Y),\rho(Y))=\sum_{i=0}^{j-2}\tilde I_{\gamma\,|\,[a_i,a_{i+1}]}(\rho(Y),\rho(Y))+I_{\gamma\,|\,[a_{j-1},a_{j+1}]}(\rho(Y),\rho(Y))$$
$$+\sum_{i=j+1}^{k}\tilde I_{\gamma\,|\,[a_i,a_{i+1}]}(\rho(Y),\rho(Y)).$$

By Proposition 3.1 in [7], we have

$$\tilde I_{\gamma\,|\,[a_i,a_{i+1}]}(Y,Y)\ge\tilde I_{\gamma\,|\,[a_i,a_{i+1}]}(\rho(Y),\rho(Y))$$

for $i=0,\dots,j-2,j+1,\dots,k$ and the equality holds if and only if Y is a Jacobi field along $\gamma\,|\,[a_i,a_{i+1}]$. By Lemma 1.7, we have

$$I_{\gamma\,|\,[a_{j-1},a_{j+1}]}(Y,Y)\ge I_{\gamma\,|\,[a_{j-1},a_{j+1}]}(\rho(Y),\rho(Y))$$

and the equality holds if and only if Y is a B-Jacobi field along $\gamma\,|\,[a_{j-1},a_{j+1}]$.

(3) If U is a subspace of $T_\gamma^\perp\Omega_{t_0}(p,q)$ on which $I_\gamma^{0,\perp}$ is negative semi-definite, then $I_\gamma^{0,\perp}$ is negative semi-definite on $\rho(U)$ by (2). Moreover, $\rho|U:U\to\rho(U)$ ($\subset L_1$) is a linear isomorphism. In fact, if $Y\in U$ and $\rho(Y)=0$, then (2) implies

$$0\ge I_\gamma(Y,Y)\ge I_\gamma(\rho(Y),\rho(Y))=0,$$

and hence $I_\gamma(Y,Y)=I_\gamma(\rho(Y),\rho(Y))$. Again by (2), we have $Y=\rho(Y)=0$. Thus $\rho|U$ is injective. It is clear that $\rho|U$ is surjective and linear. Moreover we have $a(I_\gamma^{0,\perp})\le a(I_\gamma|L_1)$. The reverse inequality is obvious. The proof for the index $i(I_\gamma^{0,\perp})$ is similar. Finally, to prove $n(I_\gamma^{0,\perp})=n(I_\gamma|L_1)$, let Y be an element of L_1 such that $I_\gamma^\perp(Y,W)=0$ for all $W\in L_1$. Since Y is a Jacobi field along $\gamma\,|\,[a_i,a_{i+1}]$ for $i=0,\dots,j-2,j+1,\dots,k$ and a B-Jacobi field along $\gamma\,|\,[a_{j-1},a_{j+1}]$, we have that

$$I_\gamma(Y,W)=\sum_{i=1}^{j-1}g_1(Y'(a_i-0)-Y'(a_i+0),W(a_i))$$
$$+\sum_{i=j+1}^{k}g_2(Y'(a_i-0)-Y'(a_i+0),W(a_i))$$

from Lemma 1.2. In the same way as we prove Lemma 1.3, we conclude that $Y'(a_i-0)=Y'(a_i+0)$ for $i=1,\dots,j-1,j+1,\dots,k$ so that Y is a B-Jacobi field along γ. This means that $n(I_\gamma^{0,\perp})\ge n(I_\gamma|L_1)$. The reverse inequality is obvious. $\square$

Proof of Theorem 2.2. Since $\dim L_1 < \infty$, (3) of Lemma 2.5 implies that both $a(I_\gamma^{0,\perp})$ and $i(I_\gamma^{0,\perp})$ are finite. The finiteness of B-conjugate points follows from the next lemma.

