The étale cohomology of the general linear group over a finite field and the Dickson algebra

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To Professor Masaharu Kaneda on the occasion of his 60th birthday

Abstract Let $p \neq \ell$ be primes. We study the étale cohomology $H^*_{\text{ét}}(\mathrm{BGL}_n(\mathbb{F}_{p^s}); \mathbb{Z}/\ell)$ by using the stratification methods from Molina-Rojas and Vistoli. To compute this cohomology, we use the Dickson algebra and the Drinfeld space.

1. Introduction

Let p and ℓ be primes with $p \neq \ell$. Let X be a smooth algebraic variety over $k = \bar{\mathbb{F}}_p$, and let $H^*_{\text{ét}}(X; \mathbb{Z}/\ell)$ be the étale cohomology of X over k. By Totaro [16], [17] and Voevodsky [20], [19], it is known that the cohomology of the classifying space BG of any algebraic group G can be approximated by smooth (quasiprojective) algebraic varieties X_i. Moreover, if G is finite, then $BG \times B\mathbb{G}_m$ can be approximated by smooth projective varieties. Hence, we can consider (see [16], [17])

$$H^*_{\text{ét}}(BG; \mathbb{Z}/\ell) = \lim_i H^*_{\text{ét}}(X_i; \mathbb{Z}/\ell).$$

Let $G_n = \mathrm{GL}_n(\mathbb{F}_q)$ be the general linear group over a finite field $\mathbb{F}_q$ with $q = p^s$. Our main computation is the following.

THEOREM 1.1

Let $\ell \neq 2$. Let r be the smallest number such that $q^r - 1 = 0 \mod \ell$. Then we have an isomorphism of graded rings

$$H^*_{\acute{e}t}(BG_n : \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_r, \dots, c_{r[n/r]}] \otimes \Lambda(e_r, \dots, e_{r[n/r]}), \tag{1.1}$$

where $\deg(c_{rj}) = 2rj$, $\deg(e_{rj}) = 2rj - 1$, *and* $\Lambda(e_r, \dots, e_{r[n/r]})$ *is the exterior algebra generated by* $e_r, \dots, e_{r[n/r]}$.

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When $\ell = 2$, we get a similar isomorphism of $\mathbb{Z}/\ell[c_r, \ldots, c_{r[n/r]}]$-modules (see the remark after Lemma 4.1 below).

By the comparison theorem (for the base change of $k = \bar{\mathbb{F}}_p$ and $\mathbb{C}$), this is just a corollary of the famous result of the topological mod ℓ cohomology $H^*(BG_n; \mathbb{Z}/\ell)$ by Quillen [14]. However, Quillen used topological arguments, for example, the Eilenberg–Moore spectral sequences and the homotopy fiber of the map $\psi^q - 1$ defined by the Adams operation. On the other hand, our proof of Theorem 1.1 is algebraic. Kroll [6] also gave a short algebraic proof of Quillen's result by using ordinary cohomology. But our proof uses the étale cohomology essentially over a field k with $\mathrm{char}(k) > 0$.

The arguments for the proof also work for the motivic cohomology. Let $H^{*,*'}(-; \mathbb{Z}/\ell)$ be the motivic cohomology over the field $\bar{\mathbb{F}}_p$, and let $0 \neq \tau \in H^{0,1}(\mathrm{Spec}(\bar{\mathbb{F}}_p); \mathbb{Z}/\ell)$.

THEOREM 1.2
Let $\ell \neq 2$. Then we have an isomorphism of graded rings

$$H^{*,*'}(BG_n; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\tau] \otimes (1.1)$$

with degree $\deg(c_{rj}) = (2rj, rj)$ *and* $\deg(e_{rj}) = (2rj - 1, rj)$.

By induction on n and the equivariant cohomology theory (stratified methods) from Molina-Rojas and Vistoli [9] and Vistoli [18], we get the above theorems. To compute the equivariant cohomology, we consider the G_n-variety

$$Q = \mathrm{Spec}\big(k[x_1, \ldots, x_n]/\big(\det(x_i^{q^{j-1}})^{q-1} = 1\big)\big)$$

and prove that $Q/G_n \cong \mathbb{A}^{n-1}$ by using the Dickson algebra. This implies the isomorphism of the equivariant (étale) cohomology rings

$$H^*_{G_n}(Q \times_{\mu_{q^n-1}} \mathbb{G}_m; \mathbb{Z}/\ell) \cong \Lambda(f), \quad \deg(f) = 1.$$

The computation of the above isomorphism is the crucial point to compute $H^*_{G_n}(pt.; \mathbb{Z}/\ell) \cong H^*_{\text{ét}}(BG_n; \mathbb{Z}/\ell)$.

The above space Q is a very particular case (first studied by Drinfeld) of the variety $\tilde{X}(\dot{w})$ defined by Deligne and Lusztig [3]. Moreover, Lusztig [7, Theorem 0.4(b)] and He and Lusztig [4, Section 4.3] proved recently that $G^F \setminus \tilde{X}(\dot{w})$ is quasi-isomorphic to the standard affine space for general G with minimal length elements w. We give here a different proof for the specific case.

The plan of this article is the following. In Section 2, we recall the Dickson algebra and show the isomorphism $Q/G_n \cong \mathbb{A}^{n-1}$ in Theorem 2.4. In Section 3, we note properties of the Chern class c_i. In Section 4, using induction and the stratification methods, we compute $H^*_{\text{ét}}(BG_n; \mathbb{Z}/\ell)$. We use Theorem 2.4 in the first step of the induction and use Proposition 3.2 in Section 3 to show the $(k+1)$st step from the kth step for the induction. Section 5 is about the special linear group SL_n, and Section 6 is a very short explanation for the motivic cohomology. In the last section we add a brief note that the quasi-isomorphism

$G_n \setminus \tilde{X}(\dot{w}) \to \mathbb{A}^{n-1}$ can be represented by the Dickson elements $c_{n,i}$ given in Section 2.

2. Dickson invariants

Throughout this article, we assume that p, ℓ are primes with $p \neq \ell$ and $q = p^s$ for some s. In this section, we define an algebraic space Q, on which $G_n = \mathrm{GL}_n(\mathbb{F}_q)$ acts with $Q/G_n \cong \mathbb{A}^{n-1}$. Here we use the fact that $Q/G_n \cong \mathrm{Spec}(A^{G_n})$ for some ring A. For the study of the invariant ring A^{G_n}, we recall the Dickson algebra (see [1], [5], [10]).

The Dickson algebra is the invariant ring of a polynomial of n variables under the usual G_n-action; namely,

$$\mathbb{F}_q[x_1, \dots, x_n]^{G_n} = \mathbb{F}_q[c_{n,0}, c_{n,1}, \dots, c_{n,n-1}],$$

where each $c_{n,i}$ is defined by

$$\sum c_{n,i} X^{q^i} = \prod_{x \in \mathbb{F}_q\{x_1,\dots,x_n\}} (X + x) = \prod_{(\lambda_1,\dots,\lambda_n)\in(\mathbb{F}_q)^n} (X + \lambda_1 x_1 + \cdots + \lambda_n x_n),$$

where $\mathbb{F}_q\{x_1, \dots, x_n\}$ is the n-dimensional $\mathbb{F}_q$-vector space generated by $x_1, \dots,$ x_n. Let us write by $|c_{n,i}|$ the degree of $c_{n,i}$ so that $|c_{n,i}| = q^n - q^i$, letting the degree $|x_i| = 1$. Let us write $e_n = c_{n,0}^{1/(q-1)}$; namely,

$$e_n = \Big(\prod_{0 \neq x \in \mathbb{F}_q\{x_1,\dots,x_n\}} x \Big)^{1/(q-1)} = \begin{vmatrix} x_1 & x_1^q & \cdots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \cdots & x_2^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^q & \cdots & x_n^{q^{n-1}} \end{vmatrix}.$$

