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ON *G*-FIBERINGS OVER THE CIRCLE WITHIN A COBORDISM CLASS

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Introduction

Conner and Floyd [1] have characterized those unoriented cobordism classes that admit a representative which fibers over the circle S^1. They have shown that a closed manifold M is cobordant to a bundle over S^1 if and only if $\chi(M) \equiv 0 \pmod 2$, where χ is the Euler characteristic. Let G be a finite abelian group of odd order and $\mathfrak{N}_*^G$ the cobordism group of unoriented closed G-manifolds. The purpose of this paper is to determine when a class β in $\mathfrak{N}_*^G$ has a representative which fibers equivariantly over S^1 such that the action of G takes place within fiber. The author [3] has discussed such a question in case where $G = \boldsymbol{Z}_{2^r}$, the cyclic group of order 2^r.

In Section 1, we first introduce an SK group SK_*^G resulting from equivariant cuttings and pastings (G-SK processes) of closed G-manifolds. The abbreviation SK stands for Schneiden und Kleben in German. Kosniowski [7] has obtained some generators of SK_*^G as a free SK_*-module, where SK_* is an SK ring of closed manifolds in Karras, Kreck, Neumann and Ossa [5] (Proposition 1.4). As an example, we perform G-SK processes on some complex projective space with G-action and write it by the above generators (Example 1.8).

In Section 2, we consider a notion of G-SK invariant studied in [5] and [7]. Let T be a map for closed G-manifolds which takes values in the ring $\boldsymbol{Z}$ of rational integers and is additive with respect to the disjoint union of G-manifolds. Such a T is said to be a G-SK invariant if it is invariant under G-SK processes. Given a G-manifold M, let M_σ be a G-submanifold of M consisting of those points whose slice types containing σ. Then a map χ_σ defined by $\chi_\sigma(M) = \chi(M_\sigma)$ is a G-SK invariant. Further, for a subgroup H of G, the map χ^H defined by $\chi^H(M) = \chi(M^H)$ is also a G-SK invariant, where $M^H = \{x \in M \mid hx = x \text{ for any } h \in H\}$. We see that $\chi^H = \sum_\sigma \chi_\sigma$ summing over all σ with H as an isotropy subgroup. The above T is considered to be an additive homomorphism $T : SK_*^G \to \boldsymbol{Z}$. We determine a form of T by using those χ_σ and have a base for a $\boldsymbol{Z}$-module $\mathscr{T}_*^G$ consisting of all G-SK invariants (Theorem 2.6).

In Section 3, we devote to a study of G-fiberings over S^1. Let $\overline{SK}_*^G$ be SK_*^G

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factored by the equivariant cobordism relation. Let $\bar{\mathscr{T}}_*^G$ be a $\boldsymbol{Z}_2$-vector space consisting of all homomorphisms $\bar{T} : \overline{SK}_*^G \to \boldsymbol{Z}_2$. Such a map $\bar{T}$ is called a G-$\overline{SK}$ invariant, namely a G-SK invariant (modulo 2) and simultaneously invariant under equivariant cobordism. We first show that a G-SK invariant T which is considered to take values in $\boldsymbol{Z}_2$ via the surjection $\boldsymbol{Z} \to \boldsymbol{Z}_2$, is always a G-$\overline{SK}$ invariants (Theorem 3.8). The kernel F_*^G of the natural surjection $j_* : \mathfrak{N}_*^G \to \overline{SK}_*^G$ is exactly generated by those classes, each of which admits a representative fibered equivariantly over S^1. We characterize the elements of F_*^G by using G-$\overline{SK}$ invariants (Theorem 3.10 and Proposition 3.12). Finally, in case $G = \boldsymbol{Z}_7$, we give a non-zero element of F_*^G by using the complex projective space with G-action treated in Example 1.8 (Example 3.14).

1. Equivariant cutting and pasting

Let G be a finite abelian group. In this paper, a G-manifold means an unoriented compact smooth manifold together with a smooth action of G. Let N_i $(i = 1, 2)$ be m-dimensional G-manifolds and $\phi, \psi : \partial N_1 \to \partial N_2$ equivariant diffeomorphisms. Pasting along their boundaries, we have closed G-manifolds $M_1 = N_1 \cup_\phi N_2$ and $M_2 = N_1 \cup_\psi N_2$. Then it is said that M_1 and M_2 are obtained from each other by an equivariant cutting and pasting (G-SK process) [5, 7]. Let $\mathscr{M}_m^G$ be the set of all m-dimensional closed G-manifolds. Then it is an abelian semigroup with respect to the disjoint union $+$ and has a zero given by the empty set $\emptyset$.

Definition 1.1. G manifolds M_1 and $M_2 \in \mathscr{M}_m^G$ are said to be G-SK equivalent, in symbols $M_1 \sim M_2$, if there is a G manifold $K \in \mathscr{M}_m^G$ such that $M_1 + K$ and $M_2 + K$ can be obtained from each other by a finite sequence of equivariant cuttings and pastings.

The G-SK equivalence $\sim$ is an equivalence relation on the set $\mathscr{M}_m^G$ and the set $\Gamma_m^G = \mathscr{M}_m^G/\sim$ of all equivalence classes is a cancellative abelian semigroup. Let denote by $[M]$ the class containing a G-manifold M. Denote by SK_m^G the Grothendieck group of Γ_m^G. We then have a graded SK_*-module $SK_*^G = \bigoplus_{m \geq 0} SK_m^G$ given by the cartesian product of manifolds. Here SK_* is an SK ring of closed manifolds which is a polynomial ring over $\boldsymbol{Z}$ with a generator α represented by the real projective plane $\boldsymbol{RP}^2$ [7; Theorem 2.5.1 (i)].

We assume for the remainder of this paper that G is an abelian group of odd order. A G-module means a finite-dimensional real vector space together with a linear action of G. For a subgroup H of G, let $C(H)$ consist of all subgroups J of H such that the quotient $H/J \cong \boldsymbol{Z}_d$, a cyclic group of odd order d. Then, for $J \in C(H)$ an irreducible H-module $V(J, j)$ is defined as follows: if $d = 1$ then $V(H, 1) = \boldsymbol{R}$ with the trivial action of H, while if $d \geq 3$ then $V(J, j)$ is the set $\mathbf{C}$ of complex numbers with a generator h of H/J acting by multiplication by $\exp(2\pi i m_j/d)$, where $\{m_j\}$ is the complete set of integers such that $0 < m_1 < m_2 < \cdots < m_{\varphi(d)} < d$ and each m_j is prime to d (φ, the Euler phi function). If M

