Regular functions on spherical nilpotent orbits in complex symmetric pairs: Classical non-Hermitian cases

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Abstract Given a classical semisimple complex algebraic group G and a symmetric pair (G, K) of non-Hermitian type, we study the closures of the spherical nilpotent K-orbits in the isotropy representation of K. For all such orbit closures, we study the normality, and we describe the K-module structure of the ring of regular functions of the normalizations.

Contents

Introduction

Let G be a connected semisimple complex algebraic group, and let K be the fixed point subgroup of an algebraic involution θ of G. Then K is a reductive group, which is connected if G is simply connected.

The Lie algebra $\mathfrak{g}$ of G splits into the sum of eigenspaces of θ,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where the Lie algebra $\mathfrak{k}$ of K is the eigenspace of eigenvalue 1, and $\mathfrak{p}$ is the eigenspace of eigenvalue -1. The adjoint action of G on $\mathfrak{g}$, once restricted to K, leaves $\mathfrak{k}$ and $\mathfrak{p}$ stable.

Therefore, $\mathfrak{p}$ provides an interesting representation of K, called the *isotropy representation*, where one may want to study the geometry of the K-orbits. With this aim, one looks at the so-called *nilpotent cone* $\mathcal{N}_\mathfrak{p} \subset \mathfrak{p}$, which consists of the

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elements whose K-orbit closure contains the origin. In this case, $\mathcal{N}_\mathfrak{p}$ actually consists of the nilpotent elements of $\mathfrak{g}$ which belong to $\mathfrak{p}$. By a fundamental result of Kostant and Rallis [21, Theorem 2], as in the case of the adjoint action of G on $\mathfrak{g}$, there are finitely many nilpotent K-orbits in $\mathfrak{p}$.

Provided K is connected, we restrict our attention to the spherical nilpotent K-orbits in $\mathfrak{p}$. Here spherical means with an open orbit for a Borel subgroup of K or, equivalently, with a ring of regular functions which affords a multiplicity-free representation of K. The classification of these orbits is known and due to King [19].

In the present article, we begin a systematic study of the closures of the spherical nilpotent K-orbits in $\mathfrak{p}$. In particular, we analyze their normality and describe the K-module structure of the coordinate rings of their normalizations. This is done by making use of the technical machinery of spherical varieties, which is recalled in Section 1.

Here we will deal with the case where (G,K) is a classical symmetric pair with K semisimple; the other cases will be treated in forthcoming articles. The semisimplicity of K is equivalent to the fact that $\mathfrak{p}$ is a simple K-module, in which case G/K is also called a *symmetric space* of non-Hermitian type.

Let $G_\mathbb{R}$ be a real form of G with Lie algebra $\mathfrak{g}_\mathbb{R}$ and Cartan decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$, so that θ is induced by the corresponding Cartan involution of $G_\mathbb{R}$. Then K is the complexification of a maximal compact subgroup $K_\mathbb{R} \subset G_\mathbb{R}$, and the Kostant–Sekiguchi–Đoković correspondence (see [14], [30]) establishes a bijection between the set of the nilpotent $G_\mathbb{R}$-orbits in $\mathfrak{g}_\mathbb{R}$ and the set of the nilpotent K-orbits in $\mathfrak{p}$. Let us briefly recall how it works. More details and references can be found in [12].

Every nonzero nilpotent element $e \in \mathfrak{g}_\mathbb{R}$ lies in an $\mathfrak{sl}(2)$-triple $\{h,e,f\} \subset \mathfrak{g}_\mathbb{R}$. Every $\mathfrak{sl}(2)$-triple $\{h,e,f\} \subset \mathfrak{g}_\mathbb{R}$ is conjugate to a *Cayley triple* $\{h',e',f'\} \subset \mathfrak{g}_\mathbb{R}$, that is, an $\mathfrak{sl}(2)$-triple with $\theta(h') = -h'$, $\theta(e') = -f'$, and $\theta(f') = -e'$. To a Cayley triple in $\mathfrak{g}_\mathbb{R}$ one can associate its *Cayley transform*

$$\{h,e,f\} \mapsto \left\{i(e-f), \frac{1}{2}(e+f+ih), \frac{1}{2}(e+f-ih)\right\}:$$

this is a normal triple in $\mathfrak{g}$, that is, an $\mathfrak{sl}(2)$-triple $\{h',e',f'\}$ with $h' \in \mathfrak{k}$ and $e', f' \in \mathfrak{p}$. By [21], any nonzero nilpotent element $e \in \mathfrak{p}$ lies in a normal triple $\{h,e,f\} \subset \mathfrak{g}$, and any two normal triples with the same nilpositive element e are conjugated under K. Then the desired bijective correspondence is constructed as follows. Let $\mathcal{O} \subset \mathfrak{g}_\mathbb{R}$ be an adjoint nilpotent orbit, choose an element $e \in \mathcal{O}$ belonging to a Cayley triple $\{h,e,f\}$, consider its Cayley transform $\{h',e',f'\}$, and let $\mathcal{O}' = Ke'$. Then $\mathcal{O}' \subset \mathfrak{p}$ is the nilpotent K-orbit corresponding to $\mathcal{O}$.

Among the nice geometric properties of the Kostant–Sekiguchi–Đoković correspondence, we just recall here one result concerning sphericality: the spherical nilpotent K-orbits in $\mathfrak{p}$ correspond to the adjoint nilpotent $G_\mathbb{R}$-orbits in $\mathfrak{g}_\mathbb{R}$ which are multiplicity free as Hamiltonian $K_\mathbb{R}$-spaces (see [18]).

In accordance with the philosophy of the orbit method (see, e.g., [1]), the unitary representations of $G_\mathbb{R}$ should be parameterized by the (co-)adjoint orbits of $G_{\mathbb{R}}$. In particular, one is interested in the so-called *unipotent representations* of $G_{\mathbb{R}}$, namely, those which should be attached to nilpotent orbits. The K-module structure of the ring of regular functions on a nilpotent K-orbit in $\mathfrak{p}$ (which we compute in our spherical cases) should give information on the corresponding unitary representation of $G_{\mathbb{R}}$. Unitary representations that should be attached to the spherical nilpotent K-orbits are studied in [17] (when G is a classical group) and [29] (when G is the special linear group). When G is the symplectic group, for particular spherical nilpotent K-orbits, such representations are constructed in [32] and [33].

The normality and the K-module structure of the coordinate ring of the closure of a spherical nilpotent K-orbit in $\mathfrak{p}$ have been studied in several particular cases, with different methods, by Nishiyama [24], [25], by Nishiyama, Ochiai, and Zhu [26], and by Binegar [2]. In Appendix A we report the list of the spherical nilpotent K-orbits in $\mathfrak{p}$ for all symmetric pairs $(\mathfrak{g},\mathfrak{k})$ of classical non-Hermitian type. In the classical cases, the adjoint nilpotent orbits in real simple algebras are classified in terms of signed partitions, as explained in [12, Chapter 9]. In the list, every orbit is labeled with its corresponding signed partition.

For every orbit we provide an explicit description of a representative $e \in \mathfrak{p}$, as an element of a normal triple $\{h, e, f\}$, and the centralizer of e, which we denote by K_e. All these data can be directly computed using King's [19] paper on the classification of the spherical nilpotent K-orbits (but we point out a missing case therein; see Remark A.1).

The first datum which is somewhat new in this work is the Luna spherical system associated with $\mathrm{N}_K(K_e)$, the normalizer of K_e in K, which is a wonderful subgroup of K. It is equal to $K_{[e]}$, the stabilizer of the line through e, and note that $K_{[e]}/K_e \cong \mathbb{C}^\times$. The Luna spherical systems are used to deduce the normality or nonnormality of the K-orbits and to compute the corresponding K-modules of regular functions.

Appendix B consists of two sets of tables, where we summarize our results on the spherical nilpotent K-orbits in $\mathfrak{p}$. Given such an orbit $\mathcal{O} = Ke$, in the first set (see Tables 2–10) we describe the normality of its closure $\overline{\mathcal{O}}$, and if $\widetilde{\mathcal{O}} \longrightarrow \overline{\mathcal{O}}$ denotes the normalization, then we describe the K-module structure of $\mathbb{C}[\widetilde{\mathcal{O}}]$ by giving a set of generators of its weight semigroup $\Gamma(\widetilde{\mathcal{O}})$ (i.e., the set of the highest weights occurring in $\mathbb{C}[\widetilde{\mathcal{O}}]$). The second set (see Tables 11–19) contains the Luna spherical systems of $\mathrm{N}_K(K_e)$.

In Section 1 we compute the Luna spherical systems. In Section 2 we study the multiplication of sections of globally generated line bundles on the corresponding wonderful varieties, which turns out to be always surjective in all cases except one. In Section 3 we deduce our results on normality and semigroups.

Notation

Simple roots of irreducible root systems are denoted by $\alpha_1, \alpha_2, \ldots$ and enumerated as in Bourbaki; when belonging to different irreducible components they are denoted by $\alpha_1, \alpha_2, \ldots, \alpha'_1, \alpha'_2, \ldots, \alpha''_1, \alpha''_2, \ldots$, and so on. For the fundamental weights we adopt the same convention, they are denoted by $\omega_1, \omega_2, \ldots, \omega'_1, \omega'_2, \ldots, \omega''_1, \omega''_2, \ldots$, and so on. In the tables for the orthogonal cases at the end of the article we use a variation of the fundamental weights $\varpi_1, \varpi_2, \ldots$, which is explained in Appendix B. By $V(\lambda)$ we denote the simple module of highest weight λ; the acting group will be clear from the context.

1. Spherical systems

In this section we compute the Luna spherical systems given in the tables at the end of the article, in Appendix B. First, let us briefly explain what a Luna spherical system is (see, e.g., [5] for a plain introduction).

1.1. Luna spherical systems

Recall that a subgroup H of K is called *wonderful* if the homogeneous space K/H admits an open equivariant embedding in a wonderful K-variety. A K-variety is called wonderful if it is smooth and complete with an open K-orbit whose complement is the union of $D_1, \ldots, D_r$ smooth prime K-stable divisors with nonempty transversal crossings such that two points x, x' lie in the same K-orbit if and only if

$$\{i : x \in D_i\} = \{i : x' \in D_i\}.$$

The wonderful embedding of K/H is unique up to equivariant isomorphism and is a projective spherical K-variety. The number r of prime K-stable divisors is called the *rank* of X.

Let us fix, inside K, a maximal torus T and a Borel subgroup B containing T. This choice yields a root system R and a set of simple roots S in R. Let us also denote by $(\cdot,\cdot)$ the scalar product in the Euclidean space spanned by R, by $\alpha^\vee$ the coroot associated with α, and by $\langle\cdot,\cdot\rangle$ the usual Cartan pairing

$$\langle \alpha^\vee, \lambda\rangle = 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}.$$

For any spherical K-variety X, the set of colors, which is denoted by Δ_X, is the set of prime B-stable, non-K-stable divisors of X. It is a finite set. In our case, if X is the wonderful embedding of K/H, then the colors of K/H are just the irreducible components of the complement of the open B-orbit, and the colors of X are just the closures of the colors of K/H, so that the two sets Δ_X and $\Delta_{K/H}$ are naturally identified.