Then each $c_{n,i}$ is written as

$$c_{n,i} = \begin{vmatrix} x_1 & \cdots & \hat{x}_1^{q^i} & \cdots & x_1^{q^n} \\ x_2 & \cdots & \hat{x}_2^{q^i} & \cdots & x_2^{q^n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n & \cdots & \hat{x}_n^{q^i} & \cdots & x_n^{q^n} \end{vmatrix} \Big/ e_n.$$

Note that the Dickson algebra for $SG_n = \mathrm{SL}_n(\mathbb{F}_q)$ is given as

$$\mathbb{F}_q[x_1, \dots, x_n]^{SG_n} = \mathbb{F}_q[e_n, c_{n,1}, \dots, c_{n,n-1}].$$

Let us write $k = \bar{\mathbb{F}}_p$. We consider the algebraic variety

$$F = \mathrm{Spec}\big(k[x_1, \dots, x_n]/(e_n)\big).$$

We want to study the G_n-space structure of $X(n) = \mathbb{A}^n - \{0\}$ and $X(1) = \mathbb{A}^n - F$. (Note that $F = \{0\}$ when $n = 1$.) For this, we consider the following variety:

$$Q = \mathrm{Spec}\big(k[x_1, \dots, x_n]/(c_{n,0} - 1)\big) = \mathrm{Spec}\big(k[x_1, \dots, x_n]/(e_n^{q-1} - 1)\big).$$

EXAMPLE

When $n = 2$, we see that

$$Q = \{(x, y) \in \mathbb{A}^2 \mid (xy^q - x^q y)^{q-1} = 1\},$$
$$F = \{(x, y) \in \mathbb{A}^2 \mid xy^q - x^q y = 0\}$$
$$= \Big\{(x, y) \in \mathbb{A}^2 \,\Big|\, x \prod_{i \in \mathbb{F}_q} (y - ix) = 0\Big\} \cong \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} F_i,$$

where the F_i's are the rational hyperplanes defined by $F_i = \{(x, ix) \in \mathbb{A}^2 \mid x \in k\}$ and $F_\infty = \{(0, y) \mid y \in k\}$.

The corresponding projective variety $\bar{Q}$ is written by

$$\bar{Q} = \mathrm{Proj}\big(k[x_0, \ldots, x_n]/(c_{n,0} = x_0^{q^n - 1})\big).$$

LEMMA 2.1

We have an isomorphism of G_n-varieties

$$Q \times_{\mu_{q^n-1}} \mathbb{G}_m \cong X(1) = \mathbb{A}^n - F.$$

Proof

We consider the map $p : Q \times \mathbb{G}_m \to X(1)$ by $(x, t) \mapsto tx$. In fact, we have

$$e_n\big(p(x, t)\big)^{q-1} = e_n(tx_1, \ldots, tx_n)^{q-1} = (t^{1+q+\cdots+q^{n-1}})^{q-1} e_n(x_1, \ldots, x_n)^{q-1}$$
$$= t^{q^n - 1} e_n(x_1, \ldots, x_n)^{q-1}.$$

Since $e_n(x)^{q-1} = 1$ for $x \in Q$, we see that $e_n(p(x, t)) \neq 0$ and $p(x, t) \in X(1)$. Let $y \in X(1)$. Then for $x = y/t$ and $t = e_n(y)^{(q-1)/(q^n-1)}$, we get that $x \in Q$ and $p(x, t) = y$. Elements in the fiber $p^{-1}(y)$ are represented as $(ax, a^{-1}t)$ in $Q \times \mathbb{G}_m$ for $a \in \mu_{q^n-1}$ since $ax \in Q$. Thus, we have the lemma. $\square$

LEMMA 2.2

We have $Q(\mathbb{F}_{q^i}) = \emptyset$ for $1 \leq i \leq n - 1$.

Proof

Let $x = (x_1, \ldots, x_n)$ be an $\mathbb{F}_{q^i}$-rational point. Then $x_j^{q^i} = x_j$ for all $1 \leq j \leq n$. Hence, $e_n(x) = 0$. $\square$

LEMMA 2.3

Stabilizer groups of the G_n-action on Q are all $\{1\}$.

Proof

Assume that there is $1 \neq g \in G_n$ such that $gx = x$ for $x \in Q \subset \mathbb{A}^n$. Then we can identify that x is an eigenvector for the (linear) action g with the eigenvalue 1. If $x = (x_1, \ldots, x_n)$ is an eigenvector of g for the eigenvalue 1, then so are $F(x) = (x_1^q, \ldots, x_n^q), F^2(x) = (x_1^{q^2}, \ldots, x^{q^2}), \ldots, F^{n-1}(x) = (x_1^{q^{n-1}}, \ldots, x_n^{q^{n-1}})$.

The property $e_n(x) \neq 0$ now ensures that $\{x, F(x), \ldots, F^{n-1}(x)\}$ is a base of $\mathbb{A}^n$, which proves that $g = 1$. This is a contradiction. $\square$

Since G_n is a finite group, the quotient Q/G_n becomes an algebraic variety. The geometric invariant theory quotient $Q//G_n$ (see explanations in [11, Section 5.1] for $\mathrm{char}(k) = 0$, [12], [13, Theorem 1, p. 111]) is defined by

$$Q//G_n = \mathrm{Spec}(A^{G_n}), \quad A = k[x_1, \ldots, x_n]/(c_{n,0} - 1).$$

THEOREM 2.4

We have a ring isomorphism $A^{G_n} \cong k[c_{n,1}, \ldots, c_{n,n-1}]$ *inducing an isomorphism* $Q/G_n \cong \mathbb{A}^{n-1}$ *of varieties. That is,*

$$\bigl(k[x_1, \ldots, x_n]/(c_{n,0} - 1)\bigr)^{G_n} \cong k[x_1, \ldots, x_n]^{G_n}/(c_{n,0} - 1).$$

Proof

We already know that $k[x_1, \ldots, x_n]^{G_n} \cong k[c_{n,0}, \ldots, c_{n,n-1}]$. Hence, it is immediate that

$$B = k[c_{n,1}, \ldots, c_{n,n-1}] \subset A^{G_n} = \bigl(k[x_1, \ldots, x_n]/(c_{n,0} - 1)\bigr)^{G_n} \subset A.$$

The projective coordinate ring $\bar{A}$ of the Zariski closure $\bar{Q}$ of Q in $\mathbb{P}^n$ is given as

$$\bar{A} = k[x_0, \ldots, x_n]/(c_{n,0} = x_0^{q^n - 1}).$$

The coordinate ring $\bar{B}$ of the closure of $\mathrm{Spec}(B) \cong \mathrm{Spec}(k[c_{n,1}, \ldots, c_{n,n-1}])$ is given as $\bar{B} = k[x_0, c_{n,1}, \ldots, c_{n,n-1}]$. Here note that $\bar{A}$ and $\bar{B}$ become graded k-algebras (projective coordinate rings have natural graded ring structures), while A does not; in fact, $c_{n,0} = 1 \in A$.