is a G-manifold and $x \in M$, then there is a G_x-module U_x which is equivariantly diffeomorphic to a G_x-neighbourhood of x. Here $G_x = \{g \in G \mid gx = x\}$ is the isotropy subgroup at x. The module U_x decomposes as $U_x = \boldsymbol{R}^p \oplus V_x$, where G_x acts trivially on $\boldsymbol{R}^p$ and $V_x^{G_x} = \{0\}$. We refer to the pair $\sigma_x = [G_x; V_x]$ as a slice type of x. By a G-slice type in general, we mean a pair $\sigma = [H; V]$ of a subgroup H and an H-module V with $V^H = \{0\}$. More precisely, V is a product of non-trivial irreducible H-modules $V(J, j)$ ($J \in C(H)$ with $H/J \cong \boldsymbol{Z}_d$ and $1 \le j < \frac{1}{2}\varphi(d) + 1$) (cf. [7; Theorem 1.6.1]). We denote by σ_0 the slice type $[\{1\}; \{0\}]$, where $\{1\}$ is the trivial group. Let $St(G)$ be the set of all G-slice types. There is a partial ordering on $St(G)$ such that $[H; V] \preceq [K; W]$ if $[H; V]$ is a slice type of G-manifold $G \times_K W$. Further, we give a total ordering on $St(G)$, which preserves the one $\preceq$, as follows. For any positive divisor k of $|G|$, let $L(k)$ be the set consisting of all subgroups H of G such that $|H| = k$. First order the elements in $L(k)$ appropriately, then this ordering gives the one $<$ on the set of all subgroups of G, preserving inclusion of subgroups, that is, if $H \subseteq K$ then $H \le K$. Moreover, for any H such an ordering leads to the one on the set of non-trivial irreducible H-modules: $V(J_1, j_1) < V(J_2, j_2)$ if $J_2 < J_1$ or $J_1 = J_2$ and $j_1 < j_2$. Finally we order the elements in $St(G)$ as follows:

(1) $[H; V] < [K; W]$ if $\dim(V) < \dim(W)$.
(2) Suppose that $\dim(V) = \dim(W)$, then $[H; V] < [K; W]$ if $H < K$.
(3) Suppose that $\dim(V) = \dim(W)$ and $H = K$, then $[H; V] < [H; W]$ if $V < W$ in the ordering of H-modules induced lexicographically from the one of irreducible H-modules (cf. [7; Section 1.7]).

Definition 1.2. Let W be a K-module and H a subgroup of K. Then denote by W_H an H-module W induced from $H \subseteq K$. Let $\{W_k\}$ be the set of all non-trivial irreducible K-modules. If $\tau = [K; W]$, $W = \prod_k W_k^{a(k)}$ ($a(k) \ge 0$) is a slice type, then we define a slice type τ_H by $\tau_H = [H; V]$, where V is the non-trivial part of the H-module $\prod_k (W_k)_H^{a(k)}$. Since $(W_k)_{\{1\}} = \boldsymbol{R}^2$, we have that $\tau_{\{1\}} = \sigma_0$ for any τ. Let $|\tau| = \dim(W)$ be the dimension of τ.

Remark 1.3. (i) More precisely, let $W_k = V(L, j)$ for some $L \subset K$ with $K/L \simeq \boldsymbol{Z}_a$ and the integer m_j such that $0 < m_j < a$, $(m_j, a) = 1$. Then $(W_k)_H = V(L \cap H, j')$ with $0 < m_{j'} < b$, $(m_{j'}, b) = 1$, where $H/(L \cap H) = LH/L \simeq \boldsymbol{Z}_b$. The integer j' is determined by the action LH/L on $(W_k)_H$ induced from the one of K/L on W_k. We see that $(W_k)_H$ is the trivial H-module $\boldsymbol{R}^2$ only if $H \subseteq L$. It follows that the difference $|\tau| - |\tau_H|$ is the sum of $\dim((W_k)_H)$ $(= 2)$ with $H \subseteq L$.

(ii) $W_H = \boldsymbol{R}^{|\tau| - |\tau_H|} \times V$ as an H-module and $W^H = (W_H)^H = \boldsymbol{R}^{|\tau| - |\tau_H|} \times \{0\}$ has slice types τ_U ($H \subseteq U \subseteq K$) as a K-invariant subspace of W. Note that $\tau_U \le \tau$ because $|\tau_U| \le |\tau|$.

Proposition 1.4 (cf. [7; Theorem 5.2.1]). *SK_*^G is a free SK_*-module with basis $\mathscr{B} = \{y[\sigma]; \sigma = [H; V] \in St(G)\}$, where $y[\sigma] = [G \times_H \boldsymbol{R}P(V \times \boldsymbol{R})]$ and $\boldsymbol{R}P(V \times \boldsymbol{R})$ denotes the real projective space of the product $V \times \boldsymbol{R}$.*

Now, by using the total ordering on $St(G)$, we rename the G-slice types: $\sigma_0 = \rho_0, \rho_1, \rho_2, \dots$ with the condition that if $i < j$ then $\rho_i < \rho_j$. Set $\mathscr{F}_k = \{\rho_j; j \le k\}$, then $\mathscr{F}_k$ is a family of G-slice types in the sense of that in [7; Section 1.2].

Corollary 1.5. *If a G-manifold M has slice types $\sigma_x \in \mathscr{F}_k$ $(x \in M)$, then the class $[M]$ is a linear combination over SK_* by the elements $y[\rho_j]$ with $\rho_j \in \mathscr{F}_k$.*

Lemma 1.6. *For G-modules U_i $(i = 1, 2)$, let $S(U_1 \times U_2)$ be a $G \times S^1$-sphere, that is the G-sphere together with the natural action of the circle group S^1. Then there is an SK equivalence*:

$$2S(U_1 \times U_2) \overset{(S^1)}{\sim} S(U_1 \times \boldsymbol{R}) \times S(U_2) + S(U_1) \times S(U_2 \times \boldsymbol{R}),$$

where we use a symbol $\overset{(S^1)}{\sim}$ instead of $\sim$ because the above G-SK process is compatible with the action of S^1.

Proof. Let $N_1 = N_2 = S(U_1) \times D(U_2) + D(U_1) \times S(U_2)$, where $S(U_i)$ and the disk $D(U_i)$ are considered to be $G \times S^1$-spaces. Then we obtain the above equivalence by pasting ∂N_1 to ∂N_2 by the natural $G \times S^1$-equivariant identifications ϕ and ψ. □

Lemma 1.7. *For G-modules V_i such that $V_i^G = \{0\}$ $(i = 1, 2)$, we have the following SK equivalences.*

(i) $S(\boldsymbol{R}^{2k+1} \times V_1) \sim 2\boldsymbol{RP}^{2k} \times \boldsymbol{RP}(V_1 \times \boldsymbol{R})$.

(ii) $\boldsymbol{RP}(V_1 \times \boldsymbol{R}) \times \boldsymbol{RP}(V_2 \times \boldsymbol{R}) \sim \boldsymbol{RP}(V_1 V_2 \times \boldsymbol{R})$.