For any spherical K-variety X one can also define another finite set, the set of spherical roots, usually denoted by Σ_X. Here we recall its definition only in the wonderful case. Suppose X is the wonderful embedding of K/H. By definition, X contains a unique closed K-orbit; therefore, every Borel subgroup of K fixes in X a unique point. Let us call z the point fixed by B^-, the opposite of the Borel subgroup B. For all K-stable prime divisors D_i, let σ_i be the T-eigenvalue occurring in the normal space of D_i at z

$$\frac{\mathrm{T}_z X}{\mathrm{T}_z D_i}.$$

Then the set of spherical roots is the set $\Sigma_X = \{\sigma_1, \dots, \sigma_r\}$, also denoted by $\Sigma_{K/H}$. The spherical roots are linearly independent, and the corresponding reflections

$$\gamma \mapsto \gamma - 2\frac{(\sigma_i, \gamma)}{(\sigma_i, \sigma_i)}\sigma_i$$

generate a finite group of orthogonal transformations which is called the *little Weyl group* of X. In our case, in which the center of K acts trivially, the spherical roots are elements of $\mathbb{N}S$, that is, linear combinations with nonnegative integer coefficients of simple roots.

The Picard group of a wonderful variety X is freely generated by the equivalence classes of the colors of X. Expressing the classes of the K-stable divisors in terms of the basis given by the classes of colors

$$[D_i] = \sum_{D \in \Delta_{K/H}} c_{K/H}(D, \sigma_i)[D],$$

we get a $\mathbb{Z}$-bilinear pairing, which is also called a *Cartan pairing*,

$$c_{K/H} \colon \mathbb{Z}\Delta_{K/H} \times \mathbb{Z}\Sigma_{K/H} \to \mathbb{Z}.$$

It is known to satisfy quite strong restrictions, as follows.

For any simple root $\alpha \in S$, the set of colors moved by α, which is denoted by $\Delta_{K/H}(\alpha)$, is the set of colors that are not stable under the action of the minimal parabolic subgroup $P_{\{\alpha\}}$. Any simple root α moves at most two colors, and more precisely, there are exactly four cases.

Case p. α moves no colors.

Case a. α moves two colors. This happens if and only if $\alpha \in \Sigma_{K/H}$, and in this case we have

(1) $\Delta_{K/H}(\alpha) = \{D \in \Delta_{K/H} : c_{K/H}(D, \alpha) = 1\}$,
(2) $c_{K/H}(D, \sigma) \le 1$ for all $D \in \Delta_{K/H}(\alpha)$ and $\sigma \in \Sigma_{K/H}$,
(3) $\sum_{D \in \Delta_{K/H}(\alpha)} c_{K/H}(D, \sigma) = \langle \alpha^\vee, \sigma \rangle$ for all $\sigma \in \Sigma_{K/H}$.

Case 2a. α moves one color and $2\alpha \in \Sigma_{K/H}$. In this case if $D \in \Delta_{K/H}(\alpha)$, then we have $c_{K/H}(D, \sigma) = \frac{1}{2}\langle \alpha^\vee, \sigma \rangle$ for all $\sigma \in \Sigma_{K/H}$.

Case b. α moves one color and $2\alpha \notin \Sigma_{K/H}$. In this case if $D \in \Delta_{K/H}(\alpha)$, then we have $c_{K/H}(D, \sigma) = \langle \alpha^\vee, \sigma \rangle$ for all $\sigma \in \Sigma_{K/H}$.

The set of simple roots moving no colors is denoted by $S^{\mathrm{p}}_{K/H}$. The set of colors $\Delta_{K/H}$ is a disjoint union of subsets $\Delta^{\mathrm{a}}_{K/H}$, $\Delta^{2\mathrm{a}}_{K/H}$, and $\Delta^{\mathrm{b}}_{K/H}$ which consist of colors moved by simple roots of type (a), (2a), and (b), respectively. The set $\Delta^{\mathrm{a}}_{K/H}$ is also denoted by $\mathrm{A}_{K/H}$.

Case a. A color in $\mathrm{A}_{K/H}$ may be moved by several simple roots.

Case 2a. A color in $\Delta^{2\mathrm{a}}_{K/H}$ is moved by a unique simple root.

Case b. A color in $\Delta^{\mathrm{b}}_{K/H}$ may be moved by at most two simple roots. In this case, two simple roots α and β move the same color if and only if α and β are orthogonal and $\alpha+\beta \in \Sigma_{K/H}$.

Note that the full Cartan pairing $c_{K/H}\colon \mathbb{Z}\Delta_{K/H} \times \mathbb{Z}\Sigma_{K/H} \to \mathbb{Z}$ is determined by its restriction to $\mathrm{A}_{K/H} \times \Sigma_{K/H}$. If H is a wonderful subgroup of K, then the triple $(S^{\mathrm{p}}_{K/H}, \Sigma_{K/H}, \mathrm{A}_{K/H})$, endowed with the map $c_{K/H}\colon \mathrm{A}_{K/H} \times \Sigma_{K/H} \to \mathbb{Z}$, is called the *spherical system* of H.

1.2. Luna diagrams

In Appendix B, we present the spherical systems of the wonderful subgroups $H = \mathrm{N}_K(K_e)$ of K by providing the sets of spherical roots $\Sigma_{K/H}$ and the Luna diagrams. The Luna diagram of a spherical system consists of the Dynkin diagram of K decorated with some extra symbols from which one can read off all the data of the spherical system. Let us briefly explain how it works. Here we only explain how to read off the missing data (the set $S^{\mathrm{p}}_{K/H}$ and the map $c_{K/H}\colon \mathrm{A}_{K/H} \times \Sigma_{K/H}$; see, e.g., [5] for a complete description).

Every circle (shadowed or not) represents a color. Circles corresponding to the same color are joined by a line. The colors moved by a simple root are close to the corresponding vertex of the Dynkin diagram.

Case p. No circle is placed in correspondence to the vertex.
Case a. Two circles are placed: one above and one below the vertex.
Case 2a. One circle is placed below the vertex.
Case b. One circle is placed around the vertex.

Therefore, the set S^{p} is given by the vertices with no circles. It is worth saying that in general S^{p} is included in $\{\alpha \in S : \langle \alpha^\vee, \sigma \rangle = 0 \ \forall \sigma \in \Sigma\}$.

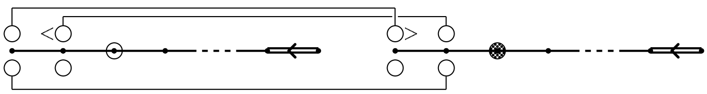
To read off the map $c\colon \mathrm{A} \times \Sigma \to \mathbb{Z}$, one has to know that an arrow (it looks more like a pointer but it has a source and a target) starting from a circle D above a vertex α and pointing toward a spherical root σ nonorthogonal to α means that $c(D,\sigma) = -1$. Vice versa, the Luna diagram is organized so that the colors D corresponding to circles that lie above the vertices have $c(D,\sigma) \geq -1$ for all $\sigma \in \Sigma$, so if there is no arrow starting from a circle D above a vertex α and pointing toward a spherical root σ nonorthogonal to α (with $D \notin \Delta(\sigma)$), then this means that $c(D,\sigma) = 0$. These together with the properties of the Cartan pairing for colors of type (a), explained above, allows us to recover the map $c\colon \mathrm{A} \times \Sigma \to \mathbb{Z}$.

The two colors moved by $\alpha \in S \cap \Sigma$ will be denoted by D^+_α and D^-_α. The former refers to the circle placed above the vertex, while the latter refers to the circle placed below. The color moved by a simple root $\alpha \notin \Sigma$ will be denoted by D_α.

As an example we show in detail how to recover the map $c\colon \mathrm{A} \times \Sigma \to \mathbb{Z}$ for the first case of the list where a nonempty set $\mathrm{A}_{K/H}$ occurs (case 4.4 with $q > 2$). The group K is of type $\mathsf{C}_p \times \mathsf{C}_q$, with p and q greater than 2. The set of spherical roots is

$$\Sigma = \{\alpha_1, \alpha_2, \alpha'_1, \alpha'_2, \alpha'_2 + 2(\alpha'_3 + \cdots + \alpha'_{q-1}) + \alpha'_q\},$$

and the Luna diagram is as follows.



Here the set S^{p} is given by the simple roots α_i for all $4 \le i \le p$ and α'_i for all $4 \le i \le q$. The elements of A, that is, the colors of type (a), are five:

$$D^-_{\alpha_2}, \qquad D^+_{\alpha_2} = D^+_{\alpha'_2}, \qquad D^-_{\alpha_1} = D^-_{\alpha'_2}, \qquad D^+_{\alpha_1} = D^+_{\alpha'_1}, \qquad D^-_{\alpha'_1}.$$

We know that, for all colors D of type (a), $c(D,\sigma) = 1$ if $\sigma \in S$ and $D \in \Delta(\sigma)$, and $c(D,\sigma) \le 0$ otherwise. Therefore, let us show how to determine $c(D^-_{\alpha_2},\sigma)$ for all $\sigma \in \Sigma$. First $D^-_{\alpha_2} \in \Delta(\alpha_2)$; then $c(D^-_{\alpha_2},\alpha_2) = 1$. Since there is an arrow from $D^+_{\alpha_2}$ to α_1, $c(D^+_{\alpha_2},\alpha_1) = -1$; furthermore, $c(D^-_{\alpha_2},\alpha_1) + c(D^+_{\alpha_2},\alpha_1) = \langle\alpha_2^\vee,\alpha_1\rangle = -1$. Thus, we have $c(D^-_{\alpha_2},\alpha_1) = 0$. The other spherical roots σ are orthogonal to α_2, so $c(D^-_{\alpha_2},\sigma) + c(D^+_{\alpha_2},\sigma) = 0$. If $c(D^-_{\alpha_2},\sigma)$ is less than 0, then $c(D^+_{\alpha_2},\sigma)$ must be greater than 0, but this happens only if $D^+_{\alpha_2} \in \Delta(\sigma)$. Therefore, $c(D^-_{\alpha_2},\alpha'_2) = -1$ while it is 0 on the other two spherical roots $c(D^-_{\alpha_2},\alpha'_1) = c(D^-_{\alpha_2},\alpha'_2 + 2(\alpha'_3 + \cdots + \alpha'_{q-1}) + \alpha'_q) = 0$. The entire map $c\colon \mathrm{A} \times \Sigma \to \mathbb{Z}$ is as follows.

	α_1	α_2	α'_1	α'_2	σ_5
$D^-_{\alpha_2}$	0	1	0	−1	0
$D^+_{\alpha_2}$	−1	1	0	1	0
$D^-_{\alpha_1}$	1	−1	−1	1	0
$D^+_{\alpha_1}$	1	0	1	−1	0
$D^-_{\alpha'_1}$	−1	0	1	0	−1

1.3. Operations on spherical systems

Here we briefly recall the definition and the essential properties of some combinatorial operations on spherical systems which correspond to geometric operations on wonderful varieties (see, e.g., [5] for some more details and references).

1.3.1. Subsystems

All (irreducible) K-subvarieties of a wonderful K-variety X are wonderful. They are exactly the K-orbit closures of X and are in correspondence with the subsets of Σ_X. If $D_1,\dots,D_r$ are the K-stable prime divisors of X, recall that the spherical roots $\sigma_1,\dots,\sigma_r$ are T-eigenvalues occurring, respectively, in the normal spaces of D_i at z, $\mathrm{T}_zX/\mathrm{T}_zD_i$. Therefore, every K-subvariety X' of X is the intersection of some K-stable prime divisors

$$X' = \bigcap_{i\in I} D_i$$

for some $I \subset \{1,\dots,r\}$. Its spherical system is thus given by

- $S^{\mathrm{p}}_{X'} = S^{\mathrm{p}}_X$,
- $\Sigma_{X'} = \{\sigma_i : i \notin I\}$,
- $\mathrm{A}_{X'} = \bigcup_{\alpha\in S\cap\Sigma_{X'}} \Delta_X(\alpha)$ with the map c_X restricted to $\mathbb{Z}\mathrm{A}_{X'} \times \mathbb{Z}\Sigma_{X'}$.