For a graded (commutative) k-algebra $R = \bigoplus_{i=0}^{\infty} R^i$, recall that the Hilbert–Poincaré series is the formal power series defined by (see, e.g., [11, Section 1.2(a)], [1])

$$PS(R) = \sum_{i=0}^{\infty} \dim_k(R^i) t^i \in \mathbb{Z}[\![t]\!].$$

Since $\bar{A}$ is generated by $n + 1$ generators of degree 1 and one relation of degree $q^n - 1$, we have

$$PS(\bar{A}) = \frac{(1 - t^{q^n - 1})}{(1 - t)^{n+1}} = \frac{(1 + t + \cdots + t^{q^n - 2})}{(1 - t)^n}.$$

The graded ring $\bar{B}$ is generated by x_0 of degree 1 and $c_{n,i}$ for $i \geq 1$. So we get

$$\begin{aligned} PS(\bar{B}) &= \frac{1}{(1 - t)(1 - t^{|c_{n,1}|}) \cdots (1 - t^{|c_{n,n-1}|})} \\ &= \frac{1}{(1 + t + \cdots + t^{|c_{n,1}| - 1}) \cdots (1 + t + \cdots + t^{|c_{n,n-1}| - 1})(1 - t)^n}. \end{aligned}$$

Hence, $PS(\bar{A})/PS(\bar{B})$ is written as

$$(1+t+\cdots+t^{|c_{n,1}|-1})\cdots(1+t+\cdots+t^{|c_{n,n-1}|-1})(1+t+\cdots+t^{q^n-2}).$$

Thus, we obtain (let $\dim_k(f(t))=\sum_i a_i$ for $f(t)=\sum_i a_i t^i$)

$$\begin{aligned}\dim_k\bigl(PS(\bar{A})/PS(\bar{B})\bigr) &= |c_{n,1}|\times\cdots\times|c_{n,n-1}|\times(q^n-1)\\ &= (q^n-q^1)\cdots(q^n-q^{n-1})(q^n-1)=|G_n|.\end{aligned}$$

On the other hand, $c_{n,1},\dots,c_{n,n-1}$ is a regular sequence in $\bar{A}$. (It is well known that $c_{n,0},\dots,c_{n,n-1}$ is a regular sequence in $k[x_1,\dots,x_n]$. This fact is proved by induction on n by using $c_{n,i}=c_{n-1,i-1}^q \bmod x_n$ and $c_{n,0}=\prod_{0\neq x\in\mathbb{A}^n}x$.) Hence, $\bar{A}$ is $\bar{B}$-free; that is, there are $y_1,\dots,y_m$ in $k[x_0,\dots,x_n]$ such that

$$\bar{A}\cong\bar{B}\{y_1,\dots,y_m\}.$$

Then $PS(\bar{A})=PS(\bar{B})\cdot(\sum_{i=1}^m t^{\deg(y_i)})$. Hence, $m=|G_n|$ from the results using the Hilbert–Poincaré series above. We can represent each element in A,B by an element in $\bar{A},\bar{B}$ letting $x_0=1$. Hence, we have

$$\operatorname{rank}_B(A)\le\operatorname{rank}_{\bar{B}}(\bar{A})=|G_n|.$$

Let $\pi:Q\to Q/G_n$ be the projection. Recall Lemma 2.3, and we see that $\pi^{-1}(y)$ is locally flat for each $y\in Q/G_n$. Since the map π is étale, for all $x\in Q$, the local ring O_x is $O_{\pi(x)}$-free, and $\operatorname{rank}_{O_{\pi(x)}}(O_x)=|G_n|$ (see [8]), namely, $\operatorname{rank}_{A^{G_n}}(A)=|G_n|$. Thus, for the inclusions $B\subset A^{G_n}\subset A$, we have $\operatorname{rank}_B(A)=\operatorname{rank}_{A^{G_n}}(A)$. Hence,

$$A^{G_n}=B\cong k[c_{n,1},\dots,c_{n,n-1}].\qquad\square$$

Similarly, we can prove the following for $SG_n=\mathrm{SL}_n(\mathbb{F}_q)$.

COROLLARY 2.5

Let $SA=k[x_1,\dots,x_n]/(e_n-1)$, and let $SQ=\operatorname{Spec}(SA)$. Then all stabilizer groups of the SG_n-action on SQ are $\{1\}$, and we have an isomorphism

$$(SA)^{SG_n}\cong k[c_{n,1},\dots,c_{n,n-1}],\quad \textit{that is, } SQ/SG_n\cong\mathbb{A}^{n-1}.$$

REMARK

Let G be an algebraic group, and let w be a Coxeter element. The space Q is related to a very particular case of the variety $\tilde{X}(\dot{w})$ (associated to G and w) defined by Deligne and Lusztig [3]. Recently, He and Lusztig [4, Section 4.3] and Lusztig [7, Theorem 0.4(b)] showed that $G^F\setminus\tilde{X}(\dot{w})$ is quasi-isomorphic to the standard affine space for G of general type with a minimal length element w.

The referee pointed out the following facts.

REMARK

The above $\tilde{X}(\dot{w})$ is defined in [3] as

$$\tilde{X}(\dot{w}) \cong \{g \in G \mid g^{-1}F(g) \in U\dot{w}U\}/(U),$$

where U is the maximal unipotent group. Let $Y(\dot{w}) = \{g \in G \mid g^{-1}F(g) \in U\dot{w}U\}$. Then the Lang map induces an isomorphism $G^F \setminus Y(\dot{w})$ to the affine space $U\dot{w}U$ by $g \mapsto g^{-1}Fg$. Hence, $H^*_{\text{ét}}(G^F \setminus Y(\dot{w}); \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$. Moreover, the projection $G \to G/U$ induces a G^F-equivariant (surjective) morphism $Y(\dot{w}) \to \tilde{X}(\dot{w})$ whose fiber is isomorphic to the affine space U. Hence, we have the spectral sequence

$$E_2^{*,*} \cong H^*_{\text{ét}}\big(G^F \setminus \tilde{X}(\dot{w}); H^*_{\text{ét}}(U;\mathbb{Z}/\ell)\big) \Longrightarrow H^*_{\text{ét}}\big(G^F \setminus Y(\dot{w}); \mathbb{Z}/\ell\big),$$

which collapses. This shows that

$$H^*_{\text{ét}}\big(G^F \setminus \tilde{X}(\dot{w}); \mathbb{Z}/\ell\big) \cong H^*_{\text{ét}}\big(G^F \setminus Y(\dot{w}); \mathbb{Z}/\ell\big) \cong \mathbb{Z}/\ell$$

for any G and w. For the proof of the main theorem in Section 4 (Lemma 4.1), only this fact is enough (instead of Theorem 2.4).

3. Chern classes and maximal torus

In this section, we prove that the polynomial ring $\mathbb{Z}/\ell[c_r, \dots, c_{[n/r]r}]$ generated by Chern classes c_{ri} is contained in $H^*_{\text{ét}}(BG_n; \mathbb{Z}/\ell)$.

For a smooth algebraic variety X over $k = \bar{\mathbb{F}}_p$, we consider the mod ℓ étale cohomology for $\ell \neq p$. Let G be a linear algebraic group (e.g., finite group). Let $W \cong \mathbb{A}^M$ for some (large) M and $\rho: G \to \mathrm{GL}(W)$ a faithful representation. For $N < M$, let $V_N = W - S$ be an open set of W such that G acts freely on V_N with $\mathrm{codim}_W S > N$. Then it is known (see [19], [16], [17]) that the cohomology $H^*_{\text{ét}}(V_N/G; \mathbb{Z}/\ell)$ does not depend on W and V_N for $* < N$. Moreover, given N, we can always take such W and V_N (see [16, Section 1] for details). In this article, we simply write

$$H^*(BG) = \lim_N H^*_{\text{ét}}(V_N/G; \mathbb{Z}/\ell).$$

REMARK

An action of an algebraic group G on an algebraic variety X is called *free* if the induced map $\mu: G \times X \to X \times X$ is a closed embedding (see [2], [10, Chapter 0, Section 3]). If each stabilizer group $G_x \cong \{1\}$ for $x \in X$ and μ is proper, then the action is free.