LEMMA 2.6. *For any finite number of conjugate points* $\gamma(t_1), \dots, \gamma(t_m)$ $(a < t_1 < \cdots < t_m < t_0)$ *to* $\gamma(a)$ *along* $\gamma \,|\, [a, t_0]$ *with multiplicity* $\mu_1, \dots, \mu_m$ *as conjugate points and B-conjugate points* $\gamma(t_{m+1}), \dots, \gamma(t_l)$ $(t_0 < t_{m+1} < \cdots < t_l < b)$ *to* $\gamma(a)$ *along* γ *with multiplicity* $\tilde{\mu}_{m+1}, \dots, \tilde{\mu}_l$, *we have*

$$a(I_\gamma^{0,\perp}) \geq \mu_1 + \cdots + \mu_m + \tilde{\mu}_{m+1} + \cdots + \tilde{\mu}_l.$$

Proof. For simplicity, we put $\mu_i := \tilde{\mu}_i$ $(i = m+1, \dots, l)$. For each i, let $\tilde{Y}_1^i, \dots, \tilde{Y}_{\mu_i}^i$ be a basis for the Jacobi fields along $\gamma \,|\, [a, t_0]$ or the B-Jacobi fields along γ which vanish at $t = a$ and $t = t_i$. We put, $j = 1, \dots, \mu_i$,

$$Y_m^i := \begin{cases} \tilde{Y}_m^i & \text{on } [a, t_i] \\ 0 & \text{on } [t_i, b] \end{cases}.$$

It suffices to prove that $\mu_1 + \cdots + \mu_l$ vector fields $Y_1^i, \dots, Y_{\mu_i}^i$, $i = 1, \dots, l$, along γ are linearly independent and that I_γ is negative semi-definite on the space spanned by them. Suppose

$$\sum_{i=1}^{l} Y^i = 0,$$

where

$$Y^i = c_1^i Y_1^i + \cdots + c_{\mu_i}^i Y_{\mu_i}^i.$$

Since $Y^1, \dots, Y^{l-1}$ vanish on $\gamma \,|\, [t_{l-1}, b]$, Y^l must vanish along $\gamma \,|\, [t_{l-1}, t_l]$. Being a B-Jacobi field or a Jacobi field along $\gamma \,|\, [a, t_l]$, Y^l must vanish identically along γ, since $\gamma(t_0)$ is not a conjugate point to $\gamma(a)$. Thus, $c_1^l = \cdots = c_{\mu_l}^l = 0$. Continuing this argument, we obtain $c_1^{l-1} = \cdots = c_{\mu_{l-1}}^{l-1} = 0$, and so on. To prove that I_γ is negative semi-definite on the space spanned by $Y_1^i, \dots, Y_{\mu_i}^i$, $i = 1, \dots, l$, let

$$Y = Y^1 + \cdots + Y^l,$$

where each Y^i is a linear combination of $Y_1^i, \dots, Y_{\mu_i}^i$ as above. Then

$$I_\gamma(Y, Y) = \sum_{i=1}^{l} I_\gamma(Y^i, Y^i) + 2 \sum_{1 \leq s < i \leq l} I_\gamma(Y^i, Y^s).$$

For each pair (i, s) with $s \leq i$, we shall show that $I_\gamma(Y^i, Y^s) = 0$. Let $\bar{\gamma} = \gamma \,|\, [a, t_i]$. Since Y^i and Y^s vanish beyond $t = t_i$, we have $I_\gamma(Y^i, Y^s) = I_{\bar{\gamma}}(Y^i, Y^s)$. As Y^i is a B-Jacobi field or a Jacobi field along $\bar{\gamma}$, $I_{\bar{\gamma}}(Y^i, Y^s) = 0$ by Lemma 1.3. Thus, $I_\gamma(Y, Y) = 0$, proving our assertion. $\square$

Let γ_r denote the restriction of γ to the interval $[a, b_r]$, where $b_r = rb + (1-r)a$ for $0 < r \leq 1$. Thus $\gamma_r : [a, b_r] \to M$ is a B-geodesic from $\gamma(a)$ to $\gamma(b_r)$ if $(t_0 - a)/(b-a) < r \leq 1$ and a geodesic in M_1 if $0 < r \leq (t_0 - a)/(b-a)$. Let I_r denote the index form associated with this B-geodesic or geodesic. Thus $i(I_1)$ is the index which we are actually trying to compute. First note that:

Assertion (1). *$i(I_r) = 0$ for small values of r.* (cf. [8])

