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UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE ONE SMALL FUNCTION WITH THEIR DERIVATIVES

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Abstract

In this paper, we study uniqueness of meromorphic functions that share one small function CM with their derivatives. We mainly obtain a uniqueness theorem which answers the questions provided by Kit-wing Yu.

1 Introduction and main results

In this paper, a meromorphic function always means a function, which is meromorphic in the whole complex plane. Let $f(z)$ and $g(z)$ be non-constant meromorphic functions, $a \in \bar{C}$. We say that $f(z)$ and $g(z)$ share the value a CM if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities. When $a = \infty$, this means that $1/f(z)$ and $1/g(z)$ share the value 0 CM. We shall use the standard notations of value distribution theory, $T(r, f), m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ (see L. Yang [1] or C. C. Yang and H. X. Yi [2]). We denote by $S(r, f)$ any function satisfying:

$$S(r, f) = o\{T(r, f)\}$$

as $r \to +\infty$, possibly outside a set (*of r*) of finite measure.

R. Brück [3] proved the following result:

Theorem A. *Let f be a nonconstant entire function satisfying $\rho_1(f) < \infty$, where $\rho_1(f)$ $(\rho_1(f) = \limsup_{r\to\infty}(\log\log T(r, f)/\log r))$ is not a positive integer. If f and f' share the value 0 CM, then $f' \equiv cf$ for some constant $c \neq 0$.*

Gary G. Gundersen and Lian-Zhong Yang proved the following result:

Theorem B [4]. *Let f be a nonconstant entire function of finite order. If f and f' share one finite value a CM then*:

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$$\frac{f' - a}{f - a} = c$$

for some non-zero constant c.

It is natural to consider whether there exit any similar results for entire functions of infinite order, or even meromorphic functions f and a small function a of f.

In [5], Kit-wing Yu answered this question by proving the following two results:

Theorem C. *Let* $k \geq 1$ *and let* f *be a non-constant entire function,* $a(z)$ *be a meromorphic function with* $a(z) \not\equiv 0, \infty$, *and* $T(r, a) = o(T(r, f))$ *as* $r \to +\infty$. *If* $f - a$ *and* $f^{(k)} - a$ *share the value* 0 *CM and* $\delta(0, f) > 3/4$, *then* $f \equiv f^{(k)}$.

Theorem D. *Let* $k \geq 1$ *and let* f *be a non-constant meromorphic function,* $a(z)$ *be a meromorphic function with* $a(z) \not\equiv 0, \infty$, f *and* $a(z)$ *do not have any common pole and* $T(r, a) = o(T(r, f))$ *as* $r \to +\infty$. *If* $f - a$ *and* $f^{(k)} - a$ *share the value* 0 *CM and* $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, *then* $f \equiv f^{(k)}$.

In the same paper, the author posed the following questions:

Question 1: Is the condition $\delta(0, f) > 3/4$ sharp in the Theorem C.

Question 2: Is the condition $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, sharp in the Theorem D.

Question 3: Can the condition "f and $a(z)$ do not have any common pole" be deleted in the Theorem D?

In this paper, we apply the different method and obtain the following result, which answers above questions.

Theorem 1. *Let* $k \geq 1$ *and let* f *be a non-constant meromorphic function,* $a(z)$ *be a meromorphic function with* $a(z) \not\equiv 0, \infty$, *and* $T(r, a) = S(r, f)$ *as* $r \to +\infty$. *If* $f - a(z)$ *and* $f^{(k)} - a(z)$ *share the value* 0 *CM and* $f^{(k)}$ *and* $a(z)$ *do not have any common poles of same multiplicity and*

$$2\delta(0, f) + 4\Theta(\infty, f) > 5 \tag{1.1}$$

then $f \equiv f^{(k)}$.

As a simple corollary, we have following result

Corollary 1. *Let* $k \geq 1$ *and let* f *be a non-constant entire function,* $a(z)$ *be a meromorphic function with* $a(z) \not\equiv 0, \infty$, *and* $T(r, a) = S(r, f)$ *as* $r \to +\infty$. *If* $f - a(z)$ *and* $f^{(k)} - a(z)$ *share the value* 0 *CM and* $\delta(0, f) > 1/2$, *then* $f \equiv f^{(k)}$.

2 Some lemmas

In this section, we have the following lemmas which will be needed in the proofs of the main results. In the following, I is a set of infinite linear measure and may not be the same each time it occurs.