Let T be a maximal torus of the algebraic group GL_n. Then the restriction map

$$H^*(\mathrm{BGL}_n) \to H^*(BT) \cong \mathbb{Z}/\ell[t_1, \dots, t_n], \quad \deg(t_i) = 2,$$

is injective and induces an isomorphism $H^*(\mathrm{BGL}_n) \cong \mathbb{Z}/\ell[t_1, \dots, t_n]^{S_n}$ mapping the Chern class c_i to the elementary symmetric function of degree i in the t_j's. Hence, we have an isomorphism (see [16], [17])

$$H^*(\mathrm{BGL}_n) \cong \mathbb{Z}/\ell[c_1, \dots, c_n].$$

The Frobenius map F acts on this cohomology by $c_i \mapsto q^i c_i$. Recall that the Lang map induces a principal G_n-bundle $G_n \to \mathrm{GL}_n \xrightarrow{L} \mathrm{GL}_n$, where $L(g) = g^{-1}F(g)$. Hence, it induces the map of classifying spaces

$$BG_n \to \mathrm{BGL}_n \xrightarrow{BL} \mathrm{BGL}_n.$$

Let r be the smallest number such that $q^r - 1 = 0 \mod \ell$. Then we have maps of graded rings

$$\mathbb{Z}/\ell[c_r, \dots, c_{[n/r]r}] \to H^*(\mathrm{BGL}_n)/\big((q^i-1)c_i\big) \to H^*(BG_n).$$

For each element $w \in S_n$, let us write $T(w)$ for the diagonal torus $T \subset \mathrm{GL}_n$ endowed with the Frobenius map $\mathrm{ad}(w)F$. For example, when $n = r$ and $w = (1, 2, \dots, r) \in S_r$, we see that, for a matrix $A = (a_{i,j}) \in \mathrm{GL}_r$, the adjoint action is given as

$$\mathrm{ad}(w)F(A) = wFw^{-1}(a_{i,j}) = (b_{i,j}) \quad \text{with } b_{i,j} = a^q_{i-1,j-1}, i, j \in \mathbb{Z}/n.$$

Hence, we have

$$\begin{aligned} T(w)^F &= \big\{t \in T \bigm| \mathrm{ad}(w)F(t) = t\big\} \\ &\cong \big\{\mathrm{diag}(x, x^q, \dots, x^{q^{r-1}}) \in T \bigm| x \in \mathbb{F}^*_{q^r}\big\} \cong \mathbb{F}^*_{q^r}. \end{aligned}$$

Write $H^*(BT) \cong \mathbb{Z}/\ell[t_1, \dots, t_r]$. Let $i : T(w)^F \subset T$. Then we can take the ring generator $t \in H^2(BT(w)^F)$ such that $i^*t_i = q^{i-1}t$.

LEMMA 3.1

The following composition map is injective:

$$\mathbb{Z}/\ell[c_r] \to H^*(\mathrm{BGL}_r)/\big((q^i-1)c_i\big) \to H^*(BG_r).$$

Proof

Let $w = (1, \dots, r)$. We consider the induced map

$$i^* : H^*(\mathrm{BGL}_r)^F \to H^*(BG_r) \to H^*\big(BT(w)^F\big) \cong H^*(\mathbb{F}^*_{q^r}).$$

Let s_i be the ith elementary symmetric function over $t_1, \dots, t_r$; that is,

$$(X - t_1)(X - t_2) \cdots (X - t_r) = X^r - s_1 X^{r-1} + \cdots + (-1)^r s_r.$$

Since $i^*(t_i) = q^{i-1}t$, we see that

$$(X - t)(X - qt) \cdots (X - q^{r-1}t) = X^n - i^*(s_1)X^{r-1} + \cdots + (-1)^r i_*(s_r).$$

On the other hand, the polynomial $X^r - t^r$ has roots $X = t, qt, \dots, q^{r-1}t$. Hence, we see that the above formula is $X^r - t^r$. Thus, we see that

$$i^*(s_1) = \cdots = i^*(s_{r-1}) = 0, \qquad t^r = (-1)^r i^*(s_r).$$

Since the Chern class c_i is represented by the symmetric function s_i in $H^*(BT)$, it implies the assertion above. $\square$

PROPOSITION 3.2

The following composition map is injective:

$$\mathbb{Z}/\ell[c_r,\dots,c_{[n/r]r}] \cong H^*(\mathrm{BGL}_n)^F \to H^*(BG_n).$$

Proof

Let $k=[n/r]$, and let us take

$$w=(1,\dots,r)(r+1,\dots,2r)\cdots\big((k-1)r+1,\dots,kr\big)\in S_n.$$

Then we see that $T(w)^F$ is isomorphic to

$$\big\{\mathrm{diag}(x_1,\dots,x_1^{q^{r-1}},\dots,x_k,\dots,x_k^{q^{r-1}})\in T \;\big|\; (x_1,\dots,x_k)\in(\mathbb{F}_{q^r}^*)^k\big\}\cong(\mathbb{F}_{q^r}^*)^k.$$

We consider the map

$$i^*: H^*(\mathrm{BGL}_n)^F \to H^*(BG_n)\to H^*\big(BT(w)^F\big)\cong H^*\big(B\big((\mathbb{F}_{q^r}^*)^k\big)\big).$$

We choose $t_i\in H^2(BT)$ $(1\le i\le n)$ and $t'_j\in H^2(BT(w)^F)$ $(1\le j\le k)$ such that

$$i^*(t_1)=t'_1,\qquad i^*(t_2)=qt'_1,\qquad \dots,$$
$$i^*(t_{r+1})=t'_2,\qquad i^*(t_{r+2})=qt'_2,\qquad \dots.$$

Then by arguments similar to those in the proof of Lemma 3.1, we have

$$X^n-i^*(c_1)X^{n-1}+\cdots+(-1)^n_*(c_n)=\big(X^r-(t'_1)^r\big)\cdots\big(X^r-(t'_k)^r\big).$$

Then we get the result as Lemma 3.1. □

4. Equivariant cohomology

In this section, using induction and the stratification methods, we compute the cohomology $H^*(BG_n)$. Recall that r is the smallest number with $q^r-1=0$ mod ℓ. We first prove the main result when $r=1$ and next show the general case.

Let X be a smooth G-variety. Recall that $V_N=\mathbb{A}^M-S$ is a G-free space with $\mathrm{codim}_{\mathbb{A}^M} S>N$ as defined in Section 3. Then we can define the equivariant cohomology (see [18], [9])

$$H_G^*(X)=\lim_N H^*_{\text{ét}}(V_N\times_G X;\mathbb{Z}/\ell).$$

In particular, $H_G^*(pt.)\cong H^*(BG)=H^*_{\text{ét}}(BG;\mathbb{Z}/\ell)$. If all stabilizer groups of a G-action on X are $\{1\}$, then we can see that $H_G^*(X)\cong H^*(X/G)$.

We recall the following localized exact sequence, which we shall use intensively throughout the proofs. Let $i:Y\subset X$ be a regular closed inclusion of G-varieties of $\mathrm{codim}_X(Y)=c$, and let $j:U=X-Y\subset X$. Then there is a long exact sequence

$$\to H_G^{*-2c}(Y)\xrightarrow{i_*}H_G^*(X)\xrightarrow{j^*}H_G^*(U)\xrightarrow{\delta}H_G^{*-2c+1}(Y)\to\cdots.$$

Now we apply the above exact sequence for concrete cases. We consider the case $G = G_n = \mathrm{GL}_n(\mathbb{F}_q)$. Recall that

$$F = \mathrm{Spec}\big(k[x_1, \dots, x_n]/(e_n^{q-1})\big) = \bigcup_{0 \neq \lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{F}_q)^n} (F_\lambda),$$

where $F_\lambda = \{(x_1, \dots, x_n) \mid \lambda_1 x_1 + \dots + \lambda_n x_n = 0\} \subset \mathbb{A}^n$.