Proof. We first consider (i). Let $SK_*^G(pt, pt)$ be an SK group resulting from cuttings and pastings of G-manifolds with boundary in [2, 4]. It follows that $[D(V_1)] = [\boldsymbol{RP}(V_1 \times \boldsymbol{R})]$ in $SK_*^G(pt, pt)$ since V_1 is a product of two-dimensional irreducible G-modules (cf. [4; Lemma 3.8 and Example 3.9 (3.3)]). Hence we obtain the equivalence in case $k = 0 : [S(V_1 \times \boldsymbol{R})] = 2[\boldsymbol{RP}(V_1 \times \boldsymbol{R})]$ by making use of the map $\mathscr{D}_* : SK_*^G(pt, pt) \to SK_*^G$ given by $\mathscr{D}_*([M]) = [M \cup M]$, the double of a G-manifold M. Further, when $k \ge 1$, set $(U_1, U_2) = (\boldsymbol{R}^{2k+1}, V_1)$, forgetting S^1-action, in the equivalence in Lemma 1.6. Then

$$2S(\boldsymbol{R}^{2k+1} \times V_1) \sim P_1 + P_2, \tag{1.7.1}$$

where $P_1 = S^{2k+1} \times S(V_1)$ and $P_2 = S^{2k} \times S(V_1 \times \boldsymbol{R})$. Since $S^{2k+1} \sim \emptyset$ and $S^{2k} \sim 2\boldsymbol{RP}^{2k}$, we have that $2S(\boldsymbol{R}^{2k+1} \times V_1) \sim P_2 \sim 2\boldsymbol{RP}^{2k} \times 2\boldsymbol{RP}(V_1 \times \boldsymbol{R})$ (cf. [7; Theorem 2.5.1 (ii)]). Thus (i) follows since SK_*^G has no torsion (cf. Proposition 1.4). Next we prove (ii). Let $(U_1, U_2) = (V_1, V_2 \times \boldsymbol{R})$, then $2S(V_1 V_2 \times \boldsymbol{R}) \sim S(V_1 \times \boldsymbol{R}) \times S(V_2 \times \boldsymbol{R}) + S(V_1) \times S(V_2 \times \boldsymbol{R}^2)$ by Lemma 1.6. It is seen that $S(V_1)$ and $S(V_2 \times \boldsymbol{R}^2) \sim \emptyset$ since they are odd-dimensional G-manifolds (cf. Proposition 1.4). Hence $4\boldsymbol{RP}(V_1 V_2 \times \boldsymbol{R}) \sim 2\boldsymbol{RP}(V_1 \times \boldsymbol{R}) \times 2\boldsymbol{RP}(V_2 \times \boldsymbol{R})$ by (i), which implies the result. □

Example 1.8. Consider the case where $G = \boldsymbol{Z}_p$ (p; odd prime). The nontrivial irreducible G-modules are $V_j = \boldsymbol{C}$ with a generator of G acting by multiplication by $\exp(2\pi ij/p)$ $(1 \le j \le t = \frac{1}{2}(p-1))$. We denote by $\langle a(1), a(2), \dots, a(t)\rangle$ a slice type $\sigma = [G; V]$ with $V = \prod_{1 \le j \le t} V_j^{a(j)}$. Let $M = \boldsymbol{C}P(\boldsymbol{C}^{a(0)} \times \sigma)$ be the associated complex projective space of the product $\boldsymbol{C}^{a(0)} \times V$ with $a(0) \ge 0$. Then $[M]$ is represented by the generators of SK_*^G in Proposition 1.4 as

$$[M] = \sum_{0 \le k \le t} a(k)\alpha^{a(k)-1} y[\sigma_{(k)}], \tag{1.8.1}$$

where $\sigma_{(k)} = \sigma$ if $k = 0$,

$$\langle a(k-1) + a(k+1), a(k-2) + a(k+2), \dots,$$
$$a(0) + a(2k), a(2k+1), \dots, a(t), 0, \dots, 0\rangle$$

if $1 \le k < \frac{1}{2}t$,

$$\langle a(k-1) + a(k+1), a(k-2) + a(k+2), \dots,$$
$$a(2k-t) + a(t), a(2k-t-1), \dots, a(0), 0, \dots, 0\rangle$$

if $\frac{1}{2}t \le k < t$ or

$$\langle a(t-1), a(t-2), \dots, a(0)\rangle$$

if $k = t$. To show (1.8.1), we use the relation in Lemma 1.6. Set $(U_1, U_2) = (V_0^{a(0)}, V)$, where $V_0 = \boldsymbol{C}$ with the natural S^1-action. Then

$$2S(V_0^{a(0)} \times V) \overset{(S^1)}{\sim} S(V_0^{a(0)} \times \boldsymbol{R}) \times S(V) + S(V_0^{a(0)}) \times S(V \times \boldsymbol{R}). \tag{1.8.2}$$

Next divide V as $V = V_1^{a(1)} \times V'$ with $V' = \prod_{2 \le j \le t} V_j^{a(j)}$ and put $(U_1, U_2) = (V_1^{a(1)}, V')$. Then

$$2S(V) \overset{(S^1)}{\sim} S(V_1^{a(1)} \times \boldsymbol{R}) \times S(V') + S(V_1^{a(1)}) \times S(V' \times \boldsymbol{R}).$$

Taking this to (1.8.2), we have

$$\begin{aligned} 2^2 S(V_0^{a(0)} \times V) \overset{(S^1)}{\sim} & S(V_0^{a(0)} \times \boldsymbol{R}) \times S(V_1^{a(1)} \times \boldsymbol{R}) \times S(V') \\ & + S(V_0^{a(0)} \times \boldsymbol{R}) \times S(V_1^{a(1)}) \times S(V' \times \boldsymbol{R}) \\ & + 2S(V_0^{a(0)}) \times S(V \times \boldsymbol{R}). \end{aligned}$$

Continuing such an SK process on $S(V')$ inductively, we have

$$2^t S(V_0^{a(0)} \times V) \overset{(S^1)}{\sim} P + \sum_{0 \le k < t} 2^{t-1-k} P_k,$$

where

$$P = \left(\prod_{0 \le j < t} S(V_j^{a(j)} \times \boldsymbol{R}) \right) \times S(V_t^{a(t)}),$$

$$P_k = \left(\prod_{0 \le j < k} S(V_j^{a(j)} \times \boldsymbol{R}) \right) \times S(V_k^{a(k)}) \times S\left(\prod_{k < j \le t} V_j^{a(j)} \times \boldsymbol{R} \right).$$

This induces an SK equivalence on the orbit spaces with respect to S^1:

$$2^t \boldsymbol{C}P(V_0^{a(0)} \times V) \sim \bar{P} + \sum_{0 \le k < t} 2^{t-1-k} \overline{P_k}. \tag{1.8.3}$$

Here, it follows from Lemma 1.7 that $\bar{P}$ fibers equivariantly over $\overline{S(V_t^{a(t)})} = \boldsymbol{C}P^{a(t)-1}$ with fiber

$$\begin{aligned} F &= \prod_{0 \le j < t} S((V_j \otimes V_t)^{a(j)} \times \boldsymbol{R}) \sim 2^t \boldsymbol{R}P\left(\prod_{0 \le j < t} (V_j \otimes V_t)^{a(j)} \times \boldsymbol{R} \right) \\ &= 2^t \boldsymbol{R}P(\sigma_{(t)} \times \boldsymbol{R}) \end{aligned} \tag{1.8.4}$$

and $\overline{P_k}$ fibers equivariantly over $\overline{S(V_k^{a(k)})} = \boldsymbol{C}P^{a(k)-1}$ with fiber

$$\begin{aligned} F_k &= \left(\prod_{0 \le j < k} S((V_j \otimes V_k)^{a(j)} \times \boldsymbol{R}) \right) \times S\left(\prod_{k < j \le t} (V_j \otimes V_k)^{a(j)} \times \boldsymbol{R} \right) \\ &\sim 2^k \prod_{0 \le j < k} \boldsymbol{R}P((V_j \otimes V_k)^{a(j)} \times \boldsymbol{R}) \times 2\boldsymbol{R}P\left(\prod_{k < j \le t} (V_j \otimes V_k)^{a(j)} \times \boldsymbol{R} \right) \\ &\sim 2^{k+1} \boldsymbol{R}P\left(\prod_{j \ne k} (V_j \otimes V_k)^{a(j)} \times \boldsymbol{R} \right) = 2^{k+1} \boldsymbol{R}P(\sigma_{(k)} \times \boldsymbol{R}). \end{aligned} \tag{1.8.5}$$