Assertion (2). *$i(I_r)$ is a monotone function of r.*

In fact, if $r < r'$ then there exists a $i(I_r)$ dimensional space $\mathscr{V}$ of vector fields along γ_r which vanish at a and b_r such that the index form I_r is negative definite on this vector space. Each vector field in $\mathscr{V}$ extends to a vector field along $\gamma_{r'}$ which vanishes identically between b_r to $b_{r'}$. Thus we obtain a $i(I_r)$ dimensional vector space of vector fields along $\gamma_{r'}$ on which $I_{r'}$ is negative definite. Hence $i(I_r) \leq i(I_{r'})$. $\square$

Now let us examine the discontinuity of the function $i(I_r)$. First note that $i(I_r)$ is continuous from the left:

Assertion (3). *For all sufficiently small $\varepsilon > 0$ we have $i(I_{r-\varepsilon}) = i(I_r)$.*

Proof. According to (3) of Lemma 2.5 the number $i(I_1)$ can be interpreted as the index of a quadratic form on a finite dimensional vector space $L_1 = L(a_0, \ldots, a_{k+1})$. If $b_r \neq t_0$, we may assume that the subdivision is chosen so that say $a_i < b_r < a_{i+1}$. Then the index $i(I_r)$ can be interpreted as the index of a corresponding quadratic form I_r on a corresponding vector space L_r of broken B-Jacobi fields or Jacobi fields along γ_r. This vector space L_r is to be constructed using the subdivision $a < a_1 < \cdots < a_i < b_r$ of $[a, b_r]$. Since a broken B-Jacobi field or a Jacobi field is uniquely determined by its values at the break points $\gamma(a_m)$, this vector space L_r is isomorphic to the direct sum

$$N_r = \begin{cases} N(a_1) \times \cdots \times N(a_{j-1}) \times N(a_{j+1}) \times \cdots \times N(a_i) & \text{if } b_r > t_0 \\ N(a_1) \times \cdots \times N(a_i) & \text{if } b_r < t_0 \end{cases},$$

by a map $\mathscr{N}_r : L_r \to N_r$ defined to be

$$\mathscr{N}_r(Y) := \begin{cases} (Y_1(a_1), \ldots, Y(a_{j-1}), Y(a_{j+1}), \ldots, Y(a_i)) & \text{if } b_r > t_0 \\ (Y_1(a_1), \ldots, Y(a_i)) & \text{if } b_r < t_0 \end{cases}.$$

Note that this vector space N_r is independent of r. Evidently, by Lemma 1.2, the quadratic form $B_r := I_r \circ \mathscr{N}_r^{-1}$ on N_r varies continuously with r.

Now B_r is negative definite on a subspace $\mathscr{V} \subset N_r$ of dimension $i(B_r)$. For all r' sufficiently close to r it follows that $B_{r'}$ is negative definite on $\mathscr{V}$. Therefore $i(B_{r'}) \geq i(B_r)$. But if $r' = r - \varepsilon < r$ then we also have $i(B_{r-\varepsilon}) \leq i(B_r)$ by Assertion (2). Hence $i(B_{r-\varepsilon}) = i(B_r)$. $\square$

Assertion (4). *For all sufficiently small $\varepsilon > 0$ we have*

$$i(I_{r+\varepsilon}) = i(I_r) + n(I_r).$$

Proof that $i(I_{r+\varepsilon}) \leq i(I_r) + n(I_r)$. Let B_r and N_r be as in the proof of Assertion (3). Since $\dim N_r < \infty$ we see that B_r is positive definite on some subspace $\mathscr{V}' \subset N_r$. For all r' sufficiently close to r, it follows that $B_{r'}$ is positive definite on $\mathscr{V}'$. Hence

$$i(B_{r'}) \leq \dim N_r - \dim \mathscr{V}' = a(B_r) = i(B_r) + n(B_r). \qquad \square$$

Proof that $i(I_{r+\varepsilon}) \geq i(I_r) + n(I_r)$. Let $V \in N_r$, with $V(a_i) \neq 0$, and denote by $V_{b_r} \in L_r$ the broken B-Jacobi field or Jacobi field which coincides with $V(a_m)$ at a_m, $m = 1, \ldots, i$, and which vanishes at the point $b_r \in (a_i, a_{i+1})$. We claim that

$$B_r(V, V) = I_r(V_{b_r}, V_{b_r}) > I_{r+\varepsilon}(V_{b_{r+\varepsilon}}, V_{b_{r+\varepsilon}}) = B_{r+\varepsilon}(V, V).$$