Let $F(1) = F$, and let $F(2)$ be the $(\mathrm{codim} = 1)$ set of singular points in $F(1)$, namely, $F(2) = \bigcup F_{\lambda,\mu}$ with

$$F_{\lambda,\mu} = \begin{cases} F_\lambda \cap F_\mu & \text{if } F_\lambda \neq F_\mu, \\ \emptyset & \text{if } F_\lambda = F_\mu. \end{cases}$$

Similarly, we define (the union of codimension i k-linear spaces)

$$F(i) = \bigcup_{(\alpha^1, \dots, \alpha^i)} (F_{\alpha^1} \cap \dots \cap F_{\alpha^i}),$$

where α^j ranges over $\alpha^j \in (\mathbb{F}_q)^n$, $1 \le j \le i$, and $\dim_k(F_{\alpha^1} \cap \dots \cap F_{\alpha^i}) = n - i$. Let us write $X(i) = \mathbb{A}^n - F(i)$. Thus, we have two sequences of the G_n-algebraic sets

$$F(1) \supset F(2) \supset \dots \supset F(n) = \{0\} \supset F(n+1) = \emptyset,$$

$$X(1) = \mathbb{A}^n - F(1) \subset \dots \subset X(n) = \mathbb{A}^n - \{0\} \subset X(n+1) = \mathbb{A}^n.$$

Let us write $F(i) - F(i+1)$ by $E(i)$. Note that the embeddings

$$Y = E(i) \subset X = X(i+1) \supset U = X(i)$$

are smooth and satisfy the condition above for Y, X, U. Therefore, we have the long exact sequences for all $1 \le i \le n$

$$\to H_{G_n}^{*-2i}\big(E(i)\big) \xrightarrow{i_*} H_{G_n}^{*}\big(X(i+1)\big) \xrightarrow{j^*} H_{G_n}^{*}\big(X(i)\big) \xrightarrow{\delta} \cdots.$$

From now on, we assume $\ell \neq 2$. (However, similar facts also hold for $\ell = 2$ (see the remark below).)

LEMMA 4.1

We have an isomorphism of graded rings

$$H_{G_n}^{*}\big(X(1)\big) \cong \Lambda(f) \quad \textit{with } \deg(f) = 1.$$

Proof

At first, we recall $H^*(\mathbb{G}_m) \cong \Lambda(f)$ with $\deg(f) = 1$, which is proved by the exact sequence (using $i_* = 0$)

$$\to H^{*-2}\big(\{0\}\big) \xrightarrow{i_*} H^*(\mathbb{A}^1) \to H^*(\mathbb{G}_m) \to \cdots.$$

Consider the map taking $t \in \mathbb{G}_m$ to $t^{q^n-1} \in \mathbb{G}_m$. It is a surjective map which induces an isomorphism $\mathbb{G}_m/\mu_{q^n-1} \cong \mathbb{G}_m$. Therefore, with μ_{q^n-1} acting freely, we have

$$H_{\mu_{q^n-1}}(\mathbb{G}_m) \cong H^*(\mathbb{G}_m/\mu_{q^n-1}) \cong H^*(\mathbb{G}_m) \cong \Lambda(f).$$

From Lemma 2.1, we have $X(1) \cong Q \times_{\mu_{q^n-1}} \mathbb{G}_m$. Then we get the equivariant cohomology from Lemma 2.3 and Theorem 2.4:

$$\begin{aligned} H^*_{G_n}\big(X(1)\big) &\cong H^*\big(X(1)/G_n\big) \cong H^*\big((Q/G_n) \times_{\mu_{q^n-1}} \mathbb{G}_m\big) \\ &\cong H^*(\mathbb{A}^{n-1} \times_{\mu_{q^n-1}} \mathbb{G}_m) \cong H^*_{\mu_{q^n-1}}(\mathbb{A}^{n-1} \times \mathbb{G}_m) \\ &\cong H^*_{\mu_{q^n-1}}(\mathbb{G}_m) \cong \Lambda(f), \quad \deg(f) = 1. \end{aligned}$$ □

REMARK

When $\ell = 2$, the above lemma also holds. All arguments in this article hold for $\ell = 2$ if we change isomorphisms $A \cong B$ of graded rings with $B = C \otimes \Lambda(a, \dots, b)$ to C-module isomorphisms.

LEMMA 4.2

For $i < n$, we have an isomorphism of graded rings

$$H^*_{G_n}\big(E(i)\big) = H^*_{G_n}\big(F(i) - F(i+1)\big) \cong H^*(BG_i) \otimes \Lambda(f).$$

Proof

Each irreducible component of $F(i)$ is a codimension i linear subspace of $\mathbb{A}^n$, which is also identified with an element of the Grassmannian. Let us write $X(1)' = \mathbb{A}^{n-i} - F(1)'$, where $F(1)'$ is a variety defined as $\mathrm{Spec}(k[x_1, \dots, x_{n-i}]/(e_{n-i}^{q-1}))$. Then we can write

$$\begin{aligned} E(i) = F(i) - F(i+1) &\cong \coprod_{\bar{g} \in G_n/(P_{n-i,i})} g\big(X(1)'\big) \\ &\cong G_n \times_{P_{n-i,i}} X(1)' \end{aligned}$$

for $g \in G_n$ and its representative element $\bar{g}$. Here $P_{n-i,i}$ is the parabolic subgroup

$$P_{n-i,i} = (G_{n-i} \times G_i) \ltimes U_{n-i,i}(\mathbb{F}_q) \cong \left\{ \begin{pmatrix} G_{n-i} & * \\ 0 & G_i \end{pmatrix} \,\middle|\, * \in U_{n-i,i}(\mathbb{F}_q) \right\}.$$

Since the stabilizer subgroup of G_n on $X(1)'$ is the parabolic subgroup $P_{n-i,i}$, we get (see [18]), by using an induction/restriction isomorphism and the fact that $U_{n-i,i}(\mathbb{F}_q)$ is a p-group,

$$H^*_{G_n}\big(E(i)\big) \cong H^*_{P_{n-i,i}}\big(X(1)'\big) \cong H^*_{G_{n-i} \times G_i}\big(X(1)'\big).$$

Hence, we can compute (for $* < N$)

$$\begin{aligned} H^*_{G_n}\big(E(i)\big) &\cong H^*\big(V'_N \times V''_N \times_{G_{n-i} \times G_i} X(1)'\big) \\ &\cong H^*\big(V'_N \times_{G_{n-i}} X(1)' \times V''_N/G_i\big) \cong H^*_{G_{n-i}}\big(X(1)'\big) \otimes H^*_{G_i}. \end{aligned}$$

Here $X(1)'$ is the $(n-i)$-dimensional version of $X(1)$, and we identify $V_N \cong V'_N \times V''_N$, where G_{n-i} acts freely on V'_N and so on. From the previous lemma, we get $H^*_{G_{n-i}}(X(1)') \cong \Lambda(f)$. □

LEMMA 4.3
If $r=1$, then we have an isomorphism of graded rings

$$H^*(BG_n)\cong \mathbb{Z}/\ell[c_1,\dots,c_n]\otimes\Lambda(e_1,\dots,e_n).$$

Proof
We prove by induction on n. Assume that

$$H^*(BG_i)\cong \mathbb{Z}/\ell[c_1,\dots,c_i]\otimes\Lambda(e_1,\dots,e_i)\quad \text{for } i<n.$$

We consider the long exact sequence

$$\to H^{*-2i}_{G_n}\big(E(i)\big)\xrightarrow{i_*} H^*_{G_n}\big(X(i+1)\big)\xrightarrow{j^*} H^*_{G_n}\big(X(i)\big)\xrightarrow{\delta}\cdots.$$

Here we use induction on i, and assume that

$$H^*_{G_n}\big(X(i)\big)\cong H^*_{G_{i-1}}\otimes\Lambda(e_i)\cong \mathbb{Z}/\ell[c_1,\dots,c_{i-1}]\otimes\Lambda(e_1,\dots,e_i).$$

(Letting $e_1=f$, we have the case $i=1$ from Lemma 4.1.) Also, from Lemma 4.2, we have $H^*_{G_n}(E(i))\cong H^*_{G_i}\otimes\Lambda(f)$.

In the above long exact sequence, we have $\delta(c_j)=\delta(e_j)=0$ for $j<i$, since $H^{<0}_{G_n}(E(i))=0$, and $\delta(e_i)\in H^0_{G_n}(E(i))\cong\mathbb{Z}/\ell$. Hence, if $\delta(e_i)=0$, then $\delta=0$ (i.e., $\delta(x)=0$ for all $x\in H^*_{G_n}(X(i))$), since $H^*_{G_n}(X(i))$ is generated by $c_1,\dots,c_{i-1}$, $e_1,\dots,e_i$ as a ring.

