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ON ENTIRE FUNCTIONS WHICH SHARE ONE SMALL FUNCTION CM WITH THEIR FIRST DERIVATIVE

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Abstract

The paper generalizes a result of R. Brück and makes an example which shows that the generalization is precise.

1. Introduction and results

In this paper the term "meromorphic" will always mean meromorphic in the complex plane. We use the standard notations and results of the Nevanlinna theory (See [2] or [3], for example). In particular, $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly for a set E of r of finite linear measure. A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$. We say that two non-constant meromorphic functions f and g share the value or small function a CM (counting multiplicities), if f and g have the same a-points with the same multiplicity.

In [1] R. Brück proved the following theorem:

Theorem A. *Let f be a non-constant entire function satisfying $N(r,1/f') = S(r,f)$. If f and f' share the value 1 CM, then $f-1 = c(f'-1)$, where c is a nonzero constant.*

It is asked naturally whether the value 1 of Theorem A can be simply replaced by small function $a(\not\equiv 0,\infty)$. We make an example which shows that the answer of this question is negative.

Example 1. Let $f(z) = 1+\exp(e^z)$ and $a(z) = 1/(1-e^{-z})$, by Lemma 1, we know that a is a small function of f. It is easy to see that f and f' share a CM and $N(r,1/f') = 0$, but $f - a \neq c(f'-a)$, for every nonzero constant c. Indeed, $f - a = e^{-z}(f'-a)$.

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In this paper we shall generalize the result in Theorem A and obtain the following theorem:

THEOREM 1. *Let f be a non-constant entire function satisfying $N(r,1/f') = S(r,f)$ and let $a(\not\equiv 0,\infty)$ be a meromorphic small function of f. If f and f' share a CM, then $f - a = (1-k/a)(f'-a)$, where $1-k/a = e^{\beta}$, k is a constant and β is an entire function.*

From Theorem 1, we immediately deduce the following corollary:

COROLLARY 1. *Let f be a non-constant entire function satisfying $N(r,1/f') = S(r,f)$ and let $a(\not\equiv 0,\infty)$ be an entire small function of f. If f and f' share a CM, then either $f = f'$ or $a \equiv const$ and $f - a = c(f'-a)$, where $c(\neq 0,1)$ is a constant.*

It is obvious that Theorem A is a special case of Theorem 1 or Corollary 1.

Remark 1. From Theorem 1, it is easy to see that $f(z) = A\exp(\int(1-k/a(z))^{-1}\,dz) + k$, where $1-k/a(z) = e^{\beta(z)}$, $A(\neq 0), k$ are constants and β is an entire function. This result includes Example 1 as a special case.

2. Some lemmas

For the proof of our results we need the following lemmas:

LEMMA 1 [2, p. 50]. *Let f and g be two transcendental entire functions. Then*

$$\lim_{r\to\infty} \frac{T(r,g)}{T(r,f(g))} = 0.$$

LEMMA 2 [4, p. 96]. *Let f_j $(j = 1,2,3,4)$ be meromorphic functions and f_k $(k = 1,2)$ are non-constants satisfying $\sum_{j=1}^{4} f_j \equiv 1$. If*

$$\sum_{j=1}^{4} N\left(r,\frac{1}{f_j}\right) + 3\sum_{j=1}^{4} \overline{N}(r,f_j) < (\lambda + o(1))T(r,f_k) \quad (r \in I, k = 1,2),$$

where $\lambda < 1$ and I is a set of infinite measure. Then $f_3 \equiv 1$, $f_4 \equiv 1$, or $f_3 + f_4 \equiv 1$.

LEMMA 3 [2]. *Let f be a non-constant meromorphic function, and a_1, a_2, a_3 be distinct small functions of f, then*

$$T(r,f) \le \sum_{j=1}^{3} \overline{N}\left(r,\frac{1}{f-a_j}\right) + S(r,f).$$

3. Proof of theorem 1

From Theorem A, we know that Theorem 1 is valid for a is a nonzero constant. Next we suppose that a is a non-constant meromorphic function.

Since f and f' share a CM, there is an entire function β such that

$$f - a = e^{\beta}(f' - a). \tag{3.1}$$

We claim that $T(r, e^{\beta}) = S(r, f)$. Differentiating (3.1) we obtain

$$\left(\frac{a}{a'}\beta' + 1\right)e^{\beta} + \frac{1}{a'}f' - \frac{\beta' e^{\beta}}{a'}f' - \frac{e^{\beta}}{a'}f'' \equiv 1. \tag{3.2}$$

In order that applying Lemma 2 to (3.2), we consider the following two cases:

Case I. $((a/a')\beta' + 1)e^{\beta} \equiv c$, where c is a constant.

If $c = 0$, $(a/a')\beta' + 1 \equiv 0$. By integration, we get $a = Ae^{-\beta}$, where A is a nonzero constant, and hence $T(r, e^{\beta}) = S(r, f)$. We also see that, if $c \neq 0$, $T(r, e^{\beta}) = S(r, f)$.

Case II. $(1/a')f' \equiv \text{const}$.

