T. C. NGUYEN
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ON BOUNDARIES OF MODULI SPACES OF NON-SINGULAR CUBIC SURFACES WITH STAR POINTS

Nguyen Chanh Tu

Abstract

Let $\boldsymbol{P}^{19}$ be the parametrizing space of cubic surfaces in $\boldsymbol{P}^3$. Let M be the moduli space of non-singular cubic surfaces and $\overline{M}$ be a suitable compactification. We study components of boundaries of the relative subspaces of non-singular cubic surfaces with star points in $\boldsymbol{P}^{19}$ and in $\overline{M}$.

1. Introduction

1.1. Let $\boldsymbol{P}^{19}$ be the parametrizing space of cubic surfaces in $\boldsymbol{P}_k^3$, where k is an algebraically closed field with characteristic 0. A hyperplane is called a *tri-tangent plane* with respect to a given cubic surface X if the intersection consists of lines. A *star point* of X is a common point of all lines of the intersection of a tri-tangent plane and X. If a smooth cubic surface X has a star point then the corresponding hyperplane intersection consists of 3 distinct lines. This triple of lines is called a *star triple*. We denote H_s for the subset of $\boldsymbol{P}^{19}$ corresponding to non-singular cubic surfaces with at least s star points. For each H_s, there is a decomposition into irreducible components $H_s^{(r)}$ where each $H_s^{(r)}$ in fact, corresponds to cubic surfaces generically containing exactly r star points. Definitions of all $H_s^{(r)}$ together results on their irreducibility, dimensions and inclusion relationships could be found in [12] or [13].

1.2. We consider the action of PGL(4) on $\boldsymbol{P}^{19}$. Let $\phi : (\boldsymbol{P}^{19})^{ss} \to \overline{M}$ be the quotient space with respect to the action of PGL(4) on $\boldsymbol{P}^{19}$, where $(\boldsymbol{P}^{19})^{ss}$ is the subset of semi-stable points in the sense of geometric invariant theory (see [8], [9] or [11]). Let $M := \mathrm{PGL}(4)\backslash(\boldsymbol{P}^{19} - \Delta)$, where Δ is the locus of singular cubic surfaces in $\boldsymbol{P}^{19}$. In fact, the space $\overline{M}$ is projective, and M is the coarse moduli

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space of non-singular cubic surfaces (see [9], [10] or Section 3.2 of [13] for proofs). The space $\overline{M}$ is a compactification of M. Let $N := \mathrm{PGL}(4)\backslash(\boldsymbol{P}^{19})^{s}$, where $(\boldsymbol{P}^{19})^{s}$ is the subset of stable points.

Let $(\Delta)^{ss} = (\boldsymbol{P}^{19})^{ss} \cap \Delta$. We denote by $\Delta H_s^{(r)}$ the intersection of the closure of $H_s^{(r)}$ with $(\Delta)^{ss}$, which is called the *boundary of* $H_s^{(r)}$ in $(\boldsymbol{P}^{19})^{ss}$. The corresponding space $\phi(\Delta H_s^{(r)})$ is called the *boundary of* $H_s^{(r)}$ in $\overline{M}$.

1.3. We denote by $i\mathscr{A}_1 j\mathscr{A}_2$ the subset of $\boldsymbol{P}^{19}$ corresponding to irreducible cubic surfaces with exactly i singular points of type A_1 and j singular points of type A_2. We refer to [1] and [2] or to [4] for general definitions of types of singularities. We use $j\mathscr{A}_2$ and $i\mathscr{A}_1$ instead of $0\mathscr{A}_1 j\mathscr{A}_2$ and $i\mathscr{A}_1 0\mathscr{A}_2$ respectively. In fact, we have $2i + 3j \leq 9$, $i \leq 4$ and $(i, j) \neq (3, 1)$. It is well-known that $(\boldsymbol{P}^{19})^{ss}$ (respectively $(\boldsymbol{P}^{19})^{s}$) consists of $\boldsymbol{P}^{19} - \Delta$ and all of $i\mathscr{A}_1 j\mathscr{A}_2$ (respectively $i\mathscr{A}_1$) (see [9], p. 51 or [13], 3.2.14 for a proof). We use the notation $\overline{i\mathscr{A}_1 j\mathscr{A}_2}$ for the closure of $i\mathscr{A}_1 j\mathscr{A}_2$ in $(\boldsymbol{P}^{19})^{ss}$.

1.4. A singular, semi-stable cubic surface can be given by a polynomial in the following form:

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 1, 2$ is a homogeneous polynomial of degree i and the point $(0:0:0:1)$ is a singular point. The type of singularity of the surface is characterized by $\mathrm{rank}(f_2)$ and the configuration of points in $V_{\boldsymbol{P}^2}(f_2, f_3)$ (see [4] for details). A closed subscheme in $\boldsymbol{P}^2$ of dimension 0 and of length 6 is called a 6-*point scheme*. Note that $V_{\boldsymbol{P}^2}(f_2, f_3)$ is a 6-point scheme with special configurations. Conversely, let $\mathscr{P}$ be a 6-point scheme in $\boldsymbol{P}^2$ with one of such configurations. Let $\mathscr{L}_{\mathscr{P}}$ be the linear space of cubic forms in $\boldsymbol{P}^2$ containing $\mathscr{P}$. Then $\mathscr{L}_{\mathscr{P}}$ has linear dimension 4. Let $\{f_1, \dots, f_4\}$ be a basis of $\mathscr{L}_{\mathscr{P}}$. Consider the morphism

$$\psi\colon \quad \boldsymbol{P}^2 - \mathscr{P} \to \boldsymbol{P}^3$$
$$P \mapsto (f_1(P) : f_2(P) : f_3(P) : f_4(P)).$$

Let X be the closure of the image of ψ. Then X is a semi-stable cubic surface. The surface X is determined uniquely by $\mathscr{P}$ up to projective transformations. The surface X is called the *csurface* of $\mathscr{P}$. In the case $\mathscr{P}$ consists of 6 points in general position, the surface X is nothing but the blowing-up of $\boldsymbol{P}^2$ at the 6 points. We denote $c(\mathscr{P})$ for the formal cycle of a given 6-point scheme $\mathscr{P}$. We denote S_{ij} for the two-dimensional linear subspace consisting of all cubic forms factoring into the linear form defining $l_{ij} = \overline{P_i P_j}$ and quadratic form passing through $\mathscr{P} - \{P_i, P_j\}$. This subspace determines uniquely a line on X which is denoted by $\tilde{l}_{ij}$. The line $\tilde{l}_{ij}$ is the closure of the image of $l_{ij} - \{P_i, P_j\}$. Similarly, we denote S_{P_i} for the two-dimensional linear subspace consisting of cubic forms singular at P_i. This determines uniquely a line on X which we denote by $\tilde{P}_i$. Also S_{C_i} is denoted for the two-dimensional linear subspace consisting of all cubic forms factoring into the quadratic form defining the conic C_i through $\{P_1, \dots, P_6\} - \{P_i\}$ and linear form vanishing at P_i. This subspace determines uniquely a line on X, which is denoted by $\tilde{C}_i$. The line $\tilde{C}_i$ is nothing but the closure of the image of $C_i - \{P_1, P_2, P_3\}$.