From these, we have $\bar{P} \sim \boldsymbol{C}P^{a(t)-1} \times F$ and $\overline{P_k} \sim \boldsymbol{C}P^{a(k)-1} \times F_k$ $(0 \le k < t)$ (cf. [7; Theorem 2.4.1 (iv)]). It is seen that $[\boldsymbol{C}P^{a(k)-1}] = a(k)\alpha^{a(k)-1}$ in SK_* since $\chi(\boldsymbol{C}P^{a(k)-1}) = a(k)$ (cf. [7; Theorem 2.5.1 (ii)]). Therefore we obtain the desired equality by taking (1.8.4) and (1.8.5) in (1.8.3).

Remark 1.9. In case of $G = \boldsymbol{Z}_{2^r}$, we have obtained a similar equality as (1.8.1) by performing an SK process on G-manifolds with boundary (cf. [2; Example 2.12 (ii)]).

2. G-SK invariants

In this section, we determine a form of G-SK invariants.

Definition 2.1. Let $\sigma = [H; V] \in St(G)$ and M a G-manifold. Then define

M_σ to be the set consisting of those points $x \in M$ such that $(\sigma_x)_H = \sigma$ in the sense of Definition 1.2.

Remark 2.2. Let M_H be M with the induced action of H, then M_σ is precisely the set $(M_H)_\sigma = \{x \in M_H; \sigma_x = \sigma\}$. Since σ is maximal in the family $\mathscr{F}(M_H) = \{\sigma_x; x \in M_H\}$ with respect to the partial ordering $\preceq$ given in Section 1, M_σ is a G-invariant submanifold of M with $\dim(M_\sigma) = \dim(M) - |\sigma|$ by the slice theorem (cf. [5; Chapter 3]). In case $\sigma = \sigma_0$, we have that $M_{\sigma_0} = M$. The submanifold M^H of M decomposes as $M^H = \sum_\sigma M_\sigma$ summing over all σ with H as an isotropy subgroup.

Example 2.3. For $\tau = [K; W] \in St(G)$, let $M = G \times_K \boldsymbol{RP}(W \times \boldsymbol{R})$ be a representative of the class $y[\tau]$ in $\mathscr{B}$ (cf. Proposition 1.4). The slice types of M are the same as those of $G \times_K (W \times \boldsymbol{R})$ (or W) because W is a complex K-module. If H is a subgroup of K, then $M_H = G/K \times \boldsymbol{RP}(W_H \times \boldsymbol{R})$ with the induced action of H given by $h(([g],[v,t])) = ([g],[hv,t])$ for $h \in H$ and $([g],[v,t]) \in M_H$. On the other hand, if H is not a subgroup of K, then $M^H = \emptyset$. Hence it follows that $[M_\sigma] = |G/K|[\boldsymbol{RP}^{|\tau|-|\tau_H|}] = |G/K|\alpha^{(|\tau|-|\tau_H|)/2}$ if $\sigma = \tau_H$ with $H \subseteq K$ or $[M_\sigma] = 0$ otherwise (cf. Remark 1.3 (ii) and [7; Theorem 1.7.1, Remark 1.7.2]). We see that $[\boldsymbol{RP}^{2m}] = \alpha^m$ in SK_{2m} by considering the SK process as in Lemma 1.7 (ii) when $(V_1, V_2) = (\boldsymbol{C}, \boldsymbol{C}^{m-1})$ (cf. [7; Theorem 2.5.1]).

Definition 2.4. Let $T : \mathscr{M}_m^G \to \boldsymbol{Z}$ be an additive map, that is, if $M = M_1 + M_2$ then $T(M) = T(M_1) + T(M_2)$. We call T a G-SK invariant or simply an invariant if $T(N_1 \cup_\phi N_2) = T(N_1 \cup_\psi N_2)$ for any G-diffeomorphisms ϕ and $\psi : \partial N_1 \to \partial N_2$ in Section 1. If $M_1 \sim M_2$, then $T(M_1) = T(M_2)$. Thus the map T induces an additive homomorphism $T : SK_m^G \to \boldsymbol{Z}$. The set $\mathscr{T}_m^G$ consisting of all these invariants is a $\boldsymbol{Z}$-module under the natural addition.

Example 2.5. Given a slice type $\sigma \in St(G)$, let χ_σ be a map defined by $\chi_\sigma(M) = \chi(M_\sigma)$ for any G-manifold M. Then it is an invariant since $M \sim M'$ implies $M_\sigma \sim M'_\sigma$ naturally. Note that $\chi_{\sigma_0} = \chi$ since $M_{\sigma_0} = M$. Further, for any subgroup H of G, the map χ^H defined by $\chi^H(M) = \chi(M^H)$ is also an invariant and the equality $\chi^H = \sum_\sigma \chi_\sigma$ holds in $\mathscr{T}_m^G$ (cf. Remark 2.2).

Let H be a subgroup of G. Then, by using the total ordering on $St(G)$, define inductively integers $n_H(K)$ for subgroups K with $H \subseteq K \subseteq G$ as follows:

$$n_H(H) = 1, \quad n_H(K) = |K/H| - \sum_{H \subseteq L \subset K} n_H(L),$$

where $L \subset K$ means that $L \subseteq K$ but $L \neq K$. If $H = \{1\}$, then the integers $n_{\{1\}}(K)$ coincide with those n_i in [6; Definition 5.3]. For $\sigma = [H; V] \in St(G)$ and a subgroup K with $H \subset K$, denote by $\mathscr{S}_K(\sigma)$ the set consisting of those slice types $\tau = [K; W]$ such that $\tau_H = \sigma$.

THEOREM 2.6. *For* $\sigma = [H; V] \in St(G)$, *define* θ_σ *by*

$$\theta_\sigma = |G/H|^{-1}\left\{\chi_\sigma + \sum_{H\subset K\subseteq G} n_H(K)\left(\sum_{\tau\in\mathscr{S}_K(\sigma)}\chi_\tau\right)\right\}.$$

Then the set $\{\theta_\sigma; |\sigma| \leq 2n\}$ *provides a basis for* $\mathscr{T}_{2n}^G$ *as a free* $\boldsymbol{Z}$*-module. On the other hand,* $\mathscr{T}_{2n+1}^G = \{0\}$.