Let $p:V\to X$ be a j-dimensional bundle, and let $i':X\to V$ be a section of p. Then it is well known that the Chern class c_j is defined as $(i')^*i'_*(1)$. Hence, we show that

$$(i')^*i'_*(1)=c_i\in H^*_{G_i}\quad \text{with } H_{G_i}(\mathbb{A}^i)\overset{(i')^*}{\cong} H^*_{G_i}\big(\{0\}\big)\cong H^*_{G_i}$$

for the G_i-embedding $i':\{0\}\subset\mathbb{A}^i$. From Proposition 3.2, we see this $c_i\neq 0$. Consider the restriction map $H^*_{G_n}(X(i+1))\to H^*_{G_i}(\mathbb{A}^i)$ which is induced from a G_i-map

$$\mathbb{A}^i\subset\mathbb{A}^i\times X(1)'=\mathbb{A}^i\times\big(\mathbb{A}^{n-i}-F(1)'\big)\subset X(i+1).$$

(Note that $\{0\}\times X(1)'\subset E(i)$.) By using the restriction, we show that

$$i_*(1)=c_i\neq 0\quad \text{in } H^*_{G_n}\big(X(i+1)\big).$$

Thus, we see that $\delta(e_i)=0$, and we get $\delta=0$ from the above argument.

Therefore, we have the short exact sequence

$$0\to H^{*-2i}_{G_i}\otimes\Lambda(f)\xrightarrow{i_*} H^*_{G_n}\big(X(i+1)\big)\xrightarrow{j^*} H^*_{G_{i-1}}\otimes\Lambda(e_i)\to 0.$$

Here $H^*_{G_{i-1}}\otimes\Lambda(e_i)$ is a free graded ring; namely, it is a tensor product of a polynomial algebra generated by even-degree elements and an exterior algebra generated by odd-degree elements (which has no relation as a graded ring). Hence, it is contained in $H_{G_n}(X(i+1))$, and j^* is split. Therefore, $H_{G_n}(X(i+1))$ is an $H_{G_{i-1}}\otimes\Lambda(e_i)$-module.

Then we have an $H^*_{G_{i-1}} \otimes \Lambda(e_i)$-module isomorphism

$$H^*_{G_n}\big(X(i+1)\big) \cong H_{G_{i-1}} \otimes \Lambda(e_i) \otimes \big(\mathbb{Z}/\ell[c_i]\{i_*(1) = c_i, i_*(f)\} \oplus \mathbb{Z}/\ell\{1\}\big)$$
$$\cong \mathbb{Z}/\ell[c_1, \dots, c_i] \otimes \Lambda(e_1, \dots, e_i) \otimes \{1, i_*(f)\}.$$

Let us write $i_*(f) = e_{i+1}$. (Note here $\deg(f) = 1$ but $\deg(i_*(f)) = 2i + 1$.) Then $H^*_{G_n}(X(i+1))$ is the desired form

$$H^*_{G_n}\big(X(i+1)\big) \cong \mathbb{Z}/\ell[c_1, \dots, c_i] \otimes \Lambda(e_1, \dots, e_i) \otimes \Lambda(e_{i+1})$$

for $i < n$. This is an isomorphism of graded rings because the right-hand side ring is a free graded ring.

When $i = n$, by the definition, $X(n+1) = \mathbb{A}^n$, $X(n) = \mathbb{A}^n - \{0\}$, and $E(n) = \{0\}$. The short exact sequence is given by

$$0 \to H^{*-2n}_{G_n}\big(\{0\}\big) \stackrel{\times c_n}{\to} H^*_{G_n}(\mathbb{A}^n) \to H^*_{G_n}\big(X(n)\big) \to 0,$$

which implies the desired isomorphism

$$H^*_{G_n} \cong H^*_{G_n}\big(X(n)\big)[c_n] \cong \mathbb{Z}/\ell[c_1, \dots, c_n] \otimes \Lambda(e_1, \dots, e_n). \qquad \square$$

THEOREM 4.4

We have an isomorphism of graded rings

$$H^*(BG_n) \cong \mathbb{Z}/\ell[c_r, \dots, c_{[n/r]r}] \otimes \Lambda(e_r, \dots, e_{[n/r]r}).$$

Proof

We prove the theorem also by induction on n. Assume that

$$H^*(BG_i) \cong \mathbb{Z}/\ell[c_r, \dots, c_{[i/r]r}] \otimes \Lambda(e_r, \dots, e_{[i/r]r}) \quad \text{for } i < n.$$

We also consider the long exact sequence

$$\to H^{*-2i}_{G_n}\big(E(i)\big) \stackrel{i_*}{\to} H^*_{G_n}\big(X(i+1)\big) \stackrel{j^*}{\to} H^*_{G_n}\big(X(i)\big) \stackrel{\delta}{\to} \cdots.$$

Here we use induction on i, and we assume that $H^*_{G_n}(X(i)) \cong H^*_{G_{i-1}} \otimes \Lambda(e_i)$.

From Lemma 4.2, we already have $H^*_{G_n}(E(i)) \cong H^*_{G_i} \otimes \Lambda(f)$. For dimensional reasons, we see that $\delta(e_i) \in H^0_{G_n}(E(i)) \cong \mathbb{Z}/\ell$.

Now we consider the case $2 \le r$ and $mr < i < (m+1)r \le n$. Note that the ℓ-Sylow subgroups of G_i and G_{i-1} are the same, and $H^*_{G_i} \cong H^*_{G_{i-1}}$. In this case we can assume that

$$H^*_{G_i} \cong H^*_{G_{i-1}} \cong \cdots \cong H^*_{G_{mr}} \cong \mathbb{Z}/\ell[c_r, \dots, c_{mr}] \otimes \Lambda(e_r, \dots, e_{mr}).$$

Hence, the above exact sequence is written as

$$\to H^*_{G_{mr}} \otimes \Lambda(f) \stackrel{i_*}{\to} H^*_{G_n}\big(X(i+1)\big) \stackrel{j^*}{\to} H^*_{G_{mr}} \otimes \Lambda(e_i) \to \cdots.$$

From Proposition 3.2, we have $c_i = 0$ in $H^*_{G_n}$. This implies that $i_*(1) = c_i = 0$ in $H^*_{G_n}(X(i+1))$, and hence, $\delta(e_i) \ne 0 \in \mathbb{Z}/\ell$.

Thus, we have the isomorphism (letting $i_*(f) = e_{i+1}$)

$$H^*_{G_n}\big(X(i+1)\big) \cong H^*_{G_{mr}}\{1, i_*(f)\} \cong H^*_{G_{mr}}\{1, e_{i+1}\} \cong H^*_{G_i} \otimes \Lambda(e_{i+1}).$$

When $i = (m+1)r$, the arguments work similarly to those in the case $r = 1$. $\square$

REMARK

Localized exact sequences (defined just before Lemma 4.1) induce the spectral sequence

$$E_1^{*',*} \cong \bigoplus_{i=1}^{n-1} H^*_{G_n}\big(E(i)\big) \Longrightarrow H^*_{G_n}\big(X(n)\big) \cong H^*_{G_n}(\mathbb{G}_m)$$

with the differential $d_r = \delta(j^*)^{-r+1} i_*$. Here, from Lemma 4.2, we have $H^*_{G_n}(E(i)) \cong H^*_{G_i} \otimes \Lambda(f_i)$ with $\deg(f_i) = 1$. When $r = 1$, the proof of Lemma 4.3 shows that $\delta = 0$, namely, $d_r = 0$, and so the above spectral sequence collapses. In fact,

$$H^*_{G_n}\big(E(i)\big) \overset{i_*}{\cong} H^*_{G_i}\{c_i, e_{i+1}\} \subset H^*_{G_n}\big(X(n)\big) \cong H^*_{G_n}/(c_n).$$