In fact, if we denote by W_{b_r} the vector field defined along $\gamma_{r+\varepsilon}$ by

$$W_{b_r}(t) = \begin{cases} V_{b_r}(t), & t \in [a, b_r] \\ 0, & t \in [b_r, b_{r+\varepsilon}] \end{cases},$$

we have, from Lemma 1.6,

$$I_r(V_{b_r}, V_{b_r}) = I_{r+\varepsilon}(W_{b_r}, W_{b_r}) > I_{r+\varepsilon}(V_{b_{r+\varepsilon}}, V_{b_{r+\varepsilon}}),$$

where the last inequality is strict, since $W_{b_r} \,|\, [a_i, b_{r+\varepsilon}]$ is neither a B-Jacobi field nor Jacobi field. Therefore, if $V \in N_r$ and $B_r(V, V) = I_r(V_{b_r}, V_{b_r}) \leq 0$, then $B_{r+\varepsilon}(V, V) = I_{r+\varepsilon}(V_{b_{r+\varepsilon}}, V_{b_{r+\varepsilon}}) < 0$. Hence, if B_r is negative definite on a subspace $\mathscr{V} \subset N_r$, $B_{r+\varepsilon}$ will still be negative definite on the direct sum of $\mathscr{V}$ with the null space of B_r. Therefore

$$i(B_{r+\varepsilon}) \geq i(B_r) + n(B_r). \qquad \square$$

The index Theorem 2.2 clearly follows from the Assertion (1), (2), (3) and (4). $\square$

3. Comparison theorem

Let (M_λ, g_λ) (resp. $(\overline{M}_\lambda, \bar{g}_\lambda)$) be Riemannian manifold with Riemannian submanifold B_λ (resp. $\overline{B}_\lambda$) for $\lambda = 1, 2$, and ψ (resp. $\bar{\psi}$) isometry from B_1 to B_2 (resp. $\overline{B}_1$ to $\overline{B}_2$). Let $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$ and $(\overline{M}, \bar{g}) = (\overline{M}_1, \bar{g}_1) \cup_{\bar{\psi}} (\overline{M}_2, \bar{g}_2)$ be glued Riemannian spaces. We put $B := B_1 \cong B_2$ and $\overline{B} := \overline{B}_1 \cong \overline{B}_2$ and assume that $\dim \overline{B} > 0$ if $\dim B > 0$. Let $\gamma \in \Omega_{t_0}$ (resp. $\bar{\gamma} \in \overline{\Omega}_{t_0}$) be a B-geodesic (resp. $\overline{B}$-geodesic) with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$ (resp. $\bar{\gamma}'(t_0 + 0) \notin T_{\bar{\gamma}(t_0)}\overline{B}$). We assume that $\gamma(t_0)$ (resp. $\bar{\gamma}(t_0)$) is not conjugate point to $\gamma(a)$ (resp. $\bar{\gamma}(a)$). For $\lambda = 1, 2$, let R^λ (resp. $\overline{R}^\lambda$) be the Riemannian curvature tensor of Riemannian manifold M_λ (resp. $\overline{M}_\lambda$). We define operators $R_t^\lambda : \{\gamma'(t)\}^\perp \to \{\gamma'(t)\}^\perp$ and $\overline{R}_t^\lambda : \{\bar{\gamma}'(t)\}^\perp \to \{\bar{\gamma}'(t)\}^\perp$ by

$$R_t^\lambda v = R^\lambda(v, \gamma'(t))\gamma'(t) \quad \text{for } v \in \{\gamma'(t)\}^\perp$$

and

$$\overline{R}_t^\lambda \bar{v} = \overline{R}^\lambda(\bar{v}, \bar{\gamma}'(t))\bar{\gamma}'(t) \quad \text{for } \bar{v} \in \{\bar{\gamma}'(t)\}^\perp,$$