Proof. First we see that $\mathscr{T}_{2n+1}^G = \{0\}$ because $SK_{2n+1}^G = \{0\}$ by Proposition 1.4. For $\sigma = [H; V]$ with $|\sigma| \leq 2n$, let $g_\sigma : SK_{2n}^G \to SK_{2n-|\sigma|}$ be a map given by $g_\sigma([M]) = [M_\sigma]$ and f_σ a map defined by

$$f_\sigma = |G/H|^{-1}\left\{g_\sigma + \sum_{H\subset K\subseteq G} n_H(K)\left(\sum_{\tau\in\mathscr{S}_K(\sigma)}\alpha^{(|\tau|-|\sigma|)/2}g_\tau\right)\right\}. \tag{2.6.1}$$

Now look at the basis elements of $\mathscr{B}$ in Proposition 1.4. Then, given $\mu = [K; W] \in St(G)$ the values $f_\sigma(y[\mu])$ which do not vanish are $f_{\mu_L}(y[\mu]) = \alpha^{(|\mu|-|\mu_L|)/2}$ $(L \subseteq K)$. In fact, if $\sigma = \mu_L$ for some L $(\subseteq K)$, then

$$\begin{aligned} f_{\mu_L}(y[\mu]) &= |G/L|^{-1}\left\{g_{\mu_L}(y[\mu]) + \sum_{L\subset U\subseteq K} n_L(U)\alpha^{(|\mu_U|-|\mu_L|)/2}g_{\mu_U}(y[\mu])\right\} \\ &= |K/L|^{-1}\left(\sum_{L\subseteq U\subseteq K} n_L(U)\right)\alpha^{(|\mu|-|\mu_L|)/2} \\ &= \alpha^{(|\mu|-|\mu_L|)/2} \end{aligned} \tag{2.6.2}$$

by Example 2.3 and the equality $\sum_{L\subseteq U\subseteq K} n_L(U) = |K/L|$. On the other hand, if $\sigma \notin \{\mu_L; L \subseteq K\}$, then $\mu_U \notin \mathscr{S}_U(\sigma)$ for $U \subseteq K$. This implies that $g_\sigma(y[\mu]) = g_\tau(y[\mu]) = 0$ in (2.6.1) and $f_\sigma(y[\mu]) = 0$ (cf. Example 2.3). Therefore each f_σ induces an SK_*-homomorphism $f_\sigma : SK_{2*}^G = \sum_n SK_{2n}^G \to SK_{2*-|\sigma|} = \sum_{n\geq(1/2)|\sigma|} SK_{2n-|\sigma|}$ of degree $-|\sigma|$. Now we recall the ordering of G-slice types: $\sigma_0 = \rho_0, \rho_1, \rho_2, \dots$ with the condition that if $i < j$ then $\rho_i < \rho_j$. This ordering ensure that if $\mu = [K; W]$ then $\mu_L < \mu$ for $L \subset K$. Let us define an SK_*-homomorphism f_* by

$$f_* = \oplus_k f_{\rho_k} : SK_{2*}^G \to A = \oplus_k SK_{2*-|\rho_k|},$$

where $f_{\rho_k}(y[\rho_k]) = [pt]_k$, the generator of $SK_{2*-|\rho_k|} \cong SK_*$ as an SK_*-module. We can totally order the basis elements of $\mathscr{B} = \{y[\rho_k]; k \geq 0\}$ and $\mathscr{B}' = \{[pt]_k; k \geq 0\}$ for A naturally. Then it follows from (2.6.2) that f_* is isomorphic because the matrix relative to the ordered bases $\mathscr{B}$ and $\mathscr{B}'$ is triangular with components 1 on the diagonal. Now let T be an element of $\mathscr{T}_{2n}^G$, then there is a factorization

$$T : SK_{2n}^G \overset{f_*}{\cong} \oplus_k SK_{2n-|\rho_k|} \overset{\oplus_k\chi}{\cong} \oplus_k \boldsymbol{Z} \overset{T'}{\to} \boldsymbol{Z} \tag{2.6.3}$$

for some T', where the direct sum is taken over all k with $|\rho_k| \le 2n$ (cf. [7; Theorem 2.5.1 (ii)]). This implies that $T = \sum_k T'(1_k)\theta_{\rho_k}$, where $\theta_{\rho_k} = \chi \circ f_{\rho_k}$ and $1_k = 1$ in the k-th copy of $\boldsymbol{Z}$ in $\oplus_k \boldsymbol{Z}$. Note that $\{\rho_k; |\rho_k| \le 2n\} = \{\sigma; |\sigma| \le 2n\}$ because the ordering on $St(G)$ preserves the dimension $|\sigma|$. Thus the set $\{\theta_\sigma; |\sigma| \le 2n\}$ provides a basis for $\mathcal{T}_{2n}^G$. □

Example 2.7. Suppose that $G = \boldsymbol{Z}_m$ (m; odd). Then, for $\sigma = [\boldsymbol{Z}_s; V] \in St(\boldsymbol{Z}_m)$ with $s|m$, we have

$$\theta_\sigma = (m/s)^{-1}\left\{\chi_\sigma + \sum_{s<t\le m,\, s|t|m} \varphi(t/s)\left(\sum_{\tau \in \mathcal{S}_{\boldsymbol{Z}_t(\sigma)}} \chi_\tau\right)\right\}$$

because $n_{\boldsymbol{Z}_s}(\boldsymbol{Z}_t) = \varphi(t/s)$ by definition. The set $\{\theta_\sigma; |\sigma| \le 2n\}$ provides a basis for $\mathcal{T}_{2n}^{\boldsymbol{Z}_m}$.

Corollary 2.8. *Let H be a subgroup of G. Then we have*

$$\sum_{H \subseteq K \subseteq G} n_H(K)\chi(M^K) \equiv 0 \pmod{|G/H|}$$

for any G-manifold M. In particular, if $H = \{1\}$, then

$$\sum_{K \subseteq G} n_{\{1\}}(K)\chi(M^K) \equiv 0 \pmod{|G|}$$

(cf. [6; Corollary 5.19]).

Proof. Consider a sum $\sum_\sigma \theta_\sigma(M)$ summing over all σ with H as an isotropy subgroup. Then it follows from Example 2.5 and Theorem 2.6 that

$$\begin{aligned}\sum_\sigma \theta_\sigma(M) &= |G/H|^{-1}\left\{\chi(M^H) + \sum_{H \subset K \subseteq G} n_H(K)\chi(M^K)\right\}\\ &= |G/H|^{-1}\sum_{H \subseteq K \subseteq G} n_H(K)\chi(M^K),\end{aligned}$$

which is an integer. This gives us the congruence. □

3. *G*-fiberings over the circle

In this section, a G-SK invariant is considered to take values in $\boldsymbol{Z}_2 = \{0, 1\}$. If m-dimensional G-manifolds M and M' are G-cobordant in the usual sense, then we write $M \overset{C}{\sim} M'$.

Lemma 3.1 (cf. [5; Lemma 1.9] and [7; Corollary 2.3.2]). *Let M and M' be m-dimensional G-manifolds.*

(i) *If* $M \sim M'$ (*SK equivalence*), *then there is a* G-*manifold* P *which fibers equivariantly over the circle* S^1 *with the trivial action of* G *such that* $M \overset{C}{\sim} M' + P$.

(ii) *If* $M \overset{C}{\sim} M'$, *then* $M \sim M' + Q$, *where*

$$Q = \sum a(H, U_1, U_2) \cdot G \times_H (S(U_1) \times S(U_2)) + \sum b(H, U) \cdot G \times_H S(U)$$

for some integers $a(H, U_1, U_2)$ *and* $b(H, U)$. *Here, the first sum is taken over all subgroups* $H \subseteq G$ *and all* H-*modules* U_i *satisfying that* $(U_1)^H = \{0\}$ *such that* $\dim(U_1) + \dim(U_2) = m + 2$, *while the second sum is taken over all* H *and all* H-*modules* U *such that* $\dim(U) = m + 1$.