We complete this section with two well-known results.

Lemma 1.1. *The subsets* $i\mathscr{A}_1 jA_2$ *are irreducible in* $\boldsymbol{P}^{19}$.

Proof. This follows from [3]. Also see [13], 3.4.1. □

Lemma 1.2. *Let* $x, y \in (\boldsymbol{P}^{19})^{ss}$ *such that the corresponding cubic surfaces contain one singular point of type* A_2. *Let* $\phi : (\boldsymbol{P}^{19})^{ss} \to \overline{M}$ *be the quotient space. Then* $\phi(x) = \phi(y)$. *Consequently, the set* $\overline{M} - N$ *consists of the singleton* s, *which is the image of all non-stable points.*

Remark 1.3. This was pointed out (without proof) in the introduction part of [10]. The reader could find a proof in [13], 3.4.2, which is formulated from a suggestion of Prof. Dr. E. Looijenga.

2. On the boundary of H_1

Recall that H_1 is the subvariety of $\boldsymbol{P}^{19} - \Delta$ parameterizing all cubic surfaces with at least one star point. The subset H_1 is irreducible of codimension 1 (see [12], p. 288).

A surface X corresponding to a point in $2\mathscr{A}_1$ has exactly 16 lines. There is one line on X containing two singular points and the unique star point of X. See [13], p. 71 for the configuration of 16 lines of X.

Proposition 2.1. *The subset* $2\mathscr{A}_1$ *is contained in the closure of* H_1. *Consequently, the subset* $\overline{2\mathscr{A}_1}$ *is a component of* $\Delta(H_1)$. *Moreover, the star point of* $2\mathscr{A}_1$ *is the specialization position of star points on surfaces corresponding to points in* H_1.

Proof. Let $x \in 2\mathscr{A}_1$. From [4] or [13], p. 71, we see that the corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{Q} = \sum_{i=1}^{6} Q_i$ where 3 points Q_2, Q_3, Q_6 lie on a line l_1; three points Q_4, Q_5, Q_6 lie on another line l_2; no 3 of the five points $Q_1, \dots, Q_5$ are collinear (Figure 1).

Let P_t be a moving point on the line $d = \overline{Q_1 Q_6}$. At a general position of P_t on d, the 6-point scheme $\mathscr{P}_t = \sum_{i=1}^{6} P_i$ where $P_i = Q_i$ for $1 \leq i \leq 5$ and $P_6 = P_t$, gives a non-singular cubic surface with at least one star point. Except for a finite number of positions, when P_t moves on the line d, we have a family in H_1. This implies that x lies on the closure of H_1. Moreover, we see that the section of star points over the family is defined by the tritangent planes $H_t = (\tilde{l}_{23}, \tilde{l}_{45}, \tilde{l}_{16})$. In the specialization position, the linear subspaces S_{23}, S_{45}, S_{C_1} and S_{Q_6} coincide. This means that $\tilde{Q}_6$ is the line connecting the 2 singular points and the section

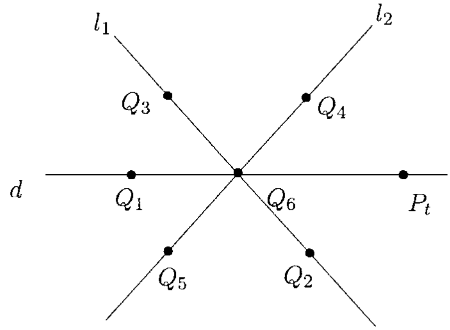

FIGURE 1. 6-point schemes giving points in $2\mathscr{A}_1$

of tritangent planes H_t contains the triple $(2\tilde{Q}_1, \tilde{l}_{16})$. This implies the last conclusion. □

Recall that each surface corresponding to a point in $\mathscr{A}_2$ has exactly 15 lines ([4], p. 255). Moreover, any singular point of type A_2 is a star point ([13], p. 82).

PROPOSITION 2.2. *Any $x \in \mathscr{A}_2$ lies on the closure of H_1. Consequently, the subset $\overline{\mathscr{A}_2}$ is a component of $\Delta(H_1)$. Moreover, the A_2 singularity of X_x, as a star point, is the specialization position of star points on surfaces corresponding to points in H_1.*

Proof. Let $\mathscr{Q}$ be a 6-point scheme where $c(\mathscr{Q}) = 2Q_1 + \sum_{i=2}^{5} Q_i$, such that three points Q_1, Q_2, Q_3 are contained in a line l; the direction at double point $2Q_1$ does not contain any Q_i for $i = 2, 3, 4, 5$; the four points Q_1, Q_2, Q_4, Q_5 as well as 4 points Q_1, Q_3, Q_4, Q_5 are in general position (FIGURE 2, (a)). We know that ([13], p. 73 and p. 86), the csurface of $\mathscr{P}$ is isomorphic to a cubic surface with exactly one A_2 singularity.

Let $x \in \mathscr{A}_2$. The surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{Q}$ where $c(\mathscr{Q}) = 2Q_1 + \sum_{i=2}^{5} Q_i$ described as above.

Let O be the intersection point of l and $\overline{Q_4Q_5}$. Let d be the direction at the double point $2Q_1$. Let m be a fixed line which contains Q_3 and does not contain any other point in $\{Q_1, \dots, Q_5\}$. Let (P_6, P_3) be a pair of moving points where $P_6 \in d$ and $P_3 \in m$ such that $\overline{P_3P_6}$ contains O. It is clear that, except for a finite number of positions, when moving (P_6, P_3), the csurfaces of 6-point schemes $\mathscr{P} = \sum_{i=1}^{6} P_i$, where $P_i = Q_i$ for $i \in \{1, 2, 4, 5\}$, are isomorphic to non-singular cubic surfaces with at least one star point. This defines a family in H_1. When $(P_6, P_3) = (Q_1, Q_3)$, we get 6-point scheme $\mathscr{Q}$ whose csurface is isomorphic to X_x. So x lies on the closure of H_1. Moreover, the star section over the family is defined by the tritangent planes $(\tilde{l}_{12}, \tilde{l}_{45}, \tilde{l}_{36})$. In the specialization position, the linear subspaces S_{12}, S_{26} and S_{Q_3} coincide; the linear subspaces S_{36}, S_{13} and S_{Q_2}

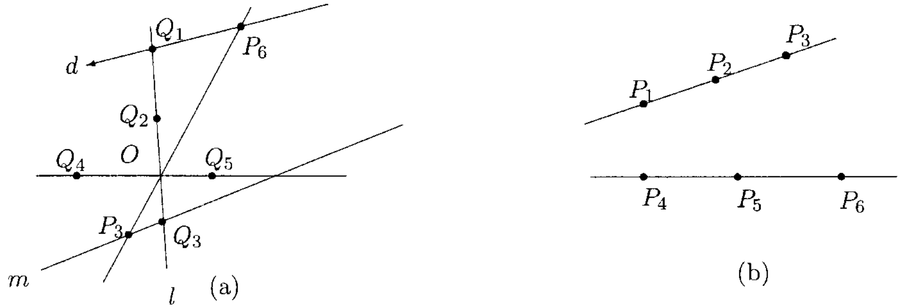

FIGURE 2. 6-point schemes giving points in $\mathscr{A}_2$

coincide. Note that the 6 lines $\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{l}_{45}, \tilde{l}_{14}$ and $\tilde{l}_{15}$ contain the A_2 singularity. It is clear that the section of star points gives a specialization to the intersection of $\tilde{Q}_2, \tilde{Q}_3$ and $\tilde{l}_{45}$, which is the A_2 singularity. □

Remark 2.3. In [13], p. 82, there is a list of all star points on semi-stable cubic surfaces. Moreover, there is a definition of *proper star point*, which is the specialization position of some section of star points on non-singular cubic surfaces. In fact, all star points on semi-stable cubic surfaces are proper star points ([13], 3.4.10).