1.3.2. Quotients

Let X_1 and X_2 be the wonderful embeddings of K/H_1 and K/H_2, respectively. If H_1 is included in H_2 with connected quotient H_2/H_1, there exists a surjective equivariant morphism from X_1 to X_2 with connected fibers. In terms of spherical systems this is equivalent to an operation called *quotient*, as follows. A subset Δ' of Δ_{X_1} is called *distinguished* if there exists a linear combination with positive coefficients

$$D' \in \sum_{D\in\Delta'} n_D D$$

such that $c_{X_1}(D',\sigma) \geq 0$ for all $\sigma \in \Sigma_{X_1}$. If Δ' is distinguished, then the monoid

$$(\mathbb{N}\Sigma_{X_1})/\Delta' = \left\{\sigma \in \mathbb{N}\Sigma_{X_1} : c_{X_1}(D,\sigma) = 0 \ \forall D \in \Delta'\right\}$$

is known to be free (see [3]). Therefore, we can consider the following triple, which is called the quotient of the spherical system of X_1 by Δ':

- $S^{\mathrm{p}}_{X_1}/\Delta' = \{\alpha \in S : \Delta_{X_1}(\alpha) \subset \Delta'\}$,
- Σ_{X_1}/Δ', the basis of $(\mathbb{N}\Sigma_{X_1})/\Delta'$,
- $\mathrm{A}_{X_1}/\Delta' = \bigcup_{\alpha\in S\cap(\Sigma_{X_1}/\Delta')} \Delta_{X_1}(\alpha)$ endowed with the map c_{X_1} restricted to $\mathbb{Z}(\mathrm{A}_{X_1}/\Delta') \times \mathbb{Z}(\Sigma_{X_1}/\Delta')$.

If X_1 and X_2 are wonderful K-varieties with a surjective equivariant morphism with connected fibers $\varphi\colon X_1 \to X_2$, then $\Delta'_\varphi = \{D \in \Delta_{X_1} : \varphi(D) = X_2\}$ is distinguished and the spherical system of X_2 is equal to the quotient of the spherical system of X_1 by Δ'_φ. If X_1 is a wonderful K-variety, then every distinguished subset Δ' of Δ_{X_1} corresponds in this way to a surjective equivariant morphism with connected fibers onto a wonderful variety whose spherical system is equal to the quotient of the spherical system of X_1 by Δ'.

1.3.3. Parabolic inductions

Let Q be a parabolic subgroup of K, with Levi decomposition $Q = LQ^{\mathrm{u}}$. A wonderful K-variety X is said to be obtained by parabolic induction from the wonderful L-variety Y if

$$X \cong K \times_Q Y,$$

where Q^{u} acts trivially on Y. Further, since Y is a wonderful L-variety, the radical of L acts trivially on Y, as well.

Clearly, if the wonderful K-variety X is obtained by parabolic induction from the wonderful embedding of L/M, then X is the wonderful embedding of $K/(MQ^{\mathrm{u}})$. In terms of spherical systems this corresponds to the following situation. Assume that Q contains B^- and L contains T, and denote by S_L the subset of S generating the root subsystem of L. The wonderful K-variety X is obtained by parabolic induction from a wonderful L-variety Y if and only if

$$S_X^{\mathrm{p}} \cup \{\operatorname{supp} \sigma : \forall \sigma \in \Sigma_X\} \subset S_L.$$

In this case, the spherical system of Y, after the above inclusion, is equal to the triple $(S_X^{\mathrm{p}}, \Sigma_X, \mathrm{A}_X)$. In plain words, the spherical system of X is obtained from the spherical system of Y by letting the extra simple roots in $S \smallsetminus S_L$ move one extra color each so that they are all of type (b).

1.3.4. Localizations
Let Q be a parabolic subgroup of K, containing B^-, and let $Q = LQ^{\mathrm{u}}$ be its Levi decomposition, with L containing T. Denote by L^{r} the radical of L, and denote by S_L the subset of S generating the root subsystem of L.

Let X be a wonderful K-variety. Consider the subset of X of points fixed by L^{r}, and take its connected component which contains z, the unique point fixed by B^-. It is a wonderful L-variety Y called the L-*localization* of X. The spherical system of Y is obtained from the spherical system of X as follows:

- $S_Y^{\mathrm{p}} = S_X^{\mathrm{p}} \cap S_L$,
- $\Sigma_Y = \{\sigma \in \Sigma_X : \operatorname{supp} \sigma \subset S_L\}$,
- $\mathrm{A}_Y = \bigcup_{\alpha \in S_L \cap \Sigma_X} \Delta_X(\alpha)$ with the map c_X restricted to $\mathbb{Z}\mathrm{A}_Y \times \mathbb{Z}\Sigma_Y$.

In this case the spherical system of Y is said to be obtained from the spherical system of X by localization in S_L.

1.4. Luna's classification of wonderful varieties

Here we recall the statement of Luna's theorem of the classification of wonderful varieties (see [23, Théorème 1], [13, Corollary 3], [9, Theorem 1.2.3]). In our case the center of K always acts trivially, so here we assume for convenience that K is a semisimple complex algebraic group of adjoint type. Let T, B, and S be as above.

Every spherical root of any wonderful K-variety is the spherical root of a wonderful K-variety of rank 1, and the wonderful varieties of rank 1 are well known. In particular, the set $\Sigma(K)$ of the spherical roots of all the wonderful K-varieties is finite and is described by the following result.

THEOREM 1.1
Every spherical root σ of any wonderful K-variety, for any semisimple complex algebraic group K of adjoint type, belongs to Table 1.

There is an abstract notion of a Luna spherical system given as follows.

DEFINITION 1.2
A triple $(S^{\mathrm{p}}, \Sigma, \mathrm{A})$, where S^{p} is a subset of S, Σ is a subset of $\Sigma(K)$ without proportional elements, and A is a finite set endowed with a map $c\colon \mathrm{A} \times \Sigma \to \mathbb{Z}$ is called a *spherical K-system* if the following axioms hold.

Table 1. Spherical roots

Type of support	Spherical root
A_1	α
A_1	2α
$\mathsf{A}_1 \times \mathsf{A}_1$	$\alpha + \alpha'$
A_m	$\alpha_1 + \cdots + \alpha_m$
A_3	$\alpha_1 + 2\alpha_2 + \alpha_3$
B_m	$\alpha_1 + \cdots + \alpha_m$
B_m	$2(\alpha_1 + \cdots + \alpha_m)$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
C_m	$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{m-1}) + \alpha_m$
D_m	$2(\alpha_1 + \cdots + \alpha_{m-2}) + \alpha_{m-1} + \alpha_m$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$2\alpha_1 + \alpha_2$
G_2	$4\alpha_1 + 2\alpha_2$
G_2	$\alpha_1 + \alpha_2$

(A1) For all $D \in \mathrm{A}$, $c(D,\sigma) \leq 1$ for all $\sigma \in \Sigma$, and $c(D,\sigma) = 1$ only if $\sigma \in S$.

(A2) For all $\alpha \in S \cap \Sigma$, $\{D \in \mathrm{A} : c(D,\alpha) = 1\}$ has cardinality 2, and for all $\sigma \in \Sigma$

$$\sum_{D:c(D,\alpha)=1} c(D,\sigma) = \langle \alpha^\vee, \sigma \rangle.$$

(A3) For all $D \in \mathrm{A}$ there exists $\alpha \in S \cap \Sigma$ with $c(D,\alpha) = 1$.

(Σ1) For all $\alpha \in S$ such that $2\alpha \in \Sigma$, $\frac{1}{2}\langle \alpha^\vee, \sigma \rangle \in \mathbb{Z}_{\leq 0}$ for all $\sigma \in \Sigma \setminus \{2\alpha\}$.

(Σ2) For all α and β in S such that α and β are orthogonal and $\alpha + \beta \in \Sigma$, $\langle \alpha^\vee, \sigma \rangle = \langle \beta^\vee, \sigma \rangle$ for all $\sigma \in \Sigma$.

(S) For all $\sigma \in \Sigma$,

- if $\sigma = \alpha_1 + \cdots + \alpha_m$ with $\operatorname{supp}\sigma$ of type B_m, then
$$\{\alpha_2, \ldots, \alpha_{m-1}\} \subset S^{\mathrm{p}} \subset \big\{\alpha \in S : \langle \alpha^\vee, \sigma \rangle = 0\big\};$$
- if $\sigma = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{m-1}) + \alpha_m$ with $\operatorname{supp}\sigma$ of type C_m, then
$$\{\alpha_3, \ldots, \alpha_m\} \subset S^{\mathrm{p}} \subset \big\{\alpha \in S : \langle \alpha^\vee, \sigma \rangle = 0\big\};$$
- otherwise
$$\big\{\alpha \in \operatorname{supp}\sigma : \langle \alpha^\vee, \sigma \rangle = 0\big\} \subset S^{\mathrm{p}} \subset \big\{\alpha \in S : \langle \alpha^\vee, \sigma \rangle = 0\big\}.$$

The following is known as Luna's theorem of classification of wonderful varieties.

THEOREM 1.3

The map which associates to a wonderful K-variety X its spherical system (S^{p}_X, Σ_X, A_X) is a bijection between the set of wonderful K-varieties up to equivariant isomorphism and the set of spherical K-systems.

1.5. The spherical systems of the list

Here we show that the spherical systems given in the tables of Appendix B are indeed the spherical systems associated with $\mathrm{N}_K(K_e)$, the normalizers of the centralizers of the representatives e given in Appendix A. For all K, every spherical system given in the tables satisfies the axioms of Definition 1.2, so by Theorem 1.3 it is equal to a spherical system associated with a (uniquely determined up to conjugation) wonderful subgroup of K. Here we compute this wonderful subgroup for any spherical system of Appendix B.

1.5.1. Parabolic inductions and trivial factors

In all the spherical systems of Appendix B the set $(\operatorname{supp}\Sigma)\cup S^{\mathrm{p}}$ is properly contained in S. Therefore, the corresponding wonderful K-varieties X can be obtained by parabolic induction from wonderful L-varieties Y, where L is properly contained in K. We set $S_L=(\operatorname{supp}\Sigma)\cup S^{\mathrm{p}}$. Furthermore, in general $\operatorname{supp}\Sigma$ and $S^{\mathrm{p}}\setminus\operatorname{supp}\Sigma$ are orthogonal, so that L is a direct product $L_1\times L_2$, where $S_{L_1}=\operatorname{supp}\Sigma$ and $S_{L_2}=S^{\mathrm{p}}\setminus\operatorname{supp}\Sigma$, with L_2 acting trivially on Y. In many cases $S^{\mathrm{p}}\setminus\operatorname{supp}\Sigma$ is nonempty. Note that the above decomposition $L=L_1\times L_2$ is not uniquely determined, but here the center of L acts trivially on Y, so we do not care which part of the center of L is contained in the two factors L_1 and L_2.

In the following we will compute, in all our cases, the wonderful subgroups associated with the spherical systems obtained by localization in $S_{L_1}=\operatorname{supp}\Sigma$.

1.5.2. Trivial cases

In cases 1.1 ($r=1$), 2.1 ($r=1$), 3.1 ($r=1$), 4.1 ($r=1$), 5.1, 6.1, 7.1 ($r=1$), 8.1 ($r=1$), and 9.1 ($r=1$), the set Σ is empty, so the spherical system obtained by localization in $\operatorname{supp}\Sigma$ is trivial. More explicitly, the parabolic subgroups Q of K given in Appendix A are the wonderful subgroups associated with the given spherical K-systems.