The relations $\sim$ and $\overset{C}{\sim}$ are commutative with each other, i.e. given M and M', the following (i) and (ii) are equivalent: (i) there is a G-manifold A such that $M \sim A \overset{C}{\sim} M'$. (ii) there is a G-manifold B such that $M \overset{C}{\sim} B \sim M'$ (cf. [3; Lemma 4.2]).

Definition 3.2. If such an A (or B) exists, then M and M' are said to be G-$\overline{\text{SK}}$ equivalent.

We note that G-$\overline{\text{SK}}$ equivalence is an equivalence relation by the above commutativity.

Definition 3.3 (cf. [5; Chapter 1]). Let $\overline{SK}{}_m^G$ be $\mathscr{M}_m^G$ factored by the G-$\overline{SK}$ equivalence. In other words, $\overline{SK}{}_m^G$ is SK_m^G factored by the relation $\overset{C}{\sim}$.

Let I_m^G be the kernel of the natural surjection $i_* : SK_m^G \to \overline{SK}{}_m^G$, that is the subgroup of SK_m^G generated by all elements $[M] - [M']$ such that $\{M\} = \{M'\}$ in $\mathfrak{R}_m^G$. Note that $\chi(x)$ is even for any $x \in I_m^G$ because so is $\chi(M) - \chi(M')$ (cf. [1; Section 1]).

Lemma 3.4. $I_{2n}^G = 2SK_{2n}^G$ *and* $I_{2n+1}^G = \{0\}$.

Proof. In case $m = 2n$, it is sufficient to show that $I_{2n}^G \subseteq 2SK_{2n}^G$. Take an element $x = [M] - [M'] \in I_{2n}^G$, then x is expressed as

$$x = \sum a(H, U_1, U_2)[G \times_H (S(U_1) \times S(U_2))] + \sum b(H, U)[G \times_H S(U)] \tag{3.4.1}$$

by Lemma 3.1 (ii). First, note that $\dim S(U_1)$ is odd by the condition $(U_1)^H = \{0\}$. This implies that the first sum of the right-hand side vanishes since $[S(U_1)] = 0$ in SK_*^H (cf. Proposition 1.4). On the other hand, since $U = \boldsymbol{R}^{2k+1} \times V$ for some slice type $\sigma = [H; V]$ $(2k + |\sigma| = 2n)$, we have that

$[G \times_H S(U)] = 2\alpha^k y[\sigma]$ by Lemma 1.7 (i) and Example 2.3. Hence $x \in 2SK_{2n}^G$. Finally, $I_{2n+1}^G = \{0\}$ since so is SK_{2n+1}^G. □

From the above, there exists an isomorphism $\overline{SK}_m^G \cong SK_m^G/2SK_m^G$. The following theorem is therefore immediate by Proposition 1.4.

THEOREM 3.5. *$\overline{SK}_{2n}^G$ is a $\boldsymbol{Z}_2$-module with basis $\{\alpha^{n-|\sigma|/2}y[\sigma]; |\sigma| \leq 2n\}$. On the other hand, $\overline{SK}_{2n+1}^G = \{0\}$.*

DEFINITION 3.6. Let $T : \mathscr{M}_m^G \to \boldsymbol{Z}_2$ be an additive map. We say that T is a G-$\overline{SK}$ invariant if $T(M) = T(M')$ for any M and $M' \in \mathscr{M}_m^G$ such that they are G-$\overline{SK}$ equivalent. A G-$\overline{SK}$ invariant T induces a homomorphism $T : \overline{SK}_m^G \to \boldsymbol{Z}_2$.

Example 3.7. Assume the M and M' are G-$\overline{SK}$ equivalent, i.e. there is a G-manifold A such that $M \sim A \overset{C}{\sim} M'$, then we have $M_\sigma \sim A_\sigma \overset{C}{\sim} M'_\sigma$ for any $\sigma \in St(G)$. This means that M_σ and M'_σ are also G-$\overline{SK}$ equivalent. Thus, $\chi_\sigma \pmod 2$ defined by $\chi_\sigma(M) = \chi(M_\sigma)$ reduced modulo 2 is a G-$\overline{SK}$ invariant.

THEOREM 3.8. *Let $\bar{\mathscr{T}}_m^G$ be the set of all G-$\overline{SK}$ invariants $T : \overline{SK}_m^G \to \boldsymbol{Z}_2$. Then $\bar{\mathscr{T}}_{2n}^G$ is a $\boldsymbol{Z}_2$-module with basis $\{\theta_\sigma \pmod 2; |\sigma| \leq 2n\}$. On the other hand, $\bar{\mathscr{T}}_{2n+1}^G = \{0\}$.*

Proof. The isomorphism in (2.6.3) induces a map

$$\oplus_\sigma \theta_\sigma \pmod 2 : SK_{2n}^G \overset{\oplus\theta_\sigma}{\cong} \oplus_\sigma \boldsymbol{Z} \xrightarrow{i} \oplus_\sigma \boldsymbol{Z}_2, \tag{3.8.1}$$

where the sums are taken over all σ with $|\sigma| \leq 2n$ and $i : \boldsymbol{Z} \to \boldsymbol{Z}_2$ is the natural surjection. Since the kernel of this map is $2SK_{2n}^G = I_{2n}^G$ by Lemma 3.4, the map $\oplus_\sigma \theta_\sigma \pmod 2$ induces the isomorphism $\overline{SK}_{2n}^G \cong \oplus_\sigma \boldsymbol{Z}_2$. This verifies that the set $\{\theta_\sigma \pmod 2; |\sigma| \leq 2n\}$ provides a basis for $\bar{\mathscr{T}}_{2n}^G$. If $m = 2n+1$, then $\bar{\mathscr{T}}_{2n+1}^G$ vanishes because so does $\overline{SK}_{2n+1}^G$. □

Let F_m^G be the kernel of the surjection $j_* : \mathfrak{N}_m^G \to \overline{SK}_m^G$, that is the subgroup of $\mathfrak{N}_m^G$ generated by all classes of the form $\{M\} + \{M'\}$ such that $[M] = [M']$ in SK_m^G. Let us consider the class β which has a representative M' fibered equivariantly over the circle S^1 with a fiber F such that the action of G takes place within F. Then $M' \sim S^1 \times F \sim \emptyset$ and $\beta \in F_m^G$ (cf. [7; Theorem 2.4.1 (i) and (ii)]). It follows from Lemma 3.1 (i) that F_m^G is precisely generated by all these classes β.

Remark 3.9. Note that $F_0^G = \{0\}$. On the other hand, we have that $F_{2n+1}^G = \mathfrak{N}_{2n+1}^G$ because $\overline{SK}_{2n+1}^G = \{0\}$. We can explain this from another point of view as follows. We see that $\mathfrak{N}_*^G$ is multiplicatively generated over the cobordism ring $\mathfrak{N}_*$ by some even-dimensinal G-manifolds (cf. [7; Theorem 4.1.1]).