Lemma 1 [2]. *Let f_1 and f_2 be two non-constant meromorphic functions and let c_1, c_2, c_3 be non-zero constant. If $c_1 f_1 + c_2 f_2 = c_3$ holds, then*

$$T(r, f_1) \le \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 2 [2]. *Let f_i $(i = 1 \cdots n)$ be n linearly independent meromorphic functions. If they satisfy:*

$$\sum_{i=1}^{n} f_i \equiv 1$$

then for $1 \le j \le n$ we have

$$T(r, f_j) \le \sum_{i=1}^{n} N\left(r, \frac{1}{f_i}\right) + N(r, f_j) + N(r, D) - \sum_{i=1}^{n} N(r, f_i) - N\left(r, \frac{1}{D}\right) + S(r),$$

where D is the Wronskian determinant $W(f_1, f_2, \dots f_n)$, $S(r) = o(T(r))$ as $r \to +\infty$, $r \in I$, and $T(r) = \max_{1 \le j \le n}\{T(r, f_j)\}$.

Lemma 3. *Let f be a meromorphic function in the complex plane. For any positive integer k such that $f^{(k)} \not\equiv 0$, we have:*

$$N\left(r, \frac{1}{f^{(k)}}\right) \le T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f).$$

Proof. By the first fundamental theorem and the lemma of logarithmic derivatives, we get:

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$$

so we can deduce

$$\begin{aligned}
N\left(r, \frac{1}{f(k)}\right) &= T(r, f^{(k)}) - m\left(r, \frac{1}{f^{(k)}}\right) + O(1) \\
&\le T(r, f^{(k)}) - \left(m\left(r, \frac{1}{f}\right) - S(r, f)\right) + O(1) \\
&\le T(r, f^{(k)}) - \left(T(r, f) - N\left(r, \frac{1}{f}\right) + O(1)\right) + S(r, f) \\
&\le T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f).
\end{aligned}$$

This proves Lemma 3.

3 Proof of Theorem 1

Proof of Theorem 1. We assume $f \not\equiv f^{(k)}$. Set:

$$\frac{f^{(k)} - a}{f - a} = h. \tag{3.1}$$

We distinguish the following two cases.

1: $h \equiv c$ $(c \neq 1)$. From (3.1) we have:

$$\frac{f^{(k)}}{a} - \frac{cf}{a} = (1 - c),$$

and hence by Lemmas 1 and 3 we have:

$$\begin{aligned} T(r, f^{(k)}) &\leq T\left(r, \frac{f^{(k)}}{a}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{a}{f^{(k)}}\right) + \overline{N}\left(r, \frac{a}{f}\right) + \overline{N}\left(r, \frac{f^{(k)}}{a}\right) + S(r, f) \\ &\leq T(r, f^{(k)}) - T(r, f) + 2N\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f), \end{aligned}$$

thus

$$T(r, f) \leq 2N\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f),$$

from which we get

$$2\delta(0, f) + \Theta(\infty, f) \leq 2,$$

This contradicts (1.1)

2: $h \not\equiv const$. From (3.1) we have;

$$\frac{f^{(k)}}{a} - \frac{hf}{a} + h = 1. \tag{3.2}$$

Set $f_1 = f^{(k)}/a$, $f_2 = -hf/a$, $f_3 = h$, then

$$\sum_{i=1}^{3} f_i \equiv 1.$$

We distinguish the following two subcases again.

Case 2.1. f_1, f_2, f_3 are three linearly independent meromorphic functions, then by lemma 2 we have:

$$\begin{aligned}T(r,f^{(k)}) &\le T\left(r,\frac{f^{(k)}}{a}\right)+S(r,f)\\ &\le N\left(r,\frac{a}{f^{(k)}}\right)+N\left(r,\frac{a}{fh}\right)+N\left(r,\frac{1}{h}\right)+N(r,D)\\ &\quad -N\left(r,\frac{hf}{a}\right)-N(r,h)+o(T(r)).\end{aligned}\tag{3.3}$$

as $r \to +\infty$, $r \in I$. On the other hand, since

$$N(r,D)=N\left(r,\left(\frac{hf}{a}\right)''h'-h''\left(\frac{hf}{a}\right)'\right)\le N\left(r,\left(\frac{hf}{a}\right)''\right)+N(r,h''),$$

we have

$$\begin{aligned}N(r,D)-N\left(r,\frac{hf}{a}\right)-N(r,h) &\le N\left(r,\left(\frac{hf}{a}\right)''\right)+N(r,h'')-N\left(r,\frac{hf}{a}\right)-N(r,h)\\ &\le 2\overline{N}\left(r,\frac{hf}{a}\right)+2\overline{N}(r,h).\end{aligned}\tag{3.4}$$