1.5.3. Symmetric cases

In cases 1.1, 2.1, 3.1, 4.1, 4.2 ($q=1$), 4.3 ($p=1$), 7.1, 7.2 ($r=0$), 7.3 ($r=0$), 8.1, 8.2 ($r=0$), 8.3 ($r=0$), 9.1, 9.2 ($r=0$), and 9.3 ($r=0$), the spherical system obtained by localization in $\operatorname{supp}\Sigma$ is the spherical system of a symmetric subgroup $\mathrm{N}_{L_1}(L_1^\theta)$ of L_1, where L_1^θ is the fixed point subgroup of an involution θ of L_1. The wonderful symmetric subgroups and their spherical systems are well known (see, e.g., [8]). More precisely,

- in case 1.1 we get [8, case 6];
- in cases 2.1, 3.1, 7.3 ($r=0,p=1$), 8.2 ($r=0,q=1$), and 8.3 ($r=0,p=1$), we get [8, case 5];
- in cases 4.1, 4.2 ($q=1$), 4.3 ($p=1$), 7.1, 7.2 ($r=0,q=2$), 8.1, 9.1, 9.2 ($r=0,q=2$), and 9.3 ($r=0,p=2$), we get [8, case 2];
- in cases 7.3 ($r=0,p>1$), 8.2 ($r=0,q>1$), and 8.3 ($r=0,p>1$), we get [8, case 9];

- in cases 7.2 ($r=0, q>2$), 9.2 ($r=0, q>2$), and 9.3 ($r=0, p>2$), we get [8, case 15].

1.5.4. Other reductive cases

In cases 4.2 ($q>1$), 4.3 ($p>1$), 4.6, and 4.7, the spherical system obtained by localization in $\operatorname{supp}\Sigma$ is the spherical system of a wonderful reductive (but not symmetric) subgroup of L_1. More precisely,

- in cases 4.2 ($q>1$) and 4.3 ($p>1$) we get [8, case 42];
- in cases 4.6 and 4.7 we get [8, case 46 ($p=5$)].

1.5.5. Morphisms of type $\mathcal{L}$

Note that in all the above cases the Levi subgroup L such that $S_L=(\operatorname{supp}\Sigma)\cup S^{\mathrm{p}}$ is equal to K_h, the centralizer of h given in the list of Appendix A. In the remaining cases this is no longer true, but we have the following situation. In the remaining cases, namely, cases 4.4, 4.5, 7.2 ($r>0$), 7.3 ($r>0$), 8.2 ($r>0$), 8.3 ($r>0$), 9.2 ($r>0$), and 9.3 ($r>0$), the given spherical K-system $(S^{\mathrm{p}}, \Sigma, \mathrm{A})$ admits a distinguished set of colors Δ' such that the corresponding quotient

$$(S^{\mathrm{p}}/\Delta', \Sigma/\Delta', \mathrm{A}/\Delta')$$

is the spherical system of a wonderful K-variety which is obtained by parabolic induction from a wonderful K_h-variety. Indeed, $S_{K_h}=(\operatorname{supp}(\Sigma/\Delta'))\cup(S^{\mathrm{p}}/\Delta')$.

Such a distinguished set of colors Δ' is minimal, that is, does not contain any proper nonempty distinguished subset. Moreover, the corresponding quotient has higher defect, which means the following. The defect of a spherical system is defined as the nonnegative integer given by the difference between the number of colors and the number of spherical roots.

In all our cases, we have

$$\operatorname{card}(\Delta\smallsetminus\Delta')-\operatorname{card}(\Sigma/\Delta')>\operatorname{card}\Delta-\operatorname{card}\Sigma. \tag{1}$$

Therefore, the set Δ' corresponds to a minimal surjective equivariant morphism with connected fibers of type $\mathcal{L}$ in the sense of [5, Proposition 2.3.5]. In particular, the minimal quotients of higher defect have been studied in [7, Section 5.3]. Let us recall their description.

Let H_1 be the wonderful subgroup associated with the spherical K-system $(S^{\mathrm{p}}, \Sigma, \mathrm{A})$, let Δ' be a distinguished subset satisfying the condition (1), and let H_2 be the wonderful subgroup of K associated with the quotient of $(S^{\mathrm{p}}, \Sigma, \mathrm{A})$ by Δ'. We can assume $H_1\subset H_2$. Recall that the quotient H_2/H_1 is connected.

Under the condition (1) we have that H_1^{u} is properly contained in H_2^{u}. Take Levi decompositions $H_1=L_{H_1}H_1^{\mathrm{u}}$ and $H_2=L_{H_2}H_2^{\mathrm{u}}$ with $L_{H_1}\subset L_{H_2}$. Then $\operatorname{Lie}H_2^{\mathrm{u}}/\operatorname{Lie}H_1^{\mathrm{u}}$ is a simple L_{H_1}-module, and L_{H_1} and L_{H_2} differ only by their connected center. The defect of a spherical system is equal to the dimension of the connected center of the associated wonderful subgroup, so the codimension of L_{H_1} in L_{H_2} is equal to

$$d = \operatorname{card}(\Delta \smallsetminus \Delta') - \operatorname{card}(\Sigma/\Delta') - (\operatorname{card}\Delta - \operatorname{card}\Sigma).$$

The quotient $\operatorname{Lie} H_2^{\mathrm{u}}/\operatorname{Lie} H_1^{\mathrm{u}}$ can be described as follows. There exist $d+1$ L_{H_2}-submodules of $\operatorname{Lie} H_2^{\mathrm{u}}$, $W_0, \ldots, W_d$, isomorphic as L_{H_1}-modules but not as L_{H_2}-modules. By denoting by V the L_{H_2}-complement of $W_0 \oplus \cdots \oplus W_d$ in $\operatorname{Lie} H_2^{\mathrm{u}}$, as an L_{H_1}-module,

$$\operatorname{Lie} H_1^{\mathrm{u}} = W \oplus V,$$

where W is a cosimple L_{H_1}-submodule of $W_0 \oplus \cdots \oplus W_d$ which projects nontrivially on every summand $W_0, \ldots, W_d$.

As stated above, in our cases we always have $H_2 \subset Q$, with $Q = K_h Q^{\mathrm{u}}$ given in the list of Appendix A, $L_{H_2} \subset K_h$, and $H_2^{\mathrm{u}} = Q^{\mathrm{u}}$.

One can say something more about the inclusion of $W_0, \ldots, W_d$ in $\operatorname{Lie} Q^{\mathrm{u}}$. One has to consider the set $S_{\Delta'}$, whose general definition involves the notion of external negative color (see [5, Section 2.3.5] and [7, Section 5.2]). Without going into technical details, in our cases it holds that

$$S_{\Delta'} = (\operatorname{supp}\Sigma) \smallsetminus \big(\operatorname{supp}(\Sigma/\Delta')\big).$$

Moreover, $\operatorname{card} S_{\Delta'} = d+1$, say, $S_{\Delta'} = \{\beta_0, \ldots, \beta_d\}$. Assuming Q contains B^-, we have that $W_0, \ldots, W_d$ are, respectively, included in the simple L-submodules $V(-\beta_0), \ldots, V(-\beta_d)$ containing the root spaces of $-\beta_0, \ldots, -\beta_d$. In our cases the integer $d+1$, the cardinality of $S_{\Delta'}$, is always equal to 2 or 3.

In the following, for all the remaining cases, we describe the quotient of $(S^{\mathrm{p}}, \Sigma, \mathrm{A})$ by Δ' and describe L_{H_2} in K_h. The knowledge of $S_{\Delta'}$ will be enough to uniquely determine the modules $W_0, \ldots, W_d$.

REMARK 1.4

Actually, the results contained in [7] allow us to reduce the computation of the wonderful subgroup associated with a spherical system to the computation of the wonderful subgroups associated with somewhat smaller spherical systems. In particular, [7, Section 5.3] allows us to reduce the computation of the wonderful subgroup associated with a spherical system with a quotient of higher defect to the computation of the wonderful subgroups associated with some spherical subsystems. Moreover, many of the spherical systems under consideration have a tail (see [7, Section 6]), and these cases can also be reduced to some smaller cases. Similar general considerations could be done for the cases obtained by "collapsing" the tails. We prefer to avoid as far as possible the technicalities and give a direct explicit description of our wonderful subgroups even if they are already somewhat known.

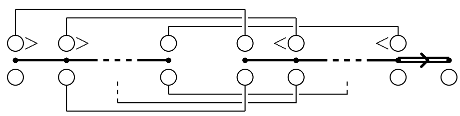
1.5.6. Type B

(a) Tail case. Localizing the spherical systems of cases 7.3 ($0 < r < p$), 8.2 ($0 < r < q$), and 8.3 ($0 < r < p$) in $\operatorname{supp}\Sigma$, we obtain the following spherical system,

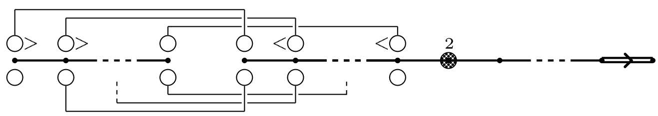
which we label as $\mathsf{a}^{\mathsf{y}}(s,s)+\mathsf{b}'(t)$, for a group of semisimple type $\mathsf{A}_s \times \mathsf{B}_{s+t}$ with $t \geq 1$:

$S^{\mathrm{p}} = \{\alpha'_{s+2}, \dots, \alpha'_{s+t}\}$,
$\Sigma = \{\alpha_1, \dots, \alpha_s, \alpha'_1, \dots, \alpha'_s, 2(\alpha'_{s+1} + \dots + \alpha'_{s+t})\}$,
$\mathrm{A} = \{D_1, \dots, D_{2s+1}\}$, with $\Delta = \mathrm{A} \cup \{D_{2s+2}\}$ and full Cartan pairing as follows:
$\alpha_1 = D_1 + D_2 - D_3$,
$\alpha_i = -D_{2i-2} + D_{2i-1} + D_{2i} - D_{2i+1}$ for $2 \leq i \leq s$,
$\alpha'_i = -D_{2i-1} + D_{2i} + D_{2i+1} - D_{2i+2}$ for $1 \leq i \leq s$,
$2(\alpha'_{s+1} + \dots + \alpha'_{s+t}) = -2D_{2s+1} + 2D_{2s+2}$.

If $t = 1$, then the Luna diagram is as follows.



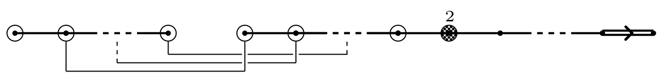
If $t > 1$, then it is as follows.



The combinatorics is the same, so from now on we just report the diagram for $t > 1$.

Consider the quotient by $\Delta' = \{D_{2i} : 1 \leq i \leq s\}$:

$$\Sigma/\Delta' = \left\{\alpha_2 + \alpha'_1, \dots, \alpha_s + \alpha'_{s-1}, 2(\alpha'_{s+1} + \dots + \alpha'_{s+t})\right\}.$$



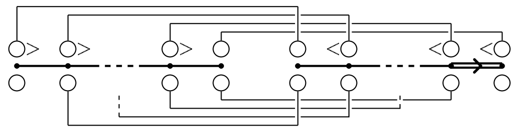
It is a spherical system obtained by parabolic induction from the direct product of case 2 and the rank 1 case 9 (resp., the rank 1 case 4) if $t > 1$ (resp., $t = 1$), the labels referring to [8]. We have $S_{\Delta'} = \{\alpha_1, \alpha'_s\}$.