Consider the set K_1 consisting of all 6-point schemes $\mathscr{P}$ where $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$ such that $\mathscr{P}$ is contained in an irreducible conic and $l_{12} \cap l_{34} \cap l_{56} = \{O\}$ (FIGURE 3).

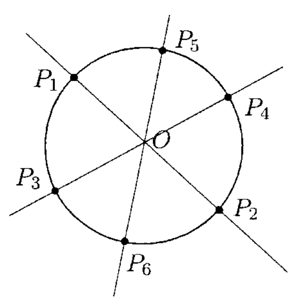

FIGURE 3. 6-point schemes of K_1

Let D_1 be the subset of $\boldsymbol{P}^{19}$ consisting of all points corresponding to the cubic surfaces, each of them is isomorphic to the csurface of an element in K_1. It is clear that $D_1 \subset \mathscr{A}_1$. We see that D_1 is contained in the closure of H_1 also. For this, let $x \in D_1$. The corresponding surface X_x is isomorphic to the csurface of some 6-point scheme $\mathscr{P} \in K$. Let $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$. Fix $P_1, \ldots, P_5$ and let P_6

move on the line $\overline{P_5O}$. This defines a family of 6-point schemes whose csurfaces are isomorphic to non-singular cubic surfaces with at least one star point. When P_6 is contained in the conic defined by $P_1, \dots, P_5$, we get $\mathscr{P}$. This implies that x lies on the closure of H_1. Moreover, we prove that:

LEMMA 2.4. *The subset D_1 is irreducible in $(\boldsymbol{P}^{19})^{ss}$.*

Proof. Let $x \in D_1$. Since $D_1 \subset \mathscr{A}_1$, by choosing coordinates, we can assume that x is given by

$$F = x_3(x_1^2 - x_0x_2) + f_3, \tag{1}$$

where

$$f_3 = a_0x_0^3 + a_1x_0^2x_1 + a_2x_0x_1^2 + a_3x_1^3 + a_4x_1^2x_2 + a_5x_1x_2^2 + a_6x_2^2,$$

such that the scheme $V_{\boldsymbol{P}^2}(x_1^2 - x_0x_2, f_3)$ consists of 6 distinct points (see [4], pp. 247–248). Furthermore, there exists a numbering of 6 points $P_1, \dots, P_6$ of $V_{\boldsymbol{P}^2}(x_1^2 - x_0x_2, f_3)$ such that $l_{12} \cap l_{34} \cap l_{56} \neq \emptyset$ (FIGURE 3).

The 6 points of $V_{\boldsymbol{P}^2}(x_1^2 - x_0x_2, f_3)$ are determined by the solutions of the equation

$$a_0\theta^6 + a_1\theta^5\psi + a_2\theta^4\psi^2 + a_3\theta^3\psi^3 + a_4\theta^2\psi^4 + a_5\theta\psi^5 + a_6\psi^6 = 0. \tag{2}$$

Let T be the projective space parameterizing all homogeneous polynomials of degree 6 in two variables.

Consider the morphism

$$(\boldsymbol{P}^1)^6 \to T$$
$$(a_1 : b_1; \cdots ; a_6 : b_6) \mapsto \prod_{i=1}^{6}(b_i\theta - a_i\psi).$$

Each solution $(\theta_i : \psi_i)$ of (2) corresponds to a point $P_i = (\theta_i^2 : \theta_i\psi_i : \psi_i^2)$ for $1 \le i \le 6$ which is contained in the conic $V_{\boldsymbol{P}^2}(x_1^2 - x_0x_2)$. We see that, the set of all elements of $(\boldsymbol{P}^1)^6 - \Delta$ such that $l_{12} \cap l_{34} \cap l_{56} \neq \emptyset$ is irreducible. This implies that the subset $D_1' \subset \boldsymbol{P}^{19}$ which consists of all elements corresponding to polynomials of the form (1) and satisfying the above condition is irreducible. We see that D_1 is the image of the morphism $\varphi : \mathrm{PGL}(4) \times D_1' \to \boldsymbol{P}^{19}$ which is induced from the action of PGL(4) on $\boldsymbol{P}^{19}$. So the set D_1 is irreducible. $\square$

THEOREM 2.5. $\Delta H_1 = \overline{D_1} \cup \overline{2\mathscr{A}_1} \cup \overline{\mathscr{A}_2}$.

Proof. It is clear that the sets $\overline{D_1}, \overline{2\mathscr{A}_1}$ and $\overline{\mathscr{A}_2}$ are irreducible components of ΔH_1. Conversely, let x be the generic point of an irreducible component W of ΔH_1. Suppose that $W \neq \overline{2\mathscr{A}_1}$ and $W \neq \overline{\mathscr{A}_2}$. Since $\dim W = 17$, we have $x \in \mathscr{A}_1$. So the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{P}$ such that $c(\mathscr{P}) = \sum_{i=1}^6 P_i$ where 6 distinct points $P_1, \dots, P_6$ are contained in an irreducible conic. Note that X_x has exactly 21 lines (see [4], p. 249). The 21 lines of X_x are $\tilde{P}_i$ and $\tilde{l}_{ij}$ for $1 \le i < j \le 6$. Note that any singular point of type A_1 is not a star point (see [13], p. 82). Therefore, the star point of X_x is determined by a triple of the form $(\tilde{l}_{ij}, \tilde{l}_{hk}, \tilde{l}_{mn})$. This implies that the 6 points $P_1, \dots, P_6$ satisfy $l_{ij} \cap l_{hk} \cap l_{mn} \neq \emptyset$. This means that $x \in D_1$ and $W = \overline{D_1}$. □

Recall that $\phi : (\boldsymbol{P}^{19})^{ss} \to \overline{M}$ is the quotient space with respect to the action of PGL(4) on $\boldsymbol{P}^{19}$.