Hence, if $\dim(M) = 2n+1$, odd, then $\{M\} = \sum_j a_j L_j$, where $a_j \in \mathfrak{N}_*$ with $\dim(a_j)$, odd and $L_j \in \mathfrak{N}_*^G$ with $\dim(L_j)$, even. Since $\chi(a_j) = 0$, we see that each a_j has a representative which fibers over the circle (cf. [1; Section 1]). This implies that $\{M\} \in F_{2n+1}^G$ and hence $F_{2n+1}^G = \mathfrak{N}_{2n+1}^G$.

Now we consider a condition that a class $\{M\}$ belongs to F_{2n}^G. Given $\{M\} \in F_{2n}^G$, let M' be a G-manifold such that $M \overset{C}{\sim} M'$ and it fibers equivariantly over S^1 with a fiber F. Then, for any $\sigma \in St(G)$ we have that $M_\sigma \overset{C}{\sim} M'_\sigma$ which also fibers equivariantly over S^1 with the fiber F_σ. Hence a necessary condition for $\{M\} \in F_{2n}^G$ is that $\chi(M_\sigma) \equiv 0 \pmod 2$ for any σ. We have the following theorem by Theorem 3.8.

Theorem 3.10. *Let M be a $2n$-dimensional G-manifold. Then $\{M\} \in F_{2n}^G$ if and only if $\theta_\sigma(M) \equiv 0 \pmod 2$ for any slice types $\sigma \in St(G)$ with $|\sigma| \le 2n$.*

The following corollary is immediate by Corollary 2.8.

Corollary 3.11. *A necessary condition for a class $\{M\} \in F_{2n}^G$ is that the following congruence*

$$\chi(M^H) + \sum_{H \subset K \subseteq G} n_H(K)\chi(M^K) \equiv 0 \pmod{2 \cdot |G/H|}$$

holds for any subgroup H of G.

Proposition 3.12. *Let $G = \boldsymbol{Z}_{p^r}$ (p; odd prime). Then $\{M\} \in F_{2n}^G$ if and only if*

$$\chi(M_\sigma) \equiv \sum_{\lambda \in \mathscr{S}_{s+1}(\sigma)} \chi(M_\lambda) \pmod{2p^{r-s}} \tag{3.12.1}$$

for any $\sigma = [\boldsymbol{Z}_{p^s}; V] \in St(G)$ with $|\sigma| \le 2n$ $(0 \le s \le r)$, where $\mathscr{S}_{s+1}(\sigma) = \mathscr{S}_{\boldsymbol{Z}_{p^{s+1}}}(\sigma)$ and $\mathscr{S}_{r+1}(\sigma) = \emptyset$.

Proof. By Theorem 3.10, in order that $\{M\} \in F_{2n}^G$, a necessary and sufficient condition is that

$$p^{r-s}\theta_\sigma(M) = \chi_\sigma + \sum_{s<t\le r}(p^{t-s} - p^{t-s-1})\left(\sum_{\tau \in \mathscr{S}_t(\sigma)} \chi_\tau\right) \equiv 0 \pmod{2p^{r-s}} \tag{3.12.2}$$

for any $\sigma = [\boldsymbol{Z}_{p^s}, V] \in St(G)$ $(0 \le s \le r)$, where $\varphi(p^{t-s}) = p^{t-s} - p^{t-s-1}$ in Example 2.7 and an integer $\chi(M_v)$ is simply written as χ_v. We define an integer $h_v(M)$ for $v = [\boldsymbol{Z}_{p^t}; V]$ by

$$h_v(M) = \chi_v - \sum_{\omega \in \mathscr{S}_{t+1}(v)} \chi_\omega.$$

Since $\mathscr{S}_{s+2}(\sigma)$ is decomposed as $\mathscr{S}_{s+2}(\sigma) = \sum_{\lambda \in \mathscr{S}_{s+1}(\sigma)} \mathscr{S}_{s+2}(\lambda)$ and so on, the right-hand side of the congruence (3.12.2) is expressed by the sum of these $h_\nu(M)$ as

$$(3.12.3)\qquad \left(\chi_\sigma - \sum_{\lambda \in \mathscr{S}_{s+1}(\sigma)} \chi_\lambda\right) + p\left(\sum_{\lambda \in \mathscr{S}_{s+1}(\sigma)} \left(\chi_\lambda - \sum_{\mu \in \mathscr{S}_{s+2}(\lambda)} \chi_\mu\right)\right)$$
$$+ p^2\left(\sum_{\mu \in \mathscr{S}_{s+2}(\lambda)} \left(\chi_\mu - \sum_{\xi \in \mathscr{S}_{s+3}(\mu)} \chi_\xi\right)\right)$$
$$+ \cdots + p^{r-s} \sum_{\tau \in \mathscr{S}_r(\rho)} \chi_\tau \equiv 0 \pmod{2p^{r-s}}.$$

If $\tau = [\boldsymbol{Z}_{p^r}; V]$, then the above congruence (when $\sigma = \tau$) implies that $h_\tau(M) = \chi_\tau \equiv 0 \pmod 2$. We assume that $h_\nu(M) \equiv 0 \pmod{2p^{r-t}}$ for any $\nu = [\boldsymbol{Z}_{p^t}; V]$ $(s < t \le r)$. Then, by induction, it follows from (3.12.3) that $h_\sigma(M) = \chi_\sigma - \sum_{\lambda \in \mathscr{S}_{s+1}(\sigma)} \chi_\lambda \equiv 0 \pmod{2p^{r-s}}$ for $\sigma = [\boldsymbol{Z}_{p^s}; V]$. Therefore the congruences (3.12.1) are obtained. Conversely, let M satisfy (3.12.1), that is $h_\sigma(M) \equiv 0 \pmod{2p^{r-s}}$ for any $\sigma = [\boldsymbol{Z}_{p^s}; V]$. Taking these in the left-hand side of (3.12.3), we have that $\theta_\sigma(M) \equiv 0 \pmod 2$. Thus $\{M\} \in F_{2n}^G$. □

Corollary 3.13. *Let $G = \boldsymbol{Z}_{p^r}$ (p; odd prime). A necessary condition for a class $\{M\} \in F_{2n}^G$ is that the following congruences*

$$\chi(M^{\boldsymbol{Z}_{p^s}}) \equiv \chi(M^{\boldsymbol{Z}_{p^{s+1}}}) \pmod{2p^{r-s}} \ (0 \le s \le r)$$

hold, where $\chi(M^{\boldsymbol{Z}_{p^{r+1}}})$ is regarded as zero.