REMARK

We can give another proof of Lemma 4.3 as follows. Let us write simply $S\Lambda = \mathbb{Z}/\ell[c_1, \ldots, c_n] \otimes \Lambda(e_1, \ldots, e_n)$. Then we have $S\Lambda \subset H^*(BG_n)$. This fact is proved by Proposition 3.2 and the restriction to the diagonal subgroup D_n of G_n so that $H^*(BD_n) \cong H^*(B(\mathbb{Z}/\ell)^n)$. Hence, for each $m \geq 0$, we get $\mathrm{rank}_{\mathbb{Z}/\ell}(H^m(BG_n)) \geq \mathrm{rank}_{\mathbb{Z}/\ell}(S\Lambda^m)$. We consider the following sum of rank:

$$s(m) = \sum_{1 \leq i \leq n-1, 2i \leq m} \mathrm{rank}_{\mathbb{Z}/\ell}\Big(H^*_{G_n}\big(E(i) \otimes \mathbb{Z}/\ell[c_n]\big)^{m-2i}\Big).$$

Then from Lemma 4.2 and the previous remark, $s(m) = \mathrm{rank}_{\mathbb{Z}/\ell}(S\Lambda^m)$. So the spectral sequence collapses; otherwise, $\mathrm{rank}_{\mathbb{Z}/\ell}(H^m(BG_n)) < s(m) = \mathrm{rank}_{\mathbb{Z}/\ell}(S\Lambda^m)$ for some m.

REMARK

When $2 \leq r$ and $mr < i < (m+1)r \leq n$, the proof of Theorem 4.4 shows that $d_1(f_i) \neq 0 \in H^0(E(i+1)) \cong \mathbb{Z}/\ell$. Hence, in $H^*(E(i))$, we see that $H^*_{G_i} \subset \mathrm{Im}(d_1)$ and $d_1 : H_{G_i}\{f_i\} \cong H^*_{G_{i+1}}$. For $i = mr$, we note that $\delta = 0$. Thus, we get

$$E_2^{i,*} \cong \begin{cases} H^*_{G_{mr}} & \text{if } i = mr, \\ H^*_{G_{(m-1)r}}\{f_i\} & \text{if } i = mr-1 \text{ or } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have $H^*_{G_{mr}} \overset{i_*}{\cong} H^*_{G_{mr}}\{c_{mr}\} \subset H^*_{G_n}(X(n))$, and $H^*_{G_{(m-1)r}}\{f_{mr-1}\} \overset{i_*}{\cong} H^*_{G_{(m-1)r}}\{e_{mr}\} \subset H^*_{G_n}(X(n))$. Therefore, this spectral sequence collapses from the E_2-term.

5. Special linear group SL_n

We consider the case $G = \mathrm{SL}_n$. Denote $\mathrm{SL}_n(\mathbb{F}_q)$ by SG_n.

PROPOSITION 5.1

For the case $r \geq 2$, the following composition map is injective:

$$\mathbb{Z}/\ell[c_r, \dots, c_{[n/r]r}] \to H^*(\mathrm{BGL}_n)^F \to H^*(\mathrm{BSG}_n).$$

When $r = 1$, the map $\mathbb{Z}/\ell[c_2, \dots, c_n] \to H^*(\mathrm{BSG}_n)$ is injective.

Proof

When $r \geq 2$, G_n and SG_n have the same Sylow ℓ-subgroup. Hence, $H^*(BG_n) \to H^*(\mathrm{BSG}_n)$ is injective, and so we have the proposition. For $r = 1$, the proposition follows from an argument similar to that for the case $r = 1$ in Section 3 by using $H^*(\mathrm{BST}) \cong \mathbb{Z}/\ell[t_1, \dots, t_n]/(\sum t_i)$. $\square$

By using Corollary 2.5 and arguments similar to those in Section 4, we get the following result.

THEOREM 5.2

Let $\ell \neq 2$. For the case $r \geq 2$, we have an isomorphism $H^*(\mathrm{BSG}_n) \cong H^*(BG_n)$ of graded rings. When $r = 1$, we have a graded ring isomorphism

$$H^*(\mathrm{BSG}_n) \cong \mathbb{Z}/\ell[c_2, \dots, c_n] \otimes \Lambda(e_2, \dots, e_n).$$

6. Motivic cohomology

In this section, we consider the motivic version of previous sections. Let $H^{*,*'}(X;\mathbb{Z}/\ell)$ be the mod ℓ motivic cohomology over $k = \bar{\mathbb{F}}_p$. Let X be a G-variety defined over k. Let us write

$$H_G^{*,*'}(X) = \lim_N H^{*,*'}(V_N \times_G X; \mathbb{Z}/\ell)$$

for the (equivariant) mod ℓ motivic cohomology over $k = \bar{\mathbb{F}}_p$. Then we have the long exact sequence

$$\to H_{G_n}^{*-2i,*'-i}(E(i)) \xrightarrow{i_*} H_{G_n}^{*,*'}(X(i+1)) \xrightarrow{j^*} H_{G_n}^{*,*'}(X(i)) \xrightarrow{\delta} \cdots.$$

In general, the Künneth formula does not hold in the mod ℓ motivic cohomology. However, it holds for $H^{*,*'}(B\mu_{q^n-1})$ by Voevodsky [20], [19]. We can easily see that, for a G_n-variety Y,

$$H_{G_n}^{*,*'}(Y \times X(1)) \cong H^{*,*'}(Y) \otimes \Lambda(f).$$

Then we can prove that Lemma 4.2 holds for the motivic cohomology. The arguments in the previous sections also work for the motivic cohomology with degree

$$\deg(c_i) = (2i, i), \qquad \deg(e_i) = (2i-1, i).$$

Thus, we get Theorem 1.2 from the Introduction.

7. Drinfeld space

For $G = \mathrm{GL}_n$ and $w = (1, \dots, n) \in S_n$, it is known from [3, Theorem 2.1] that

$$\tilde{X}(\dot{w}) \cong Q' = \mathrm{Spec}\big(k[x_1, \dots, x_n]/\big(c_{n,0} = (-1)^{n-1}\big)\big).$$

(Here $Q' \cong Q$ as varieties over $k = \bar{\mathbb{F}}_p$ by $(x) \mapsto (\zeta x)$ for the $(q^n - 1)$th root ζ of -1 (when n is even; see the proof of Lemma 2.1).) We have a quasi-isomorphism (see [3, Corollary 1.12], [7, Theorem 0.4(b)])

$$Q'/G_n \cong G_n \setminus \tilde{X}(\dot{w}) \cong U/\big(U \cap \mathrm{ad}(\dot{w})U\big) \cong \mathbb{A}^{n-1}. \tag{7.1}$$

(Quasi-isomorphisms are isomorphisms for maps generated by morphisms of varieties and (the inverse of) Frobenius maps; for a definition, see [7, Section 2.1].) In this section, we will show that the above quasi-isomorphism can be explicitly written by the Dickson elements $c_{n,i}$ given in Section 2.

Take an adequate basis of the n-dimensional vector space such that

$$w = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad U = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \,\middle|\, * \in k \right\}.$$

Let $x_{i,j}(a) = 1 + a e_{i,j}$, where $e_{i,j}$ is the elementary matrix with 1 in the (i,j)th entry and 0 otherwise. Then U is generated by $x_{i,j}(a)$,

$$U = \big\langle x_{i,j}(a) \,\big|\, 1 \le i < j \le n, a \in k \big\rangle,$$

with the relation

$$x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b), \qquad \big[x_{i,j}(a), x_{k,l}(b)\big] = \delta_{j,k} x_{i,l}(ab) \quad (\text{for } i < l).$$

Note that $\mathrm{ad}(w)x_{i,j}(a) = x_{i+1,j+1}(a)$ for $i, j \in \mathbb{Z}/n$.