Corollary 2.6. *$\phi(\Delta H_1) = \phi(\overline{D_1}) \cup \phi(\overline{2\mathscr{A}_1})$. Moreover, the components $\phi(\overline{D_1})$ and $\phi(\overline{2\mathscr{A}_1})$ contain the singleton s.*

Proof. Since $\phi : (\boldsymbol{P}^{19})^{ss} \to \overline{M}$ is a good quotient (see [11], 2.13 or [13], 3.2.8), the sets $\phi(\overline{D_1})$ and $\phi(\overline{2\mathscr{A}_1})$ are closed. Moreover, since $\overline{2\mathscr{A}_2} \subset \overline{2\mathscr{A}_1}$, we have $s \in \phi(\overline{2\mathscr{A}_1})$. The first conclusion follows from the theorem.

Let $\mathscr{P} \in K$. By definition, we have $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$ where 6 points $P_1, \dots, P_6$ are contained in an irreducible conic and $l_{12} \cap l_{34} \cap l_{56} = \{O\}$. Consider the quadratic transformation φ with respect to P_1, P_3 and P_5, we see that the 6-point scheme $\varphi(\mathscr{P})$ consists of 6 distinct points $Q_1, \dots, Q_6$ such that Q_2, Q_4, Q_6 are collinear and $\overline{Q_1 Q_2} \cap \overline{Q_3 Q_4} \cap \overline{Q_5 Q_6} = \{\varphi(O)\}$ (Figure 4). The 6-point scheme $\varphi(\mathscr{P})$ corresponds to a csurface in D_1 (see [12], 3.3.10).

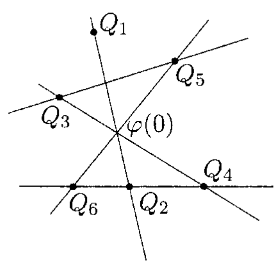

Figure 4. 6-point schemes giving points in D_1

Consider a family in D_1 given by fixing 5 points $Q_2, \dots, Q_6$ and moving Q_1 on the line $\overline{Q_2 \varphi(O)}$, where $\varphi(O) = \overline{Q_3 Q_4} \cap \overline{Q_5 Q_6}$. When Q_1 coincides with the intersection point of $\overline{Q_2 \varphi(O)}$ and $\overline{Q_3 Q_5}$, we get a 6-point scheme whose csurface is isomorphic to a surface with exactly one A_2 singularity. This implies that $s \in \phi(\overline{D_1})$. □

3. On the boundaries of $H_2^{(2)}$ and $H_2^{(3)}$

Recall that H_2 is the subset of $\boldsymbol{P}^{19} - \Delta$ corresponding to non-singular cubic surfaces with at least two star points. The space H_2 is of codimension 2 and has two irreducible components $H_2^{(2)}$ and $H_2^{(3)}$. The subset $H_2^{(2)}$ parametrizes non-singular cubic surfaces containing two star triples with one line in common. The subset $H_2^{(3)}$ parametrizes non-singular cubic surfaces containing two star triples with no line in common; these surfaces have at least 3 star points, which are collinear (see [12], p. 289 or [13], 2.3.1).

3.1. On the boundary of $H_2^{(3)}$

PROPOSITION 3.1. *The set* $3\mathscr{A}_1$ *is contained in the closure of* $H_2^{(3)}$. *Consequently, the set* $\overline{3\mathscr{A}_1}$ *is an irreducible component of* $\Delta H_2^{(3)}$.

Proof. Let $x \in 3\mathscr{A}_1$. The corresponding surface X_x can be considered as the csurface of a 6-point scheme $\mathscr{Q}$ consisting of 6 distinct points $Q_1, \dots, Q_6$ such that Q_1, Q_3, Q_6 as well as Q_3, Q_5, Q_2 and Q_4, Q_5, Q_6 are collinear; moreover $Q_1 \notin \overline{Q_2 Q_4}$ (FIGURE 5). See [13], p. 74 for other configurations of $3\mathscr{A}_1$.

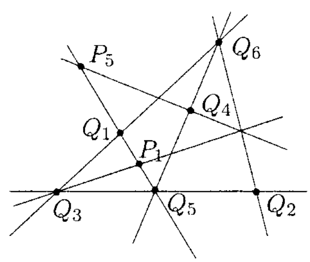

FIGURE 5. 6-point schemes giving points in $3\mathscr{A}_1$

Consider a family of 6-point schemes $\mathscr{P} = \sum_{i=1}^{6} P_i$ where $P_i = Q_i$ for $i \in \{2, 3, 4, 6\}$ and P_1, P_5 move on the line $\overline{Q_1 Q_5}$ in such a way that $\overline{P_1 Q_3} \cap \overline{Q_4 P_5} \cap \overline{Q_2 Q_6} \neq \emptyset$. Except for a finite number of positions, each pair (P_1, P_5) gives a 6-point scheme such that its csurface is isomorphic to a cubic surface in $H_2^{(3)}$. This gives a family in $H_2^{(3)}$. The three star points are given by $(\tilde{l}_{15}, \tilde{l}_{23}, \tilde{l}_{46})$, $(\tilde{l}_{13}, \tilde{l}_{45}, \tilde{l}_{26})$ and $(\tilde{l}_{34}, \tilde{l}_{16}, \tilde{l}_{25})$. When $P_1 = Q_1$ then $P_5 = Q_5$. This implies that $x \in \Delta H_2^{(3)}$. □

DEFINITION. Let K_0 be the subset of $(\boldsymbol{P}^2)^4$ consisting of 4-tuple (P_1, P_2, P_3, P_4) in general position. Let

$$K_1 = \{(P_1, P_2, P_3, P_4, P_5) \in (\boldsymbol{P}^2)^5 \mid (P_1, P_2, P_3, P_4) \in K_0; P_5 \in l_{12}; P_5 \notin l_{34}; \\ P_5 \neq P_i, \forall i = 1, 2\};$$

$$B_2^{(3)} = \{(P_1, P_2, P_3, P_4, P_5, P_6) \in (\boldsymbol{P}^2)^6 \mid (P_1, P_2, P_3, P_4, P_5) \in K_1; \\ l_{16} \cap l_{24} \cap l_{35} \neq \emptyset; l_{14} \cap l_{23} \cap l_{56} \neq \emptyset\} \quad \text{(FIGURE 6)}.$$

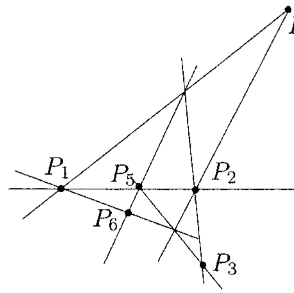

FIGURE 6. 6-point schemes giving points in $D_2^{(3)}$

Let $D_2^{(3)}$ be the subset of $\boldsymbol{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of a 6-point scheme determined by some 6-tuple in $B_2^{(3)}$. It is easy to see that $D_2^{(3)} \subset \mathscr{A}_1$.

PROPOSITION 3.2. *The closure of $D_2^{(3)}$ is an irreducible component of $\Delta H_2^{(3)}$.*

Proof. First of all, we prove that $D_2^{(3)}$ is irreducible. The set K_0 is an open subset of $(\boldsymbol{P}^2)^4$ so it is irreducible. Each fiber of the projection $p : K_1 \to K_0$ is isomorphic to an open set of $\boldsymbol{P}^1$. This implies that K_1 is irreducible. Since K_1 is isomorphic to $B_2^{(3)}$, the set $B_2^{(3)}$ is irreducible.