Example 3.14. Finally we give a non-zero element of F_{2n}^G in case $G = \boldsymbol{Z}_7$. The non-trivial irreducible G-modules are $V_k = \boldsymbol{C}$ with a generator of G acting by multiplication by $\exp(2\pi ik/7)$ $(1 \le k \le 3)$. Let η_j denote the canonical complex line bundle over $\boldsymbol{CP}^j$ and $\eta_{jk} = \eta_j \otimes_{\boldsymbol{C}} V_k$ the G-vector bundle over $\boldsymbol{CP}^j$ given by the tensor product of η_j (with the trivial G-action) and the trivial vector bundle $V_k \times \boldsymbol{CP}^j$. For convenience, we denote $\eta_{0k} = V_k$ and $\eta_{1k} = \underline{V_k}$. Now consider a G-manifold $N = \boldsymbol{CP}(\boldsymbol{C}^s \times (\underline{V_1})^t(\underline{V_2})^t(\underline{V_3})^s)$, the associated complex projective space of a product of G-vector bundles $\nu_N = \boldsymbol{C}^s \times (\underline{V_1})^t(\underline{V_2})^t(\underline{V_3})^s$ over $B_N = * \times (\boldsymbol{CP}^1)^t(\boldsymbol{CP}^1)^t(\boldsymbol{CP}^1)^s = (\boldsymbol{CP}^1)^{2t+s}$ (s, t; odd with $s < t$ and $* = \{pt\}$, the one-point set). We first show that a class $\{N\}$ is a non-zero element in $\mathfrak{N}_{2n}^G$, where $n = 3s + 4t - 1$. For each $\sigma \in St(G)$, a G-vector bundle ν is said to be of type σ if the subset $\{x \in \nu; \sigma_x = \sigma\}$ is precisely its base space B. Let $\mathfrak{N}_*^G[\sigma]$ denote the bundle bordism group of all G-vector bundles of type σ. Given a G-manifold M, the normal bundle ν over the fixed point set F^G is the direct sum of those ν_σ (of type σ) over M_σ, where the sum is taken over all σ with G as an isotropy subgroup (cf. Remark 2.2). Hence there is a well-defined homomorphism $\nu_* : \mathfrak{N}_*^G \to \sum_\sigma \mathfrak{N}_*^G[\sigma]$ given by $\nu_*(\{M\}) = \sum_\sigma \{\nu_\sigma\}$. For our element $\{N\}$, we have $\nu_*(\{N\}) = \sum_{1 \le i \le 4} \{\nu_i\}$, where each ν_i is as follows:

$$(3.14.1)\qquad \begin{aligned} v_1 &= \boldsymbol{CP}^{s-1}\cdot(\underline{V_1})^t(\underline{V_2})^t(\underline{V_3})^s \to B_1\\ &= \boldsymbol{CP}(\boldsymbol{C}^s\times\{0\}\{0\}\{0\}) = \boldsymbol{CP}^{s-1}\cdot(\boldsymbol{CP}^1)^{2t+s},\\ v_2 &= \boldsymbol{CP}((\underline{V_1})^t)\cdot(V_1)^s(\underline{V_1})^t(\underline{V_2})^s \to B_2\\ &= \boldsymbol{CP}(\{0\}\times(\underline{V_1})^t\{0\}\{0\}) = \boldsymbol{CP}((\underline{V_1})^t)\cdot *(\boldsymbol{CP}^1)^{t+s},\\ v_3 &= \boldsymbol{CP}((\underline{V_2})^t)\cdot(V_2)^s(\underline{V_1})^{t+s} \to B_3\\ &= \boldsymbol{CP}(\{0\}\times\{0\}(\underline{V_2})^t\{0\}) = \boldsymbol{CP}((\underline{V_2})^t)\cdot *(\boldsymbol{CP}^1)^{t+s},\\ v_4 &= \boldsymbol{CP}((\underline{V_3})^s)\cdot(V_3)^s(\underline{V_2})^t(\underline{V_1})^t \to B_4\\ &= \boldsymbol{CP}(\{0\}\times\{0\}\{0\}(\underline{V_3})^s) = \boldsymbol{CP}((\underline{V_3})^s)\cdot *(\boldsymbol{CP}^1)^{2t}. \end{aligned}$$

Let $\sigma = [G; V_1^t V_2^t V_3^s]$, then it is known that $\mathfrak{N}_*^G[\sigma]$ is a free $\mathfrak{N}_*$-module generated by the classes of monomials

$$\eta_{JKL} = \eta_{j(1)1}\cdots\eta_{j(t)1}\eta_{k(1)2}\cdots\eta_{k(t)2}\eta_{l(1)3}\cdots\eta_{l(s)3}$$

with $j(1)\geq\cdots\geq j(t)\geq 0$, $k(1)\geq\cdots\geq k(t)\geq 0$ and $l(1)\geq\cdots\geq l(s)\geq 0$ (cf. [7; Lemma 3.4.4 and Theorem 4.1.1]). Let $\dim(\eta_{JKL}) = s+2t+\sum j(p)+\sum k(q)+\sum l(r)$ be the complex dimension of the total space. Now go back to the image $v_*(\{N\})$. It follows from (3.14.1) that $N_\sigma = B_1+B_4$ and $v_\sigma = v_1+v_4$. From the condition that s and t are odd with $s<t$, the monomial $(\underline{V_1})^t(\underline{V_2})^t(\underline{V_3})^s$ in v_1 has the dimension $2s+4t$, which is higher than that of the monomial in v_4, and its coefficient $\{\boldsymbol{CP}^{s-1}\} = \{(\boldsymbol{RP}^{(s-1)/2})^2\}\neq 0$ in $\mathfrak{N}_*$ (cf. [8; Lemma 7]). This ensure that $\{v_\sigma\}\neq 0$ in $\mathfrak{N}_*^G[\sigma]$ and $\{N\}\neq 0$ in $\mathfrak{N}_*^G$. Next we study an SK class $[N]$. By definition, N is fibered equivariantly over the first $\boldsymbol{CP}^1$ of the base space $B_N = (\boldsymbol{CP}^1)^{2t+s}$ with fiber $F = \boldsymbol{CP}(\boldsymbol{C}^s\times V_1(\underline{V_1})^{t-1}(\underline{V_2})^t(\underline{V_3})^s)$. Hence $N\sim\boldsymbol{CP}^1\times F$ (cf. [7; Theorem 2.4.1 (iv)]). Continuing this SK processes on F inductively, we have

$$(3.14.2)\qquad N\sim(\boldsymbol{CP}^1)^{2t+s}\times M,$$

where $M = \boldsymbol{CP}(\boldsymbol{C}^s\times V_1^t V_2^t V_3^s)$. Now we apply the equality (1.8.1) for M. Note that $\sigma_{(3)} = \sigma$ and $\sigma_{(1)} = \sigma_{(2)} = [G; V_1^{s+t}V_2^s]$. Then we have that

$$[N] = [(\boldsymbol{CP}^1)^{2t+s}](2s\alpha^{s-1}y[\sigma] + 2t\alpha^{t-1}y[\sigma_{(1)}])$$

in SK_{2n}^G. Hence $[N]$ vanishes in $\overline{SK}_{2n}^G$ and $\{N\}\in F_{2n}^G$ by Lemma 3.4. The slice types of N are σ_0,σ and $\sigma_{(1)}$, and $N_{\sigma_0} = N$, $N_\sigma = B_1+B_4$ and $N_{\sigma_{(1)}} = B_2+B_3$ by (3.14.1). Thus $\chi(N) = 2^{s+2t+1}(s+t)$, $\chi(N_\sigma) = \chi(B_1)+\chi(B_4) = 2^{s+2t+1}s$ and $\chi(N_{\sigma_{(1)}}) = \chi(B_2)+\chi(B_3) = 2^{s+2t+1}t$. These imply that $\chi(N) = \chi(N_\sigma)+\chi(N_{\sigma_{(1)}}) = \chi(N^G)$, $\chi(N_\sigma)\equiv 0 \pmod 2$ and $\chi(N_{\sigma_{(1)}})\equiv 0 \pmod 2$, from which the congruences (3.12.1) are obviously satisfied.

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