And since $f-a$ and $f^{(k)}-a$ share the value 0 CM and $f^{(k)}$ and $a(z)$ do not have any common poles of same multiplicity, we know that $h \neq 0$. On the other hand, the pole of h must be the pole of f or $a(z)$, so

$$\begin{aligned}\overline{N}\left(r,\frac{hf}{a}\right) &\le \overline{N}(r,f)+S(r,f)\\ \overline{N}(r,h) &\le \overline{N}(r,f)+S(r,f).\end{aligned}\tag{3.5}$$

From (3.3), (3.4), (3.5), we can get:

$$\begin{aligned}T(r,f^{(k)}) &\le N\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{h}\right)\\ &\quad +2\overline{N}(r,h)+2\overline{N}(r,f)+S(r,f).\end{aligned}\tag{3.6}$$

According to Lemma 3 and (3.6), we have

$$T(r,f)\le 2N\left(r,\frac{1}{f}\right)+4\overline{N}(r,f)+S(r,f),$$

as $r \to +\infty$, $r \in I$, which contradicts (1.1).

Case 2.2. f_1, f_2, f_3 are three linearly dependent meromorphic functions, namely there exist three constants which are not all equal zero and satisfy that:

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0.\tag{3.7}$$

Obviously, $c_1 \neq 0$. In fact, if $c_1 = 0$, then $c_2 \neq 0$, $c_3 \neq 0$, and by (3.7) we can obtain $h(c_2(f/a) - c_3) = 0$, since $h \not\equiv const$, $f = ac_3/c_2$. From this we can deduce:

$$T(r,f) = T\left(r, \frac{ac_3}{c_2}\right) = S(r,f),$$

which is impossible.

By (3.2) and (3.7), we have

$$(c_2 - c_1)\frac{hf}{a} + (c_1 - c_3)h = c_1, \tag{3.8}$$

We consider three subcases again.

2.2.1: $(c_2 - c_1) \neq 0$, $(c_1 - c_3) \neq 0$, we obtain from (3.8)

$$\frac{(c_1 - c_2)}{c_1}\frac{f}{a} + \frac{1}{h} = \frac{c_1 - c_3}{c_1},$$

and by Lemma 1

$$\begin{aligned} T(r,f) &\leq T\left(r, \frac{f}{a}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{a}{f}\right) + \overline{N}(r,h) + \overline{N}\left(r, \frac{f}{a}\right) + S(r,f) \\ &\leq N\left(r, \frac{1}{f}\right) + 2\overline{N}(r,f) + S(r,f). \end{aligned}$$

Similarly, this is impossible because of the condition (1.1).

2.2.2: $(c_2 - c_1) = 0$, $(c_1 - c_3) \neq 0$, we deduce from (3.8),

$$h = \frac{c_1}{c_1 - c_3},$$

which is a contradiction because of $h \not\equiv const$.

2.2.3: $(c_2 - c_1) \neq 0$, $(c_1 - c_3) = 0$, from (3.8) we can obtain

$$fh = \frac{ac_1}{c_2 - c_1}, \tag{3.9}$$

by (3.2) and (3.9), we get

$$\frac{f^{(k)}}{a} + h = \frac{c_2}{c_2 - c_1}, \tag{3.10}$$

If $c_2 \neq 0$, then by Lemmas 1 and 3

$$\begin{aligned} T(r, f^{(k)}) &\leq T\left(r, \frac{f^{(k)}}{a}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{a}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}\left(r, \frac{f^{(k)}}{a}\right) + S(r,f) \\ &\leq T(r, f^{(k)}) - T(r,f) + N\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + S(r,f). \end{aligned}$$

Therefore

$$T(r,f) \leq N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f),$$

and similarly, this is impossible because of the condition (1.1).

If $c_2 = 0$, then we have

$$f^{(k)} + ah = 0 \Rightarrow fh = -a, \tag{3.11}$$

so we can deduce from (3.11)

$$ff^{(k)} = a^2. \tag{3.12}$$

We note that

$$N\left(r,\frac{1}{f}\right) \leq N\left(r,\frac{1}{ff^{(k)}}\right),$$

and

$$\begin{aligned} 2m\left(r,\frac{1}{f}\right) &= m\left(r,\frac{1}{f^2}\right) \\ &\leq m\left(r,\frac{ff^{(k)}}{f^2}\right) + m\left(r,\frac{1}{ff^{(k)}}\right) \\ &= S(r,f) + m\left(r,\frac{1}{ff^{(k)}}\right), \end{aligned}$$

so that

$$T(r,f) \leq T(r,ff^{(k)}) + S(r,f). \tag{3.13}$$

Therefore we can get from (3.12) and (3.13)

$$T(r,f) \leq T(r,ff^{(k)}) + S(r,f) = T(r,a^2) + S(r,f) = S(r,f),$$

which is impossible.

This proves Theorem 1.

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