where
$$\{\gamma'(t)\}^{\perp} := \{v \in T_{\gamma(t)}M_{\lambda} \mid g_{\lambda}(v, \gamma'(t)) = 0\}$$
and
$$\{\bar{\gamma}'(t)\}^{\perp} := \{\bar{v} \in T_{\bar{\gamma}(t)}\overline{M}_{\lambda} \mid \bar{g}_{\lambda}(\bar{v}, \bar{\gamma}'(t)) = 0\}.$$
Similarly, a bar is used to distinguish objects in $\overline{M}$ from the corresponding objects in M. We put $\Gamma_2(\gamma') := T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_2\, \gamma'(t_0+0)\}$, $\Gamma_2^{\perp}(\gamma') := \{v \in \Gamma_2(\gamma') \mid g_2(v, \gamma'(t_0+0)) = 0\}$ and $A := A_{\gamma'(t_0-0), \gamma'(t_0+0)} \mid \Gamma_2^{\perp}(\gamma')$.

We assume that $\dim M_{\lambda} \geq 2$ and $\dim \overline{M}_{\lambda} \geq 2$. Then the following assertion holds:

Proposition 3.1. *We assume that* $\dim M_{\lambda} \leq \dim \overline{M}_{\lambda}$ $(\lambda = 1, 2)$ *and the following conditions hold*:

(1) *For any* $t \in [a, b]$,
$$(\text{the maximal eigenvalue of } R_t^{\lambda}) \leq (\text{the minimal eigenvalue of } \overline{R}_t^{\lambda})$$
(2) *If* $\dim B > 0$, *then*
$$(\text{the minimal eigenvalue of } A) \geq (\text{the maximal eigenvalue of } \bar{A}).$$
Then $i(I_{\gamma}^{0,\perp}) \leq i(\bar{I}_{\bar{\gamma}}^{0,\perp})$ *holds. In particular, if one of two inequalities* (1) *and* (2) *is strict, then* $a(I_{\gamma}^{0,\perp}) = i(I_{\gamma}^{0,\perp}) + n(I_{\gamma}^{0,\perp}) \leq i(\bar{I}_{\bar{\gamma}}^{0,\perp})$ *holds.*

Proof. For $Y \in T_{\gamma}^{\perp}\Omega_{t_0}(\gamma(a), \gamma(b))$, let $e_1^-, \ldots, e_{m_1}^- := \gamma'(t_0-0)$ be an orthonormal basis of $T_{\gamma(t_0)}M_1$ and $e_1^+, \ldots, e_{m_2}^+ := \gamma'(t_0+0)$ an orthonormal basis of $T_{\gamma(t_0)}M_2$ such that $e_1^- = Y(t_0-0)/\|Y(t_0-0)\|_1$ and $e_1^+ = Y(t_0+0)/\|Y(t_0+0)\|_2$ if $Y(t_0-0) \neq 0$. Let $e_i^-(t)$ (resp. $e_i^+(t)$) be the vector field along $\gamma \mid [a, t_0]$ (resp. $\gamma \mid [t_0, b]$) obtained by parallel translation of e_i^- (resp. e_i^+) along $\gamma \mid [a, t_0]$ (resp. $\gamma \mid [t_0, b]$) for $i = 1, \ldots, m_1$ (resp. $i = 1, \ldots, m_2$). We can denote $Y(t)$ by
$$Y(t) = \sum_{i=1}^{m_1-1} y_-^i(t) e_i^-(t), \quad t \in [a, t_0]$$
and
$$Y(t) = \sum_{i=1}^{m_2-1} y_+^i(t) e_i^+(t), \quad t \in [t_0, b].$$
Let $\bar{e}_1^-, \ldots, \bar{e}_{\bar{m}_1}^- := \bar{\gamma}'(t_0-0)$ (resp. $\bar{e}_1^+, \ldots, \bar{e}_{\bar{m}_2}^+ := \bar{\gamma}'(t_0+0)$) be an orthonormal basis of $T_{\bar{\gamma}(t_0)}\overline{M}_1$ (resp. $T_{\bar{\gamma}(t_0)}\overline{M}_2$) such that if $\bar{e}_1^- \in \overline{\Gamma}_1(\gamma')$ and $\bar{e}_1^+ = \overline{Q}(\bar{e}_1^-)$ if $Y(t_0-0) \neq 0$. Let $\bar{e}_i^-(t)$ (resp. $\bar{e}_i^+(t)$) be the vector field along $\bar{\gamma} \mid [a, t_0]$ (resp. $\bar{\gamma} \mid [t_0, b]$) obtained by parallel translation of $\bar{e}_i^-$ (resp. $\bar{e}_i^+$) along $\bar{\gamma} \mid [a, t_0]$ (resp. $\bar{\gamma} \mid [t_0, b]$) for $i = 1, \ldots, \bar{m}_1$ (resp. $i = 1, \ldots, \bar{m}_2$). If we put
$$\overline{Y}(t) = \sum_{i=1}^{m_1-1} y_-^i(t) \bar{e}_i^-(t), \quad t \in [a, t_0]$$