Let

$$L = \{(\mathscr{P}, F_1, F_2, F_3, F_4) \mid \mathscr{P} \in B_2^{(3)}; F_i \text{ for } 1 \le i \le 4 \text{ is a cubic form in } \mathscr{L}_{\mathscr{P}}\},$$

$$U = \{(\mathscr{P}, F_1, F_2, F_3, F_4) \in L \mid \{F_1, F_2, F_3, F_4\} \text{ is a basis of } \mathscr{L}_{\mathscr{P}}\}.$$

Consider the following diagram:

$$\begin{array}{ccccc} U & \xhookrightarrow{\text{open}} & L & \xhookrightarrow{\text{closed}} & B_2^{(3)} \times (\boldsymbol{P}^9)^4 \\ & \searrow & \downarrow g & \swarrow p & \\ & & B_2^{(3)} & & \end{array}$$

where p is the projection. The map g is surjective and every fiber is isomorphic to $(\boldsymbol{P}^3)^4$. So L is irreducible. This implies that U is irreducible.

Let D_1 be the subset of $B_2^{(3)} \times \boldsymbol{P}^{19}$ consisting of all pairs $(\mathscr{P}, x)$ where the cubic surface corresponding to x is isomorphic to the csurface of the 6-point scheme determined by $\mathscr{P}$. Given $(\mathscr{P}, F_1, F_2, F_3, F_4) \in U$, the closure of the rational map from $\boldsymbol{P}^2$ to $\boldsymbol{P}^3$ defined by the basis $\{F_1, F_2, F_3, F_4\}$ is a cubic surface in $\boldsymbol{P}^3$. We have a morphism $\tau : U \to D_1$ which is surjective. This implies that D_1 is irreducible. The projection $D_1 \to D_2^{(3)}$ is surjective. Consequently $D_2^{(3)}$ is irreducible.

Let $x \in D_2^{(3)}$. The corresponding surface X_x can be considered as the csurface of a 6-point scheme $\mathscr{P}$ such that $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$ where $(P_1, \ldots, P_6) \in B_2^{(3)}$.

Fix P_1, P_2, P_6, P_4 and let P_3, P_5 move on the lines l_{23}, l_{56}, respectively in such a way that $l_{16} \cap l_{24} \cap l_{35} \neq \emptyset$. We obtain a family in $H_2^{(3)}$ such that x is a specialization position. In this family, the three star points are given by $(\tilde{l}_{14}, \tilde{l}_{23}, \tilde{l}_{56})$, $(\tilde{l}_{16}, \tilde{l}_{24}, \tilde{l}_{35})$ and $(\tilde{l}_{26}, \tilde{l}_{45}, \tilde{l}_{13})$. Moreover, we see that when $P_5 = P_6$ then P_3 moves to the intersection point of l_{16} and l_{23}. This gives a 6-point scheme whose csurface is isomorphic to a cubic surface with exactly one A_2 singularity (FIGURE 2, (a)). □

LEMMA 3.3. *Let x be the generic point of an irreducible component of $\Delta H_2^{(3)}$. Then $x \notin 2\mathscr{A}_1$.*

Proof. Suppose that $x \in 2\mathscr{A}_1$. Let S_1, S_2, S_3 be the three star points of X_x. By choosing coordinates, we can assume that the corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{Q}$ such that $c(\mathscr{Q}) = \sum_{i=1}^{6} Q_i$ where Q_6, Q_2, Q_3 as well as Q_6, Q_4, Q_5 are collinear, and no 3 of the five points $Q_1, \ldots, Q_5$ are collinear (see FIGURE 1). The line $\tilde{Q}_6$ contains the two singular points and has multiplicity 4. The line $\tilde{l}_{16}$ intersects $\tilde{Q}_6$ but does not contain any singular point. Note that the 9 lines of the 3 star triples of a non-singular cubic surface in $H_2^{(3)}$ are mutually different. Since the line $\tilde{l}_{16}$ has multiplicity 1, the line $\tilde{l}_{16}$ does not contain all S_1, S_2, S_3. Thus there exists a star triple whose lines are different from $\tilde{l}_{16}$. But from the configuration of lines and tritangent planes together their multiplicities (see [6], Articles 35–201 or [13], pp. 71–72), we see that there does not exist a such star triple. □

THEOREM 3.4. *Let $\phi : (\boldsymbol{P}^{19})^{ss} \to \overline{M}$ be the quotient space with respect to the action of* PGL(4) *on $\boldsymbol{P}^{19}$. Then $\phi(\Delta H_2^{(3)}) = \phi(\overline{D_2^{(3)}}) \cup \phi(\overline{3\mathscr{A}_1})$. Moreover, the components $\phi(\overline{D_2^{(3)}})$ and $\phi(\overline{3\mathscr{A}_1})$ contain the singleton s.*

Proof. By the end of the proof of Proposition 3.2, we see that the boundary of $D_2^{(3)}$ contains a point of $\mathscr{A}_2$. So $s \in \phi(\overline{D_2^3})$. Since $3\mathscr{A}_2 \subset \overline{3\mathscr{A}_1}$, we have $s \in \phi(\overline{3\mathscr{A}_1})$.

Let x be the generic point of an irreducible component W of $\phi(\Delta H_2^{(3)})$. Suppose that $W \neq \phi(\overline{3\mathscr{A}_1})$. From the previous lemma, we see that $x \notin 2\mathscr{A}_1$. Therefore $x \in \mathscr{A}_1$. By choosing coordinates, we can assume that the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{P}$ where $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$ such that the 6 points $P_1, \ldots, P_6$ are contained in an irreducible conic. Since the singular point of X_x is not a star point, the 3 star points of X_x are determined by 3 triples $(\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$, $(\tilde{l}_{im}, \tilde{l}_{jk}, \tilde{l}_{nh})$ and $(\tilde{l}_{mk}, \tilde{l}_{ih}, \tilde{l}_{jn})$, where $\{i, j, m, n, h, k\} = \{1, 2, 3, 4, 5, 6\}$. This implies that the 6 points $P_1, \ldots, P_6$ in $\boldsymbol{P}^2$ satisfy the corresponding conditions, namely $l_{ij} \cap l_{mn} \cap l_{kh} \neq \emptyset$, $l_{im} \cap l_{jk} \cap l_{nh} \neq \emptyset$ and $l_{mk} \cap l_{ih} \cap l_{jn} \neq \emptyset$. Consider the quadratic transformation with respect to P_i, P_n, P_k ([13], 3.3.10), we see that the image of $\mathscr{P}$ is a 6-point scheme $\mathscr{Q}$ where $c(\mathscr{Q}) = \sum_{i=1}^{6} Q_i$ such that 6 points $Q_1, \ldots, Q_6$, up to a permutation of 6 letters, form an element of $B_2^{(3)}$. This implies that $W = \phi(\overline{D_2^{(3)}})$. □

3.2. On the boundary of $H_2^{(2)}$

DEFINITION. Let

$$B_2^{(2)} = \{(P_1, \ldots, P_5, O) \in (\boldsymbol{P}^2)^6 \mid P_1, P_2, P_3, P_4, P_5 \text{ are in general position; } l_{24} \cap l_{35} = \{O\}; \text{ the conic determined by } P_1, \ldots, P_5 \text{ is tangent to } \overline{P_1 O} \text{ at } P_1\} \text{ (FIGURE 7 (a)).}$$

Note that each element in $B_2^{(2)}$ defines uniquely a 6-point scheme $\mathscr{P}$ such that $c(\mathscr{P}) = 2P_1 + \sum_{i=2}^{5} P_i$, where the direction at the double point $2P_1$ is determined by $\overline{P_1 O}$. Moreover, the csurface of $\mathscr{P}$ has exactly two A_1 singularities ([13], pp. 71–72). Let $D_2^{(2)}$ be the subset of $\boldsymbol{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of a 6-point scheme determined by some element of $B_2^{(2)}$. Then $D_2^{(2)} \subset 2\mathscr{A}_1$.