Let us denote by U_w the intersection $U \cap \mathrm{ad}(w)U$. Hence, $U_w \cong \langle x_{i,j} \mid x_{1,j} = 0\rangle$. We consider the U_w-action on U, which is given by (see [3, (1.11.4)])

$$\rho(u)v = \mathrm{ad}(\dot{w}^{-1})(u)vF(u^{-1}) \in U \quad \text{for } u \in U_w, v \in U.$$

LEMMA 7.1

The composition of natural maps of algebraic groups

$$\big\langle x_{in}(k) \,\big|\, i < n \big\rangle \subset U \to U/\rho(U_w)$$

induces the isomorphism $\mathbb{A}^{n-1} \cong U/\rho(U_w)$ *in (7.1), where* $\langle x_{in}(k) | i < n\rangle$ *is written as*

$$\left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & d_1 \\ \vdots & \vdots & \ddots & \vdots & * \\ 0 & 0 & \cdots & 1 & d_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in U \,\middle|\, d_1, \dots, d_{n-1} \in k \right\} \cong \mathbb{A}^{n-1}.$$

Proof
We consider the ρ-action in the case $u = x_{i,j}(a)$ for $1 < i$ and $v = x_{k,l}(b)$,

$$\begin{aligned}\rho(u)v &= \mathrm{ad}(\dot{w}^{-1})\big(x_{ij}(a)\big)x_{k,l}(b)F\big(x_{i,j}(a)^{-1}\big)\\ &= x_{i-1,j-1}(a)x_{k,l}(b)x_{i,j}(-a^q).\end{aligned}$$

For generators $x_{i,j}$ and $x_{i',j'}$, we define an order $x_{i,j} < x_{i',j'}$ if $j < j'$ or $j = j'$, $i < i'$. Then any $v \in U$ is uniquely written as the product $\prod x_{i,j}(b_{i,j})$ with respect to the order; namely,

$$\prod x_{i,j}(b_{i,j}) = x_{i_0,j_0}(b_{i_0,j_0})\cdots x_{i_s,j_s}(b_{i_s,j_s}), \quad x_{i_0,i_0} < \cdots < x_{i_s,j_s}.$$

Here, let $x_{i_0,j_0}(b_{i_0,j_0}) \neq 1$ and $j_0 < n$. Take $u = x_{\bar{i},\bar{j}}(a)$ with $\bar{i} = i_0 + 1$, $\bar{j} = j_0 + 1$, and $a = -b_{i_0,j_0}$. (Note that $x_{\bar{i},\bar{j}}(a) \in U_w$ since $\bar{i} > 1$.) Then the equation

$$\begin{aligned}\rho(u)v &= \mathrm{ad}(\dot{w}^{-1})\big(x_{\bar{i}\bar{j}}(a)\big)\Big(\prod x_{i,j}(b_{ij})\Big)F\big(x_{\bar{i},\bar{j}}(a)^{-1}\big)\\ &= x_{i_0,j_0}(-b_{i_0,j_0})\Big(\prod x_{i,j}(b_{i,j})\Big)x_{\bar{i},\bar{j}}(-a^q)\\ &= \Big(\prod_{(i_0,j_0)<(i,j)} x_{i,j}(b_{i,j})\Big)x_{i_0+1,j_0+1}(-a^q)\end{aligned}$$

implies that a nonzero minimal generator of $\rho(u)v$ is larger than x_{i_0,j_0}.

By repeating this process, for each $v \in U$, there is $u \in U_w$ such that

$$\rho(u)v \in \big\langle x_{i,n}(k) \bigm| i < n\big\rangle \cong \mathbb{A}^{n-1}.$$

Since we know that $U/\rho(U_w) \cong \mathbb{A}^{n-1}$ from (7.1), we get the lemma. $\square$

Recall that we can identify

$$\begin{aligned}Q' &= \big\{x = (x_1,\dots,x_n) \in \mathbb{A}^n \bigm| e(x)^{q-1} = (-1)^{n-1}\big\}\\ &\cong \left\{ x = \begin{pmatrix} x_1 & x_1^q & \cdots & x_1^{q^{n-1}}\\ x_2 & x_2^q & \cdots & x_2^{q^{n-1}}\\ \vdots & \vdots & \ddots & \vdots\\ x_n & x_n^q & \cdots & x_n^{q^{n-1}}\end{pmatrix} \in \mathrm{GL}_n(k) \,\middle|\, |x|^{q-1} = \det(x)^{q-1} = (-1)^{n-1} \right\}.\end{aligned}$$

THEOREM 7.2
We get the quasi-isomorphism $f : Q'/G_n \to U/(\rho(U_w))$ by $x \mapsto \dot{w}^{-1}x^{-1}Fx$. This map $f(x)$ is written as

$$f(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & (-1)^{n-2}c_{n,1}\\ \vdots & \vdots & \ddots & \vdots & *\\ 0 & 0 & \cdots & 1 & c_{n,n-1}\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix},$$

where $c_{n,i} = c_{n,i}(x_1,\dots,x_n)$ is the Dickson element defined in Section 2.

Proof
We prove only that $f(x)$ is expressed by $c_{n,i}$ above. Let us write

$$e_n \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix} = \begin{vmatrix} x_{j_1}^{q^{i_1}} & x_{j_1}^{q^{i_2}} & \cdots & x_{j_1}^{q^{i_n}} \\ x_{j_2}^{q^{i_1}} & x_{j_2}^{q^{i_2}} & \cdots & x_{j_2}^{q^{i_n}} \\ \vdots & \vdots & \ddots & \vdots \\ x_{j_n}^{q^{i_1}} & x_{j_n}^{q^{i_2}} & \cdots & x_{j_n}^{q^{i_n}} \end{vmatrix}$$

so that

$$e_n \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & n \end{pmatrix} = e(x) = |x|.$$

Then the (j,i)-cofactor of the matrix x is expressed as

$$D_{j,i} = (-1)^{i+j} e_{n-1} \begin{pmatrix} 0 & 1 & \cdots & \widehat{i-1} & \cdots & n-1 \\ 1 & 2 & \cdots & \hat{j} & \cdots & n \end{pmatrix}.$$

By Cramér's theorem, we know that

$$x^{-1} = |x|^{-1}(D_{j,i})^t = |x|^{-1}(D_{i,j}).$$

Let us write $(B_{i,j}) = |x|x^{-1}F(x)$. Then we can compute

$$\begin{aligned} B_{s,t} &= \big(DF(x)\big)_{s,t} = \sum D_{s,k} x(k,t)^q \\ &= \sum D_{s,k} x_k^{q^t} \quad \big(\text{where } x(k,t) \text{ is the } (k,t)\text{th entry of } x\big) \\ &= \begin{vmatrix} x_1 & \cdots & \overset{s}{x_1^{q^t}} & \cdots & x_1^{q^{n-1}} \\ x_2 & \cdots & x_2^{q^t} & \cdots & x_2^{q^{n-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n & \cdots & x_n^{q^t} & \cdots & x_n^{q^{n-1}} \end{vmatrix}. \end{aligned}$$

This element is nonzero only if $t = s-1$ or $t = n$. If $t = s-1$, then the above element is $|x|$. If $t = n$, then the above element is, indeed, $(-1)^{n-s}|x|c_{n,s-1}$ by the definition of the Dickson elements as stated in Section 2. Thus, we have

$$x^{-1}F(x) = |x|^{-1}(B_{st}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{n-1}c_{n,0} \\ 1 & 0 & \cdots & 0 & (-1)^{n-2}c_{n,1} \\ \vdots & \vdots & \ddots & \vdots & * \\ 0 & 0 & \cdots & 1 & c_{n,n-1} \end{pmatrix}.$$

Here $(-1)^{n-1}c_{n,0} = 1$, and acting $\dot{w}^{-1}$, we have the desired result. □

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