and

$$\bar{Y}(t) = \sum_{i=1}^{m_2-1} y_+^i(t)\bar{e}_i^+(t), \quad t \in [t_0, b],$$

then it holds that $\bar{Y} \in T_{\bar{\gamma}}^{\perp}\bar{\Omega}_{t_0}(\bar{\gamma}(a), \bar{\gamma}(b))$, since $\bar{Y}(t_0+0) = y_+^1(t_0+0)\bar{e}_1^+ = y_-^1(t_0-0)\bar{Q}(\bar{e}_1^-) = \bar{Q}(\bar{Y}(t_0-0))$ if $Y(t_0) \neq 0$. Furthermore, by the definition, we have that $\|\bar{Y}(t)\|_\lambda = \|Y(t)\|_\lambda$ and $\|\bar{Y}'(t)\|_\lambda = \|Y'(t)\|_\lambda$. From the assumption (1) and (2), we get

$$g_\lambda(R_t^\lambda Y(t), Y(t)) \leq \bar{g}_\lambda(\bar{R}_t^\lambda \bar{Y}(t), \bar{Y}(t))$$

and

$$g_2(A(Y(t_0+0)), Y(t_0+0)) \geq \bar{g}_2(\bar{A}(\bar{Y}(t_0+0)), \bar{Y}(t_0+0)).$$

Then we have that

$$I_\gamma(Y, Y) \geq \bar{I}_{\bar{\gamma}}(\bar{Y}, \bar{Y}). \tag{3.1}$$

Let $\mathscr{U}$ be the subspace of $T_\gamma^\perp\Omega_{t_0}(\gamma(a), \gamma(b))$ on which $I_\gamma^\perp$ is negative definite and $\bar{\mathscr{U}} := \{\bar{Y} \mid Y \in \mathscr{U}\}$. If $Y \in \mathscr{U}$, then $\bar{I}_{\bar{\gamma}}(\bar{Y}, \bar{Y}) < 0$. Hence, $\bar{I}_\gamma$ is negative definite on $\bar{\mathscr{U}}$ and we have $i(I_\gamma^\perp) \leq i(\bar{I}_{\bar{\gamma}}^\perp)$.

If one of two inequalities (1) and (2) is strict, then it holds that

$$I_\gamma(Y, Y) > \bar{I}_{\bar{\gamma}}(\bar{Y}, \bar{Y}). \tag{3.2}$$

Let $\mathscr{V}$ be the subspace of $T_\gamma^\perp\Omega_{t_0}(\gamma(a), \gamma(b))$ on which $I_\gamma^\perp$ is negative semi-definite and $\bar{\mathscr{V}} := \{\bar{Y} \mid Y \in \mathscr{V}\}$. If $Y \in \mathscr{V}$, then $\bar{I}_{\bar{\gamma}}(\bar{Y}, \bar{Y}) < 0$. Hence, $\bar{I}_{\bar{\gamma}}$ is negative definite on $\bar{\mathscr{V}}$ and we have $a(I_\gamma^{0,\perp}) \leq i(\bar{I}_{\bar{\gamma}}^{0,\perp})$. $\square$

The condition that $\dim M_\lambda \leq \dim \bar{M}_\lambda$ $(\lambda = 1, 2)$ is necessary. We give an example which shows that:

Example 3. Let $S^m(1)$ be the m-sphere of constant curvature 1 and γ a geodesic on $S^m(1)$. Let $e_1(t), e_2(t), \ldots, e_{m-1}(t), \gamma'(t)$ be a parallel orthonormal frame along γ. Let τ be the geodesic through $\gamma(0)$ with $\tau'(0) = e_1(0)$. We put $M_\lambda := S^m(1)$ $(\lambda = 1, 2)$, $B := \{\tau(t) \mid t \in \boldsymbol{R}\}$, $\psi = \mathrm{id}_B$ and $M = M_1 \cup_\psi M_2$. Then $\gamma : [-\pi/2, \pi] \to M$ is a B-geodesic. We set $a := -\pi/2$, $t_0 := 0$ and $b := \pi/2$. Then $\gamma(b)$ is a B-conjugate point to $\gamma(a)$, its multiplicity is $m-1$ and $i(I_\gamma^\perp) = m-1$. For $\bar{m} < m$, we set $\bar{M}_\lambda := S^{\bar{m}}(1)$, $\bar{B}, \bar{\psi}, \bar{M} = \bar{M}_1 \cup_{\bar{\psi}} \bar{M}_2$ and $\bar{\gamma}$ as above. Then, we have that $i(I_\gamma^{0,\perp}) > i(\bar{I}_{\bar{\gamma}}^{0,\perp})$.

In [11], the following assertion is given without the assumption that $\dim M_\lambda \leq \dim \bar{M}_\lambda$ $(\lambda = 1, 2)$:

Corollary 3.2. *We assume that* $\dim M_\lambda \leq \dim \bar{M}_\lambda$ $(\lambda = 1, 2)$ *and the following conditions hold*:

(1) *For any* $t \in [a, b]$,

$$(\text{the maximal eigenvalue of } R_t^\lambda) \leq (\text{the minimal eigenvalue of } \bar{R}_t^\lambda)$$

(2) *If* $\dim B > 0$, *then*

$$(\text{the minimal eigenvalue of } A) \geq (\text{the maximal eigenvalue of } \bar{A}).$$

(3) $\bar{\gamma}(t)$ *is not a conjugate point to* $\bar{\gamma}(a)$ *for any* $t \in (a, t_0]$ *and also* $\bar{\gamma}(t)$ *is not a* $\bar{B}$*-conjugate point to* $\bar{\gamma}(a)$ *for any* $t \in (t_0, b]$.

Then $\gamma(t)$ *is not a conjugate point to* $\gamma(a)$ *for any* $t \in (a, t_0]$ *and also* $\gamma(t)$ *is not B-conjugate point to* $\gamma(a)$ *for any* $t \in (t_0, b]$.

Proof. By the assumption (3), $i(\bar{I}_{\bar{\gamma}}^{0,\perp}) = 0$ holds. Hence we have that $i(I_{\gamma}^{0,\perp}) = 0$ from Proposition 3.1. □

References

[1] N. Abe and M. Takiguchi, Geodesics reflecting on a pseudo-Riemannian submanifold, SUT J. Math., **34**, No. 2, (1998), 139–168.

[2] T. Hasegawa, The index theorem of geodesics on a Riemannian manifold with boundary, Kodai Math. J., **1** (1978), 285–288.

[3] T. Hasegawa, On the position of a conjugate point of a reflected geodesic in E^2 and E^3, Yokohama Math. J., **32** (1984), 233–237.

[4] N. Innami, Convex curves whose points are vertices of billiard triangles, Kodai Math. J., **11**, (1988), 17–24.

[5] N. Innami, Integral formulas for polyhedral and spherical billiards, J. Math. Soc. Japan, **50**, No. 2, (1998), 339–357.

[6] N. Innami, Jacobi vector fields along geodesics in glued Riemannian manifolds, Nihonkai Math. J., **12**, No. 1, (2001), 101–112.

[7] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Wiley (Interscience), New York, **Vol. 2** (1969).

[8] J. Milnor, Morse Theory, Princeton University Press (1963).

[9] T. Sakai, Riemannian Geometry, Mathematical Monograph, Amer. Math. Soc. (1997).

[10] M. Takiguchi, The index form of a geodesic on a glued Riemannian space, Nihonkai Math. J., **11**, No. 2, (2000), 167–202.

[11] M. Takiguchi, An extension of Rauch comparison theorem to glued Riemannian spaces, Tsukuba J. Math., **26**, No. 2, (2002), 313–338.

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