(b) Collapsed tail. Localizing the spherical systems of cases 7.3 ($r = p$), 8.2 ($r = q$), and 8.3 ($r = p$) in $\operatorname{supp}\Sigma$, we obtain the following spherical system, which is labeled as $\mathsf{ab}^{\mathsf{y}}(s,s)$ or S-6 in [3], for a group of semisimple type $\mathsf{A}_s \times \mathsf{B}_s$:

$S^{\mathrm{p}} = \emptyset$,
$\Sigma = \{\alpha_1, \dots, \alpha_s, \alpha'_1, \dots, \alpha'_s\}$,
$\mathrm{A} = \{D_1, \dots, D_{2s+1}\} = \Delta$, with Cartan pairing as follows:
$\alpha_1 = D_1 + D_2 - D_3$,
$\alpha_i = -D_{2i-2} + D_{2i-1} + D_{2i} - D_{2i+1}$ for $2 \leq i \leq s$,
$\alpha'_i = -D_{2i-1} + D_{2i} + D_{2i+1} - D_{2i+2}$ for $1 \leq i \leq s-2$,

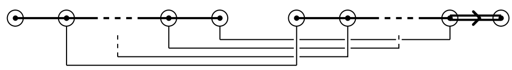
$\alpha'_{s-1} = -D_{2s-3} + D_{2s-2} + D_{2s-1} - D_{2s} - D_{2s+1}$,
$\alpha'_s = -D_{2s-1} + D_{2s} + D_{2s+1}$.

The Luna diagram is as follows.



Consider the quotient by $\Delta' = \{D_{2i} : 1 \le i \le s\}$:

$$\Sigma/\Delta' = \{\alpha_2 + \alpha'_1, \ldots, \alpha_s + \alpha'_{s-1}\}.$$



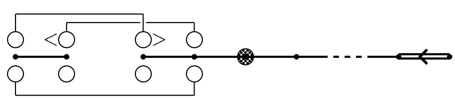
It is a spherical system obtained by parabolic induction from [8, case 2]. We have $S_{\Delta'} = \{\alpha_1, \alpha'_s\}$.

1.5.7. Type C

(a) Tail case. Localizing the spherical systems of cases 4.4 ($q > 2$) and 4.5 ($p > 2$) in $\operatorname{supp}\Sigma$, we obtain the following spherical system, which we label as $\mathsf{a}^{\mathsf{y}}(2,2) + \mathsf{c}(t)$, for a group of semisimple type $\mathsf{A}_2 \times \mathsf{C}_{t+1}$ with $t \ge 2$:

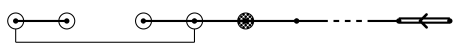
$S^{\mathrm{p}} = \{\alpha'_4, \ldots, \alpha'_{t+1}\}$,
$\Sigma = \{\alpha_1, \alpha_2, \alpha'_1, \alpha'_2, \alpha'_2 + 2(\alpha'_3 + \cdots + \alpha'_t) + \alpha'_{t+1}\}$,
$\mathrm{A} = \{D_1, \ldots, D_5\}$, with $\Delta = \mathrm{A} \cup \{D_6\}$ and full Cartan pairing as follows:
$\alpha_1 = -D_2 + D_3 + D_4 - D_5$,
$\alpha_2 = D_1 + D_2 - D_3$,
$\alpha'_1 = -D_3 + D_4 + D_5$,
$\alpha'_2 = -D_1 + D_2 + D_3 - D_4 - D_6$,
$\sigma_5 = -D_5 + D_6$.

The Luna diagram is as follows.



Consider the quotient by $\Delta' = \{D_2, D_4\}$:

$$\Sigma/\Delta' = \left\{\alpha_1 + \alpha'_2, \alpha'_2 + 2(\alpha'_3 + \cdots + \alpha'_t) + \alpha'_{t+1}\right\}.$$



It is a spherical system obtained by parabolic induction from [8, case 42], already considered in Section 1.5.4. We have $S_{\Delta'} = \{\alpha_2, \alpha'_1\}$.

(b) Collapsed tail. Localizing the spherical systems of cases 4.4 ($q=2$) and 4.5 ($p=2$) in $\operatorname{supp}\Sigma$, we obtain the spherical system $\mathsf{ab}^{\mathsf{y}}(2,2)$ for a group of semisimple type $\mathsf{A}_2 \times \mathsf{B}_2$, a particular case of the spherical system obtained above in Section 1.5.6.

1.5.8. Type D

(a) Tail case. Localizing the spherical systems of cases 7.2 ($0<r<q-1$), 9.2 ($0<r<q-1$), and 9.3 ($0<r<p-1$) in $\operatorname{supp}\Sigma$, we obtain the following spherical system for a group of semisimple type $\mathsf{A}_s \times \mathsf{D}_{s+t}$ with $t \geq 2$:

$S^{\mathrm{p}} = \{\alpha'_{s+2}, \ldots, \alpha'_{s+t}\}$,
$\Sigma = \{\alpha_1, \ldots, \alpha_s, \alpha'_1, \ldots, \alpha'_s, 2(\alpha'_{s+1} + \cdots + \alpha'_{s+t-2}) + \alpha'_{s+t-1} + \alpha'_{s+t}\}$,
$\mathrm{A} = \{D_1, \ldots, D_{2s+1}\}$, with $\Delta = \mathrm{A} \cup \{D_{2s+2}\}$ and full Cartan pairing as follows:
$\alpha_1 = D_1 + D_2 - D_3$,
$\alpha_i = -D_{2i-2} + D_{2i-1} + D_{2i} - D_{2i+1}$ for $2 \leq i \leq s$,
$\alpha'_i = -D_{2i-1} + D_{2i} + D_{2i+1} - D_{2i+2}$ for $1 \leq i \leq s$,
$\sigma_{2s+1} = -2D_{2s+1} + 2D_{2s+2}$.

It is [6, case 60], labeled as $\mathsf{a}^{\mathsf{y}}(s,s) + \mathsf{d}(t)$.

(b) Collapsed tail. Localizing the spherical systems of cases 7.2 ($r=q-1$), 9.2 ($r=q-1$), and 9.3 ($r=p-1$) in $\operatorname{supp}\Sigma$, we obtain the following spherical system for a group of semisimple type $\mathsf{A}_s \times \mathsf{D}_{s+1}$:

$S^{\mathrm{p}} = \emptyset$,
$\Sigma = \{\alpha_1, \ldots, \alpha_s, \alpha'_1, \ldots, \alpha'_s, \alpha'_{s+1}\}$,
$\mathrm{A} = \{D_1, \ldots, D_{2s+2}\} = \Delta$, with Cartan pairing as follows:
$\alpha_1 = D_1 + D_2 - D_3$,
$\alpha_i = -D_{2i-2} + D_{2i-1} + D_{2i} - D_{2i+1}$ for $2 \leq i \leq s-1$,
$\alpha_s = -D_{2s-2} + D_{2s-1} + D_{2s} - D_{2s+1} - D_{2s+2}$,
$\alpha'_i = -D_{2i-1} + D_{2i} + D_{2i+1} - D_{2i+2}$ for $1 \leq i \leq s-1$,
$\alpha'_s = -D_{2s-1} + D_{2s} + D_{2s+1} - D_{2s+2}$,
$\alpha'_{s+1} = -D_{2s-1} + D_{2s} - D_{2s+1} + D_{2s+2}$.

It is [6, case 40], labeled as $\mathsf{ad}^{\mathsf{y}}(s,s+1)$ or S-10 in [3], and considered also in [4, Section 5] as the spherical system of the comodel wonderful variety of cotype $\mathsf{D}_{2(s+1)}$.

2. Projective normality

This section is devoted to proving the following result, which we need in order to study the singularities of closures of spherical nilpotent K-orbits in $\mathfrak{p}$.

THEOREM 2.1

Let $(\mathfrak{g}, \mathfrak{k})$ be a classical symmetric pair of non-Hermitian type, and let $\mathcal{O} \subset \mathfrak{p}$ be a spherical nilpotent K-orbit. If $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sp}(2p+2q), \mathfrak{sp}(2p) + \mathfrak{sp}(2q))$, assume that the signed partition of $\mathcal{O}$ is neither $(+3^4, +1^{2p-8})$ nor $(-3^4, -1^{2q-8})$ (Cases 4.6 *and 4.7 in Appendix A). Let X be the wonderful K-variety associated to $\mathcal{O}$. Then the multiplication of sections*

$$m_{\mathcal{L},\mathcal{L}'}\colon \Gamma(X,\mathcal{L})\otimes\Gamma(X,\mathcal{L}')\longrightarrow\Gamma(X,\mathcal{L}\otimes\mathcal{L}')$$

is surjective for all globally generated line bundles $\mathcal{L},\mathcal{L}'\in\mathrm{Pic}(X)$.

We point out that multiplication is not surjective if $(\mathfrak{g},\mathfrak{k})=(\mathfrak{sp}(2p+2q),\mathfrak{sp}(2p)+\mathfrak{sp}(2q))$ and $\mathcal{O}$ is the spherical nilpotent orbit corresponding to the signed partitions $(+3^4,+1^{2p-8})$ or $(-3^4,-1^{2q-8})$) (see Example 2.7 below). These cases will be treated separately in Section 3.1 with an ad hoc argument.

Let us briefly recall here some generalities about the multiplication of sections of line bundles on a wonderful variety (for more details and references, see [4]). Let X be a wonderful K-variety with set of spherical roots Σ and set of colors Δ. The classes of colors form a free basis for the Picard group of X and for the semigroup of globally generated line bundles. Therefore, the Picard group of X is identified with $\mathbb{Z}\Delta$, and the semigroup of globally generated line bundles is identified with $\mathbb{N}\Delta$. Given $E,F\in\mathbb{N}\Delta$ we will also write $m_{E,F}$ meaning $m_{\mathcal{L}_E,\mathcal{L}_F}$.

Given $D\in\mathbb{Z}\Delta$ we denote by $\mathcal{L}_D\in\mathrm{Pic}(X)$ the corresponding line bundle, and we fix $s_D\in\Gamma(X,\mathcal{L}_D)$ a section whose associated divisor is D. Recall that every line bundle on X has a unique K-linearization. Then s_D is a highest weight vector, and we denote by $V_D\subset\Gamma(X,\mathcal{L}_D)$ the K-submodule generated by s_D. Since X is a spherical variety, $\Gamma(X,\mathcal{L}_D)$ is a multiplicity-free K-module; hence, V_D is uniquely determined and s_D is uniquely determined up to a scalar factor.

By identifying Σ with the set of K-stable prime divisors of X, every $\sigma\in\mathbb{Z}\Sigma$ determines a line bundle $\mathcal{L}_\sigma\in\mathrm{Pic}(X)$, and the map $\mathbb{Z}\Sigma\longrightarrow\mathrm{Pic}(X)$ is injective. The line bundle $\mathcal{L}_\sigma$ is effective if and only if $\sigma\in\mathbb{N}\Sigma$, and for all $\sigma\in\mathbb{Z}\Sigma$ we fix a section $s^\sigma\in\Gamma(X,\mathcal{L}_\sigma)$ whose associated divisor is σ. Such a section is a highest weight vector of weight 0 and is uniquely determined up to a scalar factor.

By identifying $\mathrm{Pic}(X)$ with $\mathbb{Z}\Delta$, we regard $\mathbb{Z}\Sigma$ as a sublattice of $\mathbb{Z}\Delta$. This defines a partial order $\leq_\Sigma$ on $\mathbb{Z}\Delta$ as follows: if $D,E\in\mathbb{Z}\Delta$, then $D\leq_\Sigma E$ if and only if $E-D\in\mathbb{N}\Sigma$. This allows us to describe the space of global sections of $\mathcal{L}_E$ as

$$\Gamma(X,\mathcal{L}_E)=\bigoplus_{F\in\mathbb{N}\Delta\colon F\leq_\Sigma E} s^{E-F}V_F.$$

In particular, if $E\in\mathbb{N}\Delta$, then we have that $\Gamma(X,\mathcal{L}_E)$ is an irreducible K-module if and only if E is minuscule in $\mathbb{N}\Delta$ with respect to $\leq_\Sigma$ or zero; that is, if $F\in\mathbb{N}\Delta$ and $F\leq_\Sigma E$, then it must be $F=E$.