Let

$$C_2^{(2)} = \{(P_1, \ldots, P_6) \in (\boldsymbol{P}^2)^6 \mid P_i \neq P_j \ \forall i \neq j; l_{12} \cap l_{34} \cap l_{56} \neq \emptyset; l_{14} \cap l_{23} \cap l_{56} \neq \emptyset; P_1, P_2, P_3, P_4, P_5, P_6 \text{ are contained in an irreducible conic}\}$$

(FIGURE 7 (b)).

Let $E_2^{(2)}$ be the subset of $\boldsymbol{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of a 6-point scheme determined by some 6-tuple in $C_2^{(2)}$. Then $E_2^{(2)} \subset \mathscr{A}_1$.

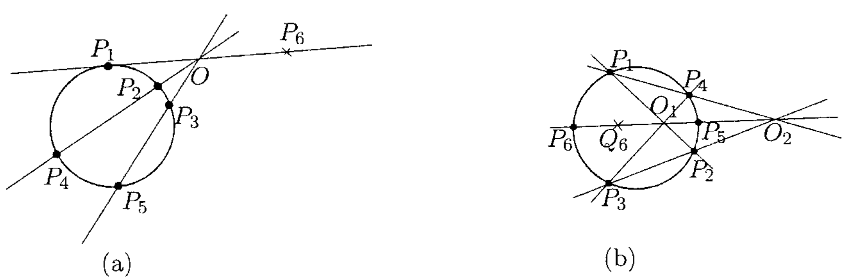

FIGURE 7. 6-point schemes of $B_2^{(2)}$ and $C_2^{(2)}$, respectively

PROPOSITION 3.5. *The closures of* $D_2^{(2)}$ *and* $E_2^{(2)}$ *are irreducible components of* $\Delta H_2^{(2)}$.

Proof. Let $\boldsymbol{P}^5$ be the projective space parameterizing the non-zero quadratic forms in three variables. Let

$$K_0 = \{(P_1, P_2, P_3, P_4) \in (\boldsymbol{P}^2)^4 \mid P_1, P_2, P_3, P_4 \text{ are in general position}\};$$

$$K_1 = \{(P_1, P_2, P_3, P_4, O, \mathscr{C}) \in (\boldsymbol{P}^2)^5 \times \boldsymbol{P}^5 \mid (P_1, P_2, P_3, P_4) \in K_0; O \in l_{24}; O \notin \{P_2, P_4\}; O \notin l_{13}; \mathscr{C} \text{ is the conic containing } P_1, P_2, P_3, P_4 \text{ and tangent to } \overline{P_1 O} \text{ at } P_1\};$$

$$K_2 = \{(P_1, P_2, P_3, P_4, P_5, O, \mathscr{C}) \in (\boldsymbol{P}^2)^6 \times \boldsymbol{P}^5 \mid (P_1, P_2, P_3, P_4, O, \mathscr{C}) \in K_1; P_5 \in \mathscr{C} \cap \overline{P_3 O}\}.$$

It is clear that the set K_0 is irreducible. Every fiber of the projection $p_1 : K_1 \to K_0$ is isomorphic to an open set of $\boldsymbol{P}^1$. This implies that K_1 is irreducible. The projections $K_2 \to K_1$ and $K_2 \to B_2^{(2)}$ are isomorphisms. Therefore $B_2^{(2)}$ is irreducible.

Similarly, we prove that $C_2^{(2)}$ is irreducible.

By a similar argument as used in the proof of (3.2), we see that $D_2^{(2)}$ and $E_2^{(2)}$ are irreducible.

Suppose $x \in D_2^{(2)}$. The corresponding surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{P}_0$ determined by an element $(P_1, \ldots, P_5, O) \in B_2^{(2)}$. Consider a family of $H_2^{(2)}$ given by 6-point schemes $\mathscr{P}$ such that $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$ where P_6 is a moving point on the line $\overline{P_1 O}$ (Figure 7 (a)). The two star points are given by $(\tilde{P}_1, \tilde{C}_6, \tilde{l}_{16})$ and $(\tilde{l}_{16}, \tilde{l}_{24}, \tilde{l}_{35})$. This implies that $x \in \Delta H_2^{(2)}$.

Let x be an element in $E_2^{(2)}$. The cubic surface X_x corresponding to x is isomorphic to the csurface of a 6-point scheme $\mathscr{P}_0$ determined by a 6-tuple $(P_1, \ldots, P_6)$ in $C_2^{(2)}$. Consider a family of $H_2^{(2)}$ given by 6-point schemes $\mathscr{Q}$ such that $c(\mathscr{Q}) = \sum_{i=1}^{6} Q_i$ where $Q_i = P_i$ for $1 \le i \le 5$ and Q_6 is a moving point on the line $\overline{O_1 O_2}$ (Figure 7 (b)). The two star points are given by $(\tilde{l}_{12}, \tilde{l}_{34}, \tilde{l}_{56})$ and $(\tilde{l}_{56}, \tilde{l}_{14}, \tilde{l}_{23})$. This implies that $x \in \Delta H_2^{(2)}$. □

Proposition 3.6. *Let x be the generic point of an irreducible component W of $\Delta H_2^{(2)}$. If $x \in \mathscr{A}_1$, then $W = \overline{E_2^{(2)}}$.*

Proof. By choosing coordinates, we can assume that the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{P}$ such that $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$ where 6 points $P_1, \ldots, P_6$ are contained in an irreducible conic. Since the A_1 singularity is not a star point, the two star points of X_x are defined by triples $(\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$ and $(\tilde{l}_{ij}, \tilde{l}_{mk}, \tilde{l}_{nh})$ where $\{i, j, m, n, h, k\} = \{1, 2, 3, 4, 5, 6\}$. This implies that the 6 points $P_1, \ldots, P_6$ in $\boldsymbol{P}^2$ satisfy the corresponding conditions, namely $l_{ij} \cap l_{mn} \cap l_{kh} \neq \emptyset$ and $l_{ij} \cap l_{mk} \cap l_{nh} \neq \emptyset$. Up to a permutation of 6 letters, the six points $P_1, \ldots, P_6$ define a 6-tuple in $C_2^{(2)}$. This means that $x \in E_2^{(2)}$ and therefore $W = \overline{E_2^{(2)}}$. □

Remark 3.7. How many components does the boundary of $H_2^{(2)}$ have? We do not know. We just know that, if $\phi(\Delta(H_2^{(2)}))$ has another component then it is a subset of $\phi(\overline{2\mathscr{A}_2})$. For this, a cubic surface corresponding to a general point of $\phi(\overline{3\mathscr{A}_1})$ contains 3 star points and there is no line containing more than one star point.