Then $T(r, f') = S(r, f)$. It follows that $N(r, 1/(f - a)) = N(r, 1/(f' - a)) = S(r, f)$, and

$$\begin{aligned} m\left(r, \frac{1}{f - a}\right) &\le m\left(r, \frac{f' - a'}{f - a}\right) + m\left(r, \frac{1}{f' - a'}\right) \\ &\le T(r, f') + S(r, f) = S(r, f). \end{aligned}$$

Thus, we have $T(r, f) = S(r, f)$ which is a contradiction.

Now suppose $((a/a')\beta' + 1)e^{\beta}$ and $(1/a')f$ are non-constants. Note that

$$\begin{aligned} N\left(r, \frac{1}{f''}\right) &\le N\left(r, \frac{f'}{f''}\right) + N\left(r, \frac{1}{f'}\right) \le T\left(r, \frac{f''}{f'}\right) + N\left(r, \frac{1}{f'}\right) + O(1) \\ &\le 2N\left(r, \frac{1}{f'}\right) + \overline{N}(r, f) + S(r, f) = S(r, f). \end{aligned}$$

Applying Lemma 2 to (3.2), we divide into the following three cases:

Case 1. $-(\beta' e^{\beta}/a')f' \equiv 1$.

Substituting this into (3.2) gives

$$\frac{f''}{a\beta' + a'} - \frac{e^{-\beta}f'}{a\beta' + a'} \equiv 1.$$

From this and the second fundamental theorem for $H = f''/(a\beta' + a')$

$$\begin{aligned}
T(r,H) &\le N\left(r,\frac{1}{H}\right) + N\left(r,\frac{1}{H-1}\right) + \overline{N}(r,H) + S(r,H) \\
&\le N\left(r,\frac{1}{f''}\right) + N(r,a\beta'+a') + N\left(r,\frac{1}{f'}\right) + N(r,a\beta'+a') \\
&\quad + \overline{N}(r,f'') + \overline{N}\left(r,\frac{1}{a\beta'+a'}\right) + S(r,f) \\
&\le N\left(r,\frac{1}{f''}\right) + N\left(r,\frac{1}{f'}\right) + \overline{N}(r,f'') \\
&\quad + 3(T(r,a) + T(r,a') + T(r,\beta')) + S(r,f) = S(r,f).
\end{aligned}$$

It follows that $T(r,f'') = S(r,f)$, and so

$$\begin{aligned}
T(r,f') = T\left(r,\frac{f'}{f''}\cdot f''\right) &\le T\left(r,\frac{f'}{f''}\right) + T(r,f'') \\
&\le T\left(r,\frac{f''}{f'}\right) + T(r,f'') + O(1) \\
&\le T\left(r,\frac{f''}{f'}\right) + S(r,f) = S(r,f),
\end{aligned}$$

giving a contradiction.

Case 2. $-(e^{\beta}/a')f'' \equiv 1$.
Similarly as the Case 1, we arrive at a contradiction.

Case 3. $-(\beta' e^{\beta}/a')f' - (e^{\beta}/a')f'' \equiv 1$.
Substitution of this identical equation in (3.2) gives

$$f' = -(a\beta' + a')e^{\beta}. \tag{3.3}$$

Differentiating (3.3) we find that

$$f'' = -e^{\beta}(a'' + 2a'\beta' + a\beta'' + a\beta'^2). \tag{3.4}$$

Substituting (3.3) and (3.4) into above identical equation gives

$$e^{-2\beta} = \frac{2a'\beta'^2}{a'} + 3\beta' + \frac{a}{a'}\beta'' + \frac{a''}{a'}.$$

This implies that $T(r,e^{\beta}) = S(r,f)$, and this proves the claim.

Now (3.1) can be written

$$f' = e^{-\beta}(f-b), \tag{3.5}$$

where $b = a(1-e^{\beta})$ is a small function of f. Since $N(r,1/f') = S(r,f)$, we see from (3.5) that

$$N\left(r,\frac{1}{f-b}\right) = S(r,f). \tag{3.6}$$

From (3.6) and the second fundamental theorem for $F = f - b$

$$\begin{aligned} T(r,F) &\le N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{F-1}\right) + \overline{N}(r,F) - N\left(r,\frac{1}{F'}\right) + S(r,F) \\ &\le N\left(r,\frac{1}{F-1}\right) - N\left(r,\frac{1}{F'}\right) + S(r,f) \\ &\le T(r,F) - N\left(r,\frac{1}{F'}\right) + S(r,f). \end{aligned}$$

It follows that

$$N\left(r,\frac{1}{f'-b'}\right) = S(r,f). \tag{3.7}$$

From (3.7) and Lemma 3 $(a_1 = 0, a_2 = b', a_3 = \infty)$, we deduce that if $b' \not\equiv 0$,

$$T(r,f') \le \overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{f'-b'}\right) + \overline{N}(r,f') + S(r,f') = S(r,f),$$

which is a contradiction. Therefore, we have $b' \equiv 0$ and so $a(1 - e^{\beta}) \equiv k$, where k is a constant. Combining with (3.1), we get $f - a = (1 - k/a)(f' - a)$. ■

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