To any line bundle $\mathcal{L}_E$ on X, we attach two characters ξ_E and ω_E as follows. Let H be the stabilizer of a point x_0 in the open orbit of X, fix a maximal torus T and a Borel subgroup B such that $T\subset B$, and let y_0 be the point fixed by the opposite of the Borel subgroup B. Then we denote by $\xi_E\in\mathrm{Hom}(H,\mathbb{C}^\times)$ the character given by the action of H over the fiber $\mathcal{L}_{E,x_0}$, and we denote by $\omega_E\in\mathrm{Hom}(T,\mathbb{C}^\times)$ the character given by the action of T over the fiber $\mathcal{L}_{E,y_0}$.

If $E \in \mathbb{N}\Delta$, then the set of sections $V_E \subset \Gamma(X, \mathcal{L}_E)$ does not vanish on the closed orbit of X, so it defines a regular map $\phi_E \colon X \longrightarrow \mathbb{P}(V_E^*)$. We choose a nonzero element $h_E \in V_E^*$ in the line $\phi_E(x_0)$. Note that V_E is the irreducible module of highest weight ω_E and that h_E is determined by the condition $g \cdot h_E = \xi_E(g) h_E$ for all $g \in H$.

For $D \in \Delta$, the weight ω_D is combinatorially described as follows: if $D \in \Delta^{2\mathrm{a}}$ and $\alpha \in S$ is such that $D \in \Delta(\alpha)$, then $\omega_D = 2\omega_\alpha$; otherwise $\omega_D = \sum \omega_\alpha$ for all $\alpha \in S$ such that $D \in \Delta(\alpha)$ (see [31, Lemma 30.24]).

2.1. General reductions

By making use of quotients and parabolic inductions, it is possible to reduce the study of the multiplication maps. We recall such reductions from [4].

LEMMA 2.2 ([4, COROLLARY 1.4])

Let X be a wonderful variety with set of colors Δ, let X' be a quotient of X by a distinguished subset $\Delta_0 \subset \Delta$ with set of colors Δ', and identify Δ' with $\Delta \smallsetminus \Delta_0$. If $D \in \mathbb{N}\Delta$ and $\operatorname{supp}(D) \cap \Delta_0 = \varnothing$ and if $\mathcal{L}_D \in \operatorname{Pic}(X)$ and $\mathcal{L}'_D \in \operatorname{Pic}(X')$ are the line bundles corresponding to D regarded as an element in $\mathbb{N}\Delta$ and in $\mathbb{N}\Delta'$, respectively, then $\Gamma(X, \mathcal{L}_D) = \Gamma(X', \mathcal{L}'_D)$. In particular, if $m_{D,E}$ is surjective for all $D, E \in \mathbb{N}\Delta$, then $m_{D',E'}$ is surjective for all $D', E' \in \mathbb{N}\Delta'$.

LEMMA 2.3 ([4, PROPOSITION 1.6])

Let X be a wonderful variety, and suppose that X is the parabolic induction of a wonderful variety X'. Then for all $\mathcal{L}, \mathcal{L}'$ in $\operatorname{Pic}(X)$ the multiplication $m_{\mathcal{L},\mathcal{L}'}$ is surjective if and only if the multiplication $m_{\mathcal{L}|_{X'},\mathcal{L}'|_{X'}}$ is surjective.

We now explain how to reduce the study of the multiplication maps with respect to wonderful subvarieties.

LEMMA 2.4

Let X be a wonderful variety, and let $X' \subset X$ be a wonderful subvariety. If $m_{\mathcal{L},\mathcal{L}'}$ is surjective for all globally generated $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic}(X)$, then $m_{\mathcal{L},\mathcal{L}'}$ is surjective for all globally generated $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic}(X')$.

Proof

Denote by Σ and Δ the set of spherical roots and the set of colors of X, respectively, and denote by Σ' and Δ' those of X'. The restriction of line bundles induces a map $\rho \colon \mathbb{N}\Delta \longrightarrow \mathbb{N}\Delta'$, and the restriction of sections $\Gamma(X, \mathcal{L}_D) \longrightarrow \Gamma(X', \mathcal{L}_{\rho(D)})$ is surjective for all $D \in \mathbb{N}\Delta$. Given $E, F \in \mathbb{N}\Delta$, the surjectivity of $m_{\rho(E),\rho(F)}$ follows then from the surjectivity of $m_{E,F}$.

Set

$$\Delta'_0 = \{ D \in \Delta' : c(D, \sigma) \leq 0 \ \forall \sigma \in \Sigma' \}.$$

Note that every $D \in \mathbb{N}\Delta'_0$ is minuscule with respect to $\leq_{\Sigma'}$ or zero; namely, $\Gamma(X', \mathcal{L}_D) = V_D$ for all $D \in \mathbb{N}\Delta'$. Indeed, if $D \in \mathbb{N}\Delta'_0$ and $D - \sigma \in \mathbb{N}\Delta$ for some $\sigma \in \mathbb{N}\Sigma$, then it follows that $-\sigma \in \mathbb{N}\Delta$; hence, both σ and $-\sigma$ define effective divisors on X'. On the other hand the cone of effective divisors of X' contains no line, since X' is complete; therefore, it must be $\sigma = 0$.

Let $D \in \Delta$. By reasoning as in [15, Section 1.13] with the combinatorial description of ρ, it follows that for all $D \in \Delta$ there exists $D' \in (\Delta' \smallsetminus \Delta'_0) \cup \{0\}$ such that $\rho(D) - D' \in \mathbb{N}\Delta'_0$, and conversely for all $D' \in \Delta' \smallsetminus \Delta'_0$ there exists $D \in \Delta$ with $\rho(D) - D' \in \mathbb{N}\Delta'_0$.

Now let $E, F \in \mathbb{N}\Delta'$. Then by the previous discussion there exist $E', F' \in \mathbb{N}\Delta'_0$ such that $E + E', F + F' \in \rho(\mathbb{N}\Delta)$. On the other hand since $E', F' \in \mathbb{N}\Delta'_0$ we have $\Gamma(X, \mathcal{L}_{E+E'+F+F'}) = \Gamma(X, \mathcal{L}_{E+F})V_{E'+F'}$ and

$$\mathrm{Im}(m_{E+E',F+F'}) = \mathrm{Im}(m_{E,F})V_{E'}V_{F'} = \mathrm{Im}(m_{E,F})V_{E'+F'}.$$

Therefore, the surjectivity of $m_{E,F}$ follows from that of $m_{E+E',F+F'}$. $\square$

A strategy to prove the surjectivity of the multiplication map was described in [11] for wonderful symmetric varieties and in [4] for general wonderful varieties. Such a strategy reduces the proof of the surjectivity of the multiplication maps for all pairs of globally generated line bundles to a finite number of computations, which arise in correspondence to the so-called *fundamental low triples*.

Recall from [4] that a triple $(D, E, F) \in (\mathbb{N}\Delta)^3$ with $F \leq_\Sigma D + E$ is called a *low triple* if, for all $D', E' \in \mathbb{N}\Delta$ such that $D' \leq_\Sigma D$, $E' \leq_\Sigma E$, and $F \leq_\Sigma D' + E'$, it holds that $D' = D$ and $E' = E$. The triple (D, E, F) is called a *fundamental triple* if $D, E \in \Delta$.

To determine the low triples, the notion of covering difference is useful. Let $E, F \in \mathbb{N}\Delta$ with $E <_\Sigma F$, and suppose that E is maximal in $\mathbb{N}\Delta$ with this property: then we say that F covers E and we call $F - E$ a *covering difference* in $\mathbb{N}\Delta$.

For all $E = \sum_{D\in\Delta} k_D D \in \mathbb{Z}\Delta$, define its *positive part* $E^+ = \sum_{k_D>0} k_D D$, its *negative part* $E^- = E^+ - E$, and its *height* $\mathrm{ht}(E) = \sum_{D\in\Delta} k_D$. Note that $\gamma \in \mathbb{N}\Sigma$ is a covering difference in $\mathbb{N}\Delta$ if and only if γ^+ covers γ^-.

As noted in [4, Section 2.1, Remark], the covering differences in $\mathbb{N}\Delta$ are finitely many; therefore, there is always a bound for the height of the positive part of a covering difference. In all the examples we know (including those we will deal with in the present article) this bound can be taken to be 2.

Let (D, E, F) be a low triple, and suppose that $m_{D,E}$ is surjective. Then it is a straightforward consequence of the definition that $s^{D+E-F}V_F \subset V_D V_E$. On the other hand we have the following.

LEMMA 2.5 ([4, LEMMA 2.3])

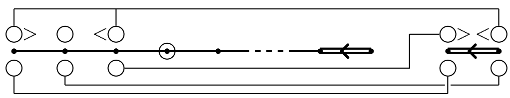
Let X be a wonderful variety, and let n be such that $\mathrm{ht}(\gamma^+) \leq n$ for every covering difference γ. If $s^{D+E-F}V_F \subset V_D V_E$ for all low triples (D, E, F) with $\mathrm{ht}(D+E) \leq n$, then the multiplication maps $m_{D,E}$ are surjective for all $D, E \in \mathbb{N}\Delta$.

To verify that $s^{D+E-F}V_F \subset V_D V_E$ we will make use of the following.

LEMMA 2.6 ([10, LEMMA 19])
Let $D, E, F \in \mathbb{N}\Delta$ be such that $D \leq_\Sigma E + F$. Then $s^{E+F-D}V_D \subset V_E V_F$ if and only if the projection of $h_E \otimes h_F \in V(\omega_E^*) \otimes V(\omega_F^*)$ onto the isotypic component of highest weight ω_D^* is nonzero.

EXAMPLE 2.7
Let $\mathfrak{g} = \mathfrak{sp}(2p + 2q)$ and $\mathfrak{k} = \mathfrak{sp}(2p) + \mathfrak{sp}(2q)$. If $p \geq 4$, consider the spherical nilpotent K-orbit $\mathcal{O}$ defined by the signed partition $(+3^4, +1^{2p-8})$ (or similarly the one defined by $(-3^4, -1^{2q-8})$ if $q \geq 4$). Let X be the corresponding wonderful K-variety. Then there are elements $D, E \in \mathbb{N}\Delta$ such that $m_{D,E}$ is not surjective.

Indeed, the spherical system of X is the following.



Label the spherical roots and the colors of X as

$$\sigma_1 = \alpha_2, \qquad \sigma_2 = \alpha_2', \qquad \sigma_3 = \alpha_1, \qquad \sigma_4 = \alpha_1', \qquad \sigma_5 = \alpha_3,$$
$$D_1 = D_{\alpha_2}^+, \qquad D_2 = D_{\alpha_2}^-, \qquad D_3 = D_{\alpha_1}^+, \qquad D_4 = D_{\alpha_1}^-,$$
$$D_5 = D_{\alpha_3}^-, \qquad D_6 = D_{\alpha_4}.$$

Then the Cartan pairing of X is expressed as

$$\sigma_1 = D_1 + D_2 - D_3,$$
$$\sigma_2 = -D_1 + D_2 + D_3 - D_4 - D_5,$$
$$\sigma_3 = -D_2 + D_3 + D_4 - D_5,$$
$$\sigma_4 = -D_3 + D_4 + D_5,$$
$$\sigma_5 = -D_2 + D_3 - D_4 + D_5 - D_6.$$

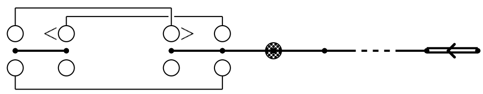
Consider the triple $(D_3, D_3, D_1 + D_2 + D_6)$. Then $2D_3 - D_1 - D_2 - D_6 = \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5$, and the triple is easily shown to be low. On the other hand if $V_{D_1+D_2+D_6} \subset V_{D_3}^2$, then it would be $V(2\omega_2 + \omega_4 + \omega_2') \subset V(\omega_1 + \omega_3 + \omega_2')^{\otimes 2}$, which is not the case. Therefore, m_{D_3,D_3} is not surjective.