4. On the boundaries of $H_4^{(4)}$ and $H_4^{(6)}$

DEFINITION. Let X be a non-singular cubic surface having two star triples with no lines in common. It follows that X has a third star triple such that three corresponding star points are collinear. A such three star triples is called a *Star-Steiner set*.

From [12], p. 289 or [13], p. 24, we have $H_4 = H_4^{(4)} \cup H_4^{(6)} \cup H_4^{(9)}$ where the irreducible component $H_4^{(6)}$ $(H_4^{(4)}, H_4^{(9)}$ respectively) parametrizes non-singular cubic surfaces, each of which possesses a pair (S, T) where S is a Star-Steiner set and T is another star triple with exactly one line (all three lines, no line respectively) in common with the lines of S.

4.1. On the boundary of $H_4^{(6)}$

PROPOSITION 4.1. *The set* $4\mathscr{A}_1$ *is contained in the closure of* $H_4^{(6)}$. *Consequently, the closure of* $4\mathscr{A}_1$ *in* $(\boldsymbol{P}^{19})^{ss}$ *is an irreducible component of* $\Delta H_4^{(6)}$.

Proof. Let $x \in 4\mathscr{A}_1$. By choosing coordinates, the corresponding surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{Q}$, where $c(\mathscr{Q}) = 2Q_1 + 2Q_4 + Q_2 + Q_5$ such that Q_1, Q_2, Q_3, Q_4 are in general position and the directions at Q_1 and Q_4 contain Q_5 (FIGURE 8, (b)). See [13], p. 96 for another configuration for $4\mathscr{A}_1$ (FIGURE 8, (a)).

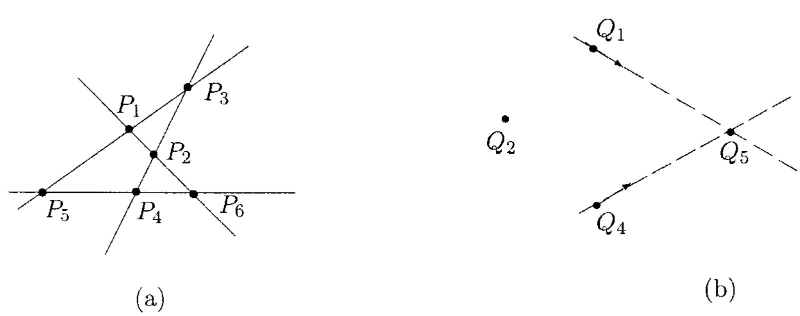

FIGURE 8. 6-point schemes giving points in $4\mathscr{A}_1$

Let C be the conic containing Q_1, Q_2, Q_4 and being tangent to the directions at Q_1 and Q_4. Consider a 6-point scheme $\mathscr{P}$ such that $c(\mathscr{P}) = \sum_{i=1}^{6} P_6$ where $P_i = Q_i$ for $i \in \{1, 2, 4, 5\}$, two points P_3, P_6 are contained in C and $\overline{P_1P_4} \cap \overline{P_2P_5} \cap \overline{P_3P_6} \neq \emptyset$ (FIGURE 9).

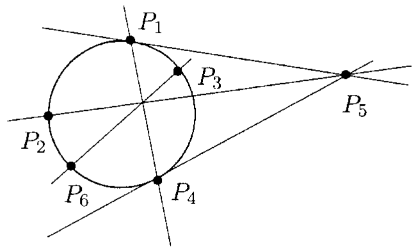

FIGURE 9. 6-point schemes giving points in $H_4^{(6)}$

Let P_3 and P_6 move on the conic C in such a way that $\overline{P_1P_4} \cap \overline{P_2P_5} \cap \overline{P_3P_6} \neq \emptyset$. We have a family of cubic surfaces in $H_4^{(6)}$. The 6 star points are given by $(\tilde{P}_1, \tilde{C}_5, \tilde{l}_{15})$, $(\tilde{P}_4, \tilde{C}_5, \tilde{l}_{45})$, $(\tilde{l}_{14}, \tilde{l}_{25}, \tilde{l}_{36})$, $(\tilde{C}_4, \tilde{P}_2, \tilde{l}_{24})$, $(\tilde{P}_5, \tilde{C}_2, \tilde{l}_{25})$ and $(\tilde{C}_1, \tilde{P}_2, \tilde{l}_{12})$. When $(P_3, P_6) = (P_1, P_4)$, we get the 6-point scheme $\mathscr{Q}$. This implies that $x \in \Delta H_4^{(6)}$. □

Remark 4.2. We know that four A_1 singularities on a cubic surface corresponding to a point in $4\mathscr{A}_1$ form a tetrahedron. Each edge of the tetrahedron contains one star point. Two star points on opposite edges lie on another line of X, which has multiplicity one. Therefore, the 6 star points of X lie on a hyperplane spanned by the three lines of multiplicity 1.

Recall that $\phi : (\boldsymbol{P}^{19})^{ss} \to \overline{M}$ be the quotient space with respect to the action of PGL(4) on $\boldsymbol{P}^{19}$.

THEOREM 4.3. *The set $\phi(\Delta H_4^{(6)})$ consists of two points, one is the singleton s and another is the image of $4\mathscr{A}_1$ in $\overline{M}$.*

Proof. Let $K_4^{(6)}$ be the set consisting of all 6-point schemes $\mathscr{P}$ in general position such that $c(\mathscr{P}) = \sum_{i=1}^{6} P_6$ where the conic C_5 is tangent to l_{15} and l_{45} at P_1 and P_4 respectively; the lines l_{14}, l_{25} and l_{36} have one point in common (FIGURE 9).

The blowing-up of $\boldsymbol{P}^2$ at $\mathscr{P} \in K_4^{(6)}$ is isomorphic to a non-singular cubic surface in $H_4^{(6)}$. Conversely, for any $x \in H_4^{(6)}$, the corresponding cubic surface X_x is isomorphic to the csurface of some 6-point scheme in $K_4^{(6)}$. Let $\mathscr{P} \in K_4^{(6)}$. Fix P_1, P_2, P_4, P_5 and let P_3, P_6 move on the conic C_5 in such a way that $\overline{P_3P_6}$ contains the intersection point of l_{14} and l_{25}. Except for two positions determined when $\overline{P_3P_6} = l_{25}$ and $\overline{P_3P_6} = l_{14}$, the 6 points $P_1, \ldots, P_6$ define a 6-point scheme in $K_4^{(6)}$. This defines a surjective morphism from an open set of $\boldsymbol{P}^1$ to $\phi(H_4^{(6)})$. This extends to a surjective morphism $\xi : \boldsymbol{P}^1 \to \phi(\overline{H_4^{(6)}})$. It is clear that when $\overline{P_3P_6} = l_{14}$, we get a point $t_1 \in \boldsymbol{P}^1$ such that $\xi(t_1) = \phi(4\mathscr{A}_1)$ (see (4.1)). When $\overline{P_3P_5} = l_{25}$, we get a point $t_2 \in \boldsymbol{P}^1$. The point $\xi(t_2)$ corresponds to the csurface of a 6-point scheme $\mathscr{P}_0$ such that $c(\mathscr{P}_0) = 2P_2 + P_1 + P_4 + P_3 + P_5 + P_6$

where P_2, P_3, P_5 are collinear. The csurface of 6-point scheme $\mathscr{P}_0$ has exactly one singular point of type A_2. Therefore $\xi(t_2) = s$. □

Since $H_4^{(6)} \subset H_2^{(2)}$, we have:

Corollary 4.4. *The set* $\phi(\Delta H_2^{(2)})$ *contains the singleton* s.

4.2. On the boundary of $H_4^{(4)}$

Recall that $N = \mathrm{PGL}(4)\backslash(\boldsymbol{P}^{19})^s$. We know that $\overline{M} = N \cup \{s\}$. Let

$$B_4^{(4)} = \{(P_1, \dots, P_6, O_1) \in (\boldsymbol{P}^2)^6 \mid P_1, P_2, P_3, P_4 \text{ are in general position}; \\ l_{12} \cap l_{34} = \{O_1\}; P_5 \in l_{13} \cap l_{24}; P_6 \in l_{23} \cap \overline{P_5 O_1}\} \text{ (Figure 10 (a))}.$$

Note that each element in $B_4^{(4)}$ defines uniquely a 6-point scheme $\mathscr{P}$ such that $c(\mathscr{P}) = \sum_{i=2}^{6} P_i$. Moreover, the csurface of $\mathscr{P}$ is isomorphic to a cubic surface with exactly three A_1 singularities (see [13], pp. 74–75). Let $D_4^{(4)}$ be the subset of $\boldsymbol{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of a 6-point scheme determined by some element in $B_4^{(4)}$. We see that $D_4^{(4)} \subset 3\mathscr{A}_1$.

Let

$$C_4^{(4)} = \{(P_1, \dots, P_6, O_1, O_2) \in (\boldsymbol{P}^2)^6 \mid P_1, P_2, P_3, P_4 \text{ are in general position}; \\ l_{12} \cap l_{34} \cap l_{56} = \{O_1\}; l_{13} \cap l_{24} \cap l_{56} = \{O_2\}; l_{12} \cap l_{36} \cap l_{45} \neq \emptyset; P_5 \in l_{23}\}$$

(Figure 10 (b)). Each element in $C_4^{(4)}$ defines uniquely a 6-point scheme $\mathscr{P}$ such that $c(\mathscr{P}) = \sum_{i=1}^{6} P_i$. Moreover, the csurface of $\mathscr{P}$ is isomorphic to a cubic surface with exactly one A_1 singularity. Let $E_4^{(4)}$ be the subset of $\boldsymbol{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of a 6-point scheme determined by some element in $C_4^{(4)}$. We see that $E_4^{(4)} \subset \mathscr{A}_1$.

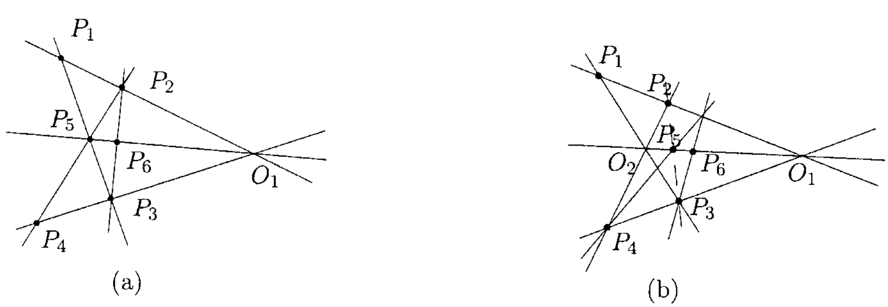

Figure 10. 6-point schemes of $B_4^{(4)}$ and $C_4^{(4)}$, respectively

Proposition 4.5. *The closures of* $D_4^{(4)}$ *and* $E_4^{(4)}$ *in* $(\boldsymbol{P}^{19})^{ss}$ *are irreducible components of* $\Delta H_4^{(4)}$.

Proof. Let $K_0 = \{(P_1, P_2, P_3, P_4) \in (\boldsymbol{P}^2)^4 \mid P_1, P_2, P_3, P_4$ are in general position$\}$. Consider the projection $p : B_4^{(4)} \to K_0$. We see that p is an isomorphism. So $B_4^{(4)}$ is irreducible. Similarly, the projection $C_4^{(4)} \to K_0$ is an isomorphism. So the set $C_4^{(4)}$ is irreducible. By a similar argument as used in the proof of (3.2), we see that $D_4^{(4)}$ and $E_4^{(4)}$ are irreducible.

Let $x \in D_4^{(4)}$. By definition, the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathscr{P}$ determined by an element $(P_1, \ldots, P_6, O_1)$ of $B_4^{(4)}$. Fix 4 points P_1, P_2, P_3, P_4. Let $l_{13} \cap l_{24} = \{O_2\}$. Let P_5^t, P_6^t move on the line $\overline{O_1 O_2}$ in such a way that $l_{45} \cap l_{36} \cap l_{12} \neq \emptyset$. Except for a finite number of positions of (P_5^t, P_6^t), the 6 points $P_1, \ldots, P_4, P_5^t, P_6^t$ form a 6-point scheme $\mathscr{P}$ such that the csurface of $\mathscr{P}$ is isomorphic to a cubic surface in $H_4^{(4)}$ ([12], p. 289). The four star points are given by $(\tilde{l}_{12}, \tilde{l}_{34}, \tilde{l}_{56})$, $(\tilde{l}_{12}, \tilde{l}_{36}, \tilde{l}_{45})$, $(\tilde{l}_{13}, \tilde{l}_{24}, \tilde{l}_{56})$ and $(\tilde{l}_{34}, \tilde{l}_{15}, \tilde{l}_{26})$. We obtain a family in $H_4^{(4)}$. It is clear that x is a specialization position of this family.

Similarly, if $x \in E_4^{(4)}$, we consider a family defined as above. The point x is a specialization position which is determined when $l_{23} \cap \overline{O_1 O_2} = \{P_5\}$. □

Remark 4.6. It is not clear that if the boundary of $H_4^{(4)}$ in $\overline{M}$ contains the singleton. However, we can prove the following corollary.

COROLLARY 4.7. *The set $\phi(\Delta H_4^{(4)}) \cap N$ consists of two points which are the image of $D_4^{(4)}$ and $E_4^{(4)}$.*

Proof. See [13], 3.4.24. □

5. Acknowledgement

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Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-0397
Japan
e-mail: chanhtu@math.metro-u.ac.jp