2.2. Basic cases

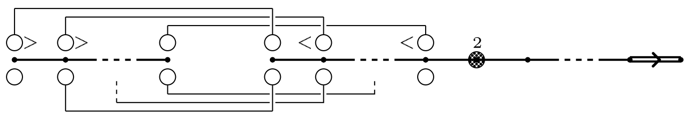
We show in this section that, in order to prove Theorem 2.1, we are reduced to the study of three special families of wonderful varieties. By following Section 1.5.1 and by Lemma 2.3, the surjectivity of the multiplications on X is reduced to that one on a wonderful L_1-variety Y, where L_1 is the Levi subgroup of K corresponding to the set of simple roots in $\operatorname{supp}\Sigma$. More precisely, Y is the localization of X at the subset $\operatorname{supp}\Sigma \subset S$, and the wonderful varieties arising in this way are described in Sections 1.5.2, 1.5.3, 1.5.4 (only cases 4.2 and 4.3), 1.5.6, 1.5.7, and 1.5.8.

Analyzing all the possible cases, we now show that to prove the surjectivity of the multiplications for Y we are reduced to the following three families:

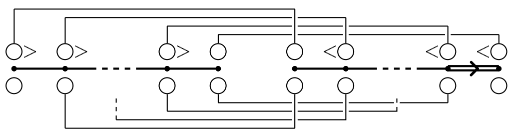
- $\mathsf{a}^{\mathsf{y}}(2,2)+\mathsf{c}(t)$, $t \geq 2$,



- $\mathsf{a}^{\mathsf{y}}(s,s)+\mathsf{b}'(t)$, $s,t \geq 1$,



- $\mathsf{ab}^{\mathsf{y}}(s,s)$, $s \geq 2$.



In the cases of Section 1.5.2 the wonderful variety X is a flag variety. Therefore, the surjectivity of the multiplication of globally generated line bundles holds trivially, since the space of sections of a globally generated line bundle on a flag variety is an irreducible K-module.

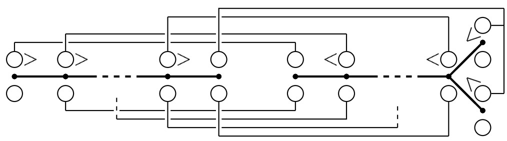
In the cases of Section 1.5.3 the wonderful variety Y is the wonderful compactification of an adjoint symmetric variety, and the surjectivity of the multiplication of globally generated line bundles holds thanks to [11].

In cases 4.2 and 4.3 of Section 1.5.4 (up to switching the two factors of K) the surjectivity of the multiplications of Y is reduced to that one of the wonderful variety Z with spherical system $\mathsf{a}^{\mathsf{y}}(2,2)+\mathsf{c}(t)$ where $t \geq 2$. More precisely, start with Z and consider the set of colors $\{D^+_{\alpha_1}, D^+_{\alpha_2}\}$. It is distinguished, and the corresponding quotient is a parabolic induction of Y. Therefore, the surjectivity of the multiplications of Y follows from that of Z thanks to Lemmas 2.2 and 2.3.

In the cases of Section 1.5.6(a) Y is the wonderful variety with spherical system $\mathsf{a}^{\mathsf{y}}(s,s)+\mathsf{b}'(t)$, where $s \geq 0$ and $t \geq 1$, but if $s=0$, then it is just an adjoint symmetric variety. In the cases of Section 1.5.6(b) Y is the wonderful variety with spherical system $\mathsf{ab}^{\mathsf{y}}(s,s)$, where $s \geq 2$.

In the cases of Section 1.5.7(a) Y is the wonderful variety with spherical system $\mathsf{a}^{\mathsf{y}}(2,2)+\mathsf{c}(t)$, where $t \geq 2$, whereas in the cases of Section 1.5.7(b) Y is the wonderful variety with spherical system $\mathsf{ab}^{\mathsf{y}}(2,2)$.

In the cases of Section 1.5.8(a) Y is the wonderful variety with spherical system $\mathsf{a}^{\mathsf{y}}(s,s)+\mathsf{d}(t)$, where $s\geq 0$ and $t\geq 2$. The surjectivity of the multiplications in this case can be reduced to that of a comodel wonderful variety, which is known by [4, Theorem 5.2]. Let indeed Z be the comodel wonderful variety of cotype $\mathsf{D}_{2(s+t)}$. This is the wonderful variety with the following spherical system for a group of semisimple type $\mathsf{A}_{s+t-1}\times\mathsf{D}_{s+t}$.

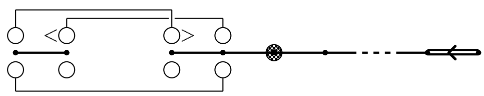


Consider the wonderful subvariety of Z associated to $\Sigma\smallsetminus\{\alpha_{s+1},\dots,\alpha_{s+t-1}\}$. Then the set of colors $\{D^-_{\alpha'_{s+1}}, D^\pm_{\alpha'_{s+2}},\dots,D^\pm_{\alpha'_{s+t}}\}$ is distinguished, and the corresponding quotient is a parabolic induction of Y. Therefore, the surjectivity of the multiplications of Y follows from that of Z thanks to Lemmas 2.2 and 2.3.

Finally, in the cases of Section 1.5.8(b) Y is the comodel wonderful variety of cotype $\mathsf{D}_{2(s+1)}$, and the surjectivity of the multiplications for this variety follows by [4, Theorem 5.2].

2.3. Projective normality of $\mathsf{a}^{\mathsf{y}}(2,2)+\mathsf{c}(t)$

Consider the wonderful variety X for a semisimple group G of type $\mathsf{A}_2\times\mathsf{C}_{t+1}$ with $t\geq 2$ defined by the following spherical system.



The spherical system associated to this Luna diagram is described in Section 1.5.7. For convenience we number the five spherical roots as

$$\sigma_1=\alpha_2,\qquad \sigma_2=\alpha'_2,\qquad \sigma_3=\alpha_1,\qquad \sigma_4=\alpha'_1,\qquad \sigma_5=\alpha'_2+\sum_{i=3}^{t}2\alpha'_i+\alpha'_{t+1}.$$

There are six colors that we label as

$$D_1=D^-_{\alpha_2},\qquad D_2=D^+_{\alpha_2},\qquad D_3=D^-_{\alpha_1},$$
$$D_4=D^+_{\alpha_1},\qquad D_5=D^-_{\alpha'_1},\qquad D_6=D_{\alpha'_3}.$$

The weights of these colors are

$$\omega_{D_1}=\omega_2,\qquad \omega_{D_2}=\omega_2+\omega'_2,\qquad \omega_{D_3}=\omega_1+\omega'_2,$$
$$\omega_{D_4}=\omega_1+\omega'_1,\qquad \omega_{D_5}=\omega'_1,\qquad \omega_{D_6}=\omega'_3.$$

Note that the G-stable divisor of X corresponding to σ_5 is a parabolic induction of a comodel wonderful variety of cotype A_5 (see [4, Section 5]). Therefore, we can restrict our study to the covering differences and the low triples of X which contain σ_5.

LEMMA 2.8

Let $\gamma \in \mathbb{N}\Sigma$ be a covering difference in $\mathbb{N}\Delta$ with $\sigma_5 \in \operatorname{supp}_\Sigma \gamma$. Then either $\gamma = \sigma_5 = -D_5 + D_6$ or $\gamma = \sigma_2 + \sigma_4 + \sigma_5 = -D_1 + D_2$. Every other covering difference $\gamma \in \mathbb{N}\Sigma$ verifies $\operatorname{ht}(\gamma^+) = 2$.

Proof

Denote $\gamma = \sum a_i \sigma_i$. Then we have

$$\begin{aligned}\gamma = {} & (a_1 - a_2)D_1 + (a_1 + a_2 - a_3)D_2 + (-a_1 + a_2 + a_3 - a_4)D_3 \\ & + (-a_2 + a_3 + a_4)D_4 + (-a_3 + a_4 - a_5)D_5 + (-a_2 + a_5)D_6.\end{aligned} \tag{2}$$

Suppose that $a_5 \neq 0$. If $D_5 \in \operatorname{supp}(\gamma^-)$, then $\gamma^- + \sigma_5 \in \mathbb{N}\Delta$, and if $D_6 \in \operatorname{supp}(\gamma^+)$, then $\gamma^+ - \sigma_5 \in \mathbb{N}\Delta$. Therefore, if $\gamma \neq \sigma_5$, then it must be $D_5 \notin \operatorname{supp}(\gamma^-)$ and $D_6 \notin \operatorname{supp}(\gamma^+)$; namely, $a_3 + a_5 \leq a_4$ and $a_5 \leq a_2$. It follows that $a_2 > 0$ and $a_4 > 0$. Suppose that $\sigma \neq \sigma_2 + \sigma_4 + \sigma_5 = -D_1 + D_2$. Then $a_1 + a_4 \leq a_2 + a_3$ since $\gamma^- + \sigma_4 \notin \mathbb{N}\Delta$, and $a_2 \leq a_1$ since $\gamma^- + \sigma_2 + \sigma_4 + \sigma_5 \notin \mathbb{N}\Delta$. Therefore, we get $a_1 + (a_4 - a_3) \leq a_2 \leq a_1$, which is absurd since $a_4 - a_3 \geq a_5 > 0$.

As already noted, the G-stable divisor of X corresponding to σ_5 is a parabolic induction of a comodel wonderful variety of cotype A_5. Therefore, the covering differences γ with $\sigma_5 \notin \operatorname{supp}_\Sigma \gamma$ coincide with those studied in [4, Proposition 3.2], and they all satisfy $\operatorname{ht}(\gamma^+) = 2$. $\square$

LEMMA 2.9

Let (D, E, F) be a low fundamental triple, denote $\gamma = D + E - F$, and suppose that $\sigma_5 \in \operatorname{supp}_\Sigma \gamma$. Then we have the following possibilities:

- $(D_2, D_3, D_1 + D_4 + D_5)$, $\gamma = \sigma_2 + \sigma_5$;
- $(D_3, D_3, D_1 + 2D_5)$, $\gamma = \sigma_2 + \sigma_3 + \sigma_5$;
- $(D_2, D_2, D_4 + D_5)$, $\gamma = \sigma_1 + \sigma_2 + \sigma_5$;
- $(D_2, D_3, 2D_5)$, $\gamma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_5$;
- $(D_3, D_4, D_1 + D_5)$, $\gamma = \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5$;
- (D_4, D_4, D_1), $\gamma = \sigma_2 + \sigma_3 + 2\sigma_4 + \sigma_5$.

Proof

By Lemma 2.8, $\sigma_5 = -D_5 + D_6$ and $\sigma_2 + \sigma_4 + \sigma_5 = -D_1 + D_2$ are the unique covering differences γ with $\operatorname{ht}(\gamma^+) = 1$. Therefore, D_1, D_3, D_4, D_5 are minuscule in $\mathbb{N}\Delta$.

Let (D, E, F) be a fundamental triple with $\operatorname{supp}(F) \cap \operatorname{supp}(D + E) = \varnothing$, denote $\gamma = D + E - F = \sum a_i \sigma_i$, and suppose $a_5 > 0$. Note that if (D, E, F) is a low triple, then $D_6 \notin \operatorname{supp}(\gamma^+)$. Suppose that indeed $D = D_6$. Then $D_5 <_\Sigma D$ and $F \leq_\Sigma D_5 + E$. Therefore, if (D, E, F) is a low triple, then (2) implies $0 < a_5 \leq a_2$.

Suppose $a_4 = 0$. Then for every covering difference $\sigma \leq \gamma$ it holds that $\operatorname{ht}(\sigma^+) = 2$. Therefore, (D, E, F) is necessarily a low triple.

To classify such fundamental triples, suppose $D_2 \notin \operatorname{supp}(\gamma^+)$. Then $c(D_2, \gamma) \leq 0$; hence, $a_1 + a_2 \leq a_3$ and we get $2 \leq 2a_2 \leq a_2 + a_3 - a_1$. Since

$\mathrm{ht}(\gamma^+) = 2$, it follows that $D = E = D_3$. Equivalently, we have the equality $c(D_3, \gamma) = -a_1 + a_2 + a_3 = 2$, and the inequalities $c(D_2, \gamma) \le 0$, $c(D_4, \gamma) \le 0$ imply $2a_1 - a_3 + 2 \le a_3 \le a_1 - a_3 + 2$. It follows that $a_1 = 0$ and $a_2 = a_3 = 1$, and the inequality $0 < a_5 \le a_2$ implies $a_5 = 1$. Therefore, $\gamma = \sigma_2 + \sigma_3 + \sigma_5$ and $F = D_1 + 2D_5$.

Similarly, suppose that $a_4 = 0$ and $D_3 \notin \mathrm{supp}(\gamma^+)$. Then $c(D_3, \gamma) \le 0$. Hence, $a_2 + a_3 \le a_1$, and we get $2 \le 2a_2 \le a_1 + a_2 - a_3$. Since $\mathrm{ht}(\gamma^+) = 2$, it follows that $D = E = D_2$. Equivalently, $c(D_2, \gamma) = a_1 + a_2 - a_3 = 2$, and the inequalities $c(D_1, \gamma) \le 0$, $c(D_3, \gamma) \le 0$ imply $a_2 + a_3 \le a_1 \le a_2$. It follows that $a_3 = 0$ and $a_1 = a_2 = 1$, and the inequality $0 < a_5 \le a_2$ implies $a_5 = 1$. Therefore, $\gamma = \sigma_1 + \sigma_2 + \sigma_5$ and $F = D_4 + D_5$.

Suppose now that $a_4 = 0$ and $\gamma^+ = D_2 + D_3$. Then the equalities $c(D_2, \gamma) = c(D_3, \gamma) = 1$ imply $a_3 - a_1 = a_2 - 1 = 1 - a_2$, and it follows that $a_1 = a_3$ and $a_2 = 1$. Therefore, the inequality $0 < a_5 \le a_2$ implies $a_5 = 1$, and the inequality $c(D_1, \gamma) \le 0$ implies $a_1 \le a_2$. Therefore, either $\gamma = \sigma_2 + \sigma_5$ and $F = D_1 + D_4 + D_5$, or $\gamma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_5$ and $F = 2D_5$.

Suppose finally that $a_4 > 0$. Note that if (D, E, F) is a low triple, then $D_2 \notin \mathrm{supp}(\gamma^+)$. Indeed, $\sigma_2 + \sigma_4 + \sigma_5 \le \gamma$, and if, for example, $D = D_2$, then $D_1 <_\Sigma D$ and $F \le_\Sigma D_1 + E$. Therefore, $c(D_2, \gamma) \le 0$; hence, $0 < a_1 + a_2 \le a_3$. It follows that $c(D_4, \gamma) = -a_2 + a_3 + a_4 \ge a_1 + a_4 > 0$; therefore, $D_4 \in \mathrm{supp}(\gamma^+)$. Since $\mathrm{ht}(\gamma^+) = 2$, in particular, it must be that $a_4 \le 2$.

Suppose that $a_4 = 1$. Then $c(D_3, \gamma) = -a_1 + a_2 + a_3 - a_4 \ge 2a_2 - a_4 > 0$; hence, $\gamma^+ = D_3 + D_4$. Therefore, $c(D_3, \gamma) = c(D_4, \gamma) = 1$ and we get the equalities $a_2 + a_3 = a_1 + 2$ and $a_2 = a_3$. The inequality $c(D_2, \gamma) \le 0$ then implies that $a_1 + a_2 \le a_2$; hence, $a_1 = 0$, $a_2 = a_3 = 1$, and $a_5 = 1$ thanks to the inequality $0 < a_5 \le a_2$. Therefore, $\gamma = \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5$ and $F = D_1 + D_5$, and (D, E, F) is a low triple since D_3, D_4 are both minuscule.

Suppose that $a_4 = 2$. Then $c(D_4, \gamma) = -a_2 + a_3 + a_4 \ge a_1 + a_4 \ge 2$, and since $\mathrm{ht}(\gamma^+) = 2$ it follows that $\gamma^+ = 2D_4$. Moreover, we get $a_1 = 0$ and $a_2 = a_3$. By the inequalities $c(D_2, \gamma) \le 0$, $c(D_3, \gamma) \le 0$ we then get that $a_1 + a_2 \le a_3$ and $-a_1 + a_2 + a_3 - a_4 \le 0$. On the other hand $c(D_3, \gamma) = -a_1 + a_2 + a_3 - a_4 \ge 2a_2 - a_4 = 2a_2 - 2 \ge 0$. Therefore, $c(D_3, \gamma) = 0$ and it follows that $a_2 = 1$. Thanks to the inequality $0 < a_5 \le a_2$, we have $a_5 = 1$ as well. Therefore, $\gamma = \sigma_2 + \sigma_3 + 2\sigma_4 + \sigma_5$ and $F = D_1$, and (D, E, F) is a low triple since D_4 is minuscule. $\square$

To prove the projective normality of X we now apply Lemma 2.6. This requires some computations. We first need an explicit description of the invariants. Let $V = \mathbb{C}^3$ with standard basis given by e_1, e_2, e_3. Let $W = \mathbb{C}^{2n}$, where $n = t + 1$. We choose a basis $e'_1, \ldots, e'_n, e'_{-n}, \ldots, e'_{-1}$ and fix a symplectic form such that $\omega(e'_i, e'_j) = \delta_{i,-j}$ for $i > 0$.

We set $\Lambda^2_0 W = \{\alpha \in \Lambda^2 W : \langle \omega, \alpha \rangle = 0\}$ and $\omega^* = \sum_{i=1}^n e'_i \wedge e'_{-i}$. Let $\varphi_1, \varphi_2, \varphi_3$ be the basis of V^* dual to e_1, e_2, e_3. Note that the isomorphism from $\Lambda^2 V$ to V^* sending $e_1 \wedge e_2$ to φ_3, $e_1 \wedge e_3$ to $-\varphi_2$, and $e_2 \wedge e_3$ to φ_1 is G-equivariant. We set $G = \mathrm{SL}(V^*) \times \mathrm{Sp}(W, \omega)$, so that we can take H as the stabilizer of the line spanned by the vector $e = e_1 \otimes e'_{-2} - e_2 \otimes e'_2 - e_3 \otimes e'_1$. We denote by h_i the vector $h_{D_i} \in V^*_{D_i}$. In coordinates the vectors h_i are given as

- $V^*_{D_4} = V \otimes W$ and $h_4 = e$;
- $V^*_{D_5} = W$ and $h_5 = e'_1$;
- $V^*_{D_3} = V \otimes \Lambda^2_0 W$ and $h_3 = e_1 \otimes (e'_1 \wedge e'_{-2}) - e_2 \otimes (e'_1 \wedge e'_2)$;
- $V^*_{D_2} = V^* \otimes \Lambda^2_0 W$ and
$$h_2 = \varphi_3 \otimes \left(e'_2 \wedge e'_{-2} - \frac{1}{n}\omega^*\right) - \varphi_2 \otimes (e'_1 \wedge e'_{-2}) - \varphi_1 \otimes (e'_1 \wedge e'_2);$$
- $V^*_{D_1} = V^*$ and $h_1 = \varphi_3$.

We can now prove the following result.

PROPOSITION 2.10

The multiplication $m_{D,E}$ is surjective for all $D, E \in \mathbb{N}\Delta$.

Proof

By Lemma 2.8 every covering difference $\gamma \in \mathbb{N}\Sigma$ satisfies $\mathrm{ht}(\gamma^+) \leq 2$. Therefore, by Lemma 2.5 it is enough to check that $s^{D+E-F} V_F \subset V_D \cdot V_E$ for all low fundamental triples (D, E, F).

Suppose that $\sigma_5 \notin \mathrm{supp}_\Sigma(D + E - F)$, and let X' be the G-stable divisor of X corresponding to σ_5. Then X' is a parabolic induction of a comodel wonderful variety of cotype A_5; hence, the inclusion $s^\gamma V_F \subset V_D \cdot V_E$ follows by Lemma 2.3 together with [4, Theorem 5.2].

By Lemma 2.6 we are reduced to proving that for all low triples (D_i, D_j, F) listed in Lemma 2.9 the projection of $h_i \otimes h_j$ onto the isotypic component of type V^*_F in $V^*_{D_i} \otimes V^*_{D_j}$ is nonzero.

$(D_2, D_3, D_1 + D_4 + D_5)$. We have $V^*_{D_1+D_4+D_5} = \mathfrak{sl}(V) \otimes \mathsf{S}^2 W$, the equivariant map

$$\pi\colon (V^* \otimes \Lambda^2_0 W) \otimes (V \otimes \Lambda^2_0 W) \longrightarrow \mathfrak{sl}(V) \otimes \mathsf{S}^2 W$$

given by

$$\begin{aligned}&\pi\big((\varphi \otimes a \wedge b) \otimes (v \otimes c \wedge d)\big)\\ &\quad = \left(\varphi \otimes v - \frac{1}{3}\varphi(v)\mathrm{Id}\right) \otimes \big(\omega(a,c)bd - \omega(b,c)ad - \omega(a,d)bc + \omega(b,d)ac\big),\end{aligned}$$

and

$$\pi(h_2 \otimes h_3) = (\varphi_3 \otimes e_1) \otimes e'_1 e'_{-2} - (\varphi_3 \otimes e_2) \otimes e'_1 e'_2 + (\varphi_1 \otimes e_1 + \varphi_2 \otimes e_2) \otimes (e'_1)^2 \neq 0.$$

$(D_3, D_3, D_1 + 2D_5)$. We have $V^*_{D_1+2D_5} = \Lambda^2 V \otimes \mathsf{S}^2 W$, the equivariant map

$$\pi\colon (V \otimes \Lambda^2_0 W) \otimes (V \otimes \Lambda^2_0 W) \longrightarrow \Lambda^2 V \otimes \mathsf{S}^2 W$$

given by

$$\begin{aligned}&\pi\big((u \otimes a \wedge b) \otimes (v \otimes c \wedge d)\big)\\ &\quad = (u \wedge v) \otimes \big(\omega(a,c)bd - \omega(b,c)ad - \omega(a,d)bc + \omega(b,d)ac\big),\end{aligned}$$

and

$$\pi(h_3 \otimes h_3) = 2(e_1 \wedge e_2) \otimes (e_1')^2 \neq 0.$$

$(D_2, D_2, D_4 + D_5)$. We have $V^*_{D_4+D_5} = \mathsf{\Lambda}^2 V^* \otimes \mathsf{S}^2 W$, the equivariant map

$$\pi\colon (V^* \otimes \mathsf{\Lambda}^2_0 W) \otimes (V^* \otimes \mathsf{\Lambda}^2_0 W) \longrightarrow \mathsf{\Lambda}^2 V^* \otimes \mathsf{S}^2 W$$

given by

$$\begin{aligned}&\pi\big((\varphi \otimes a \wedge b) \otimes (\psi \otimes c \wedge d)\big)\\&\quad = (\varphi \wedge \psi) \otimes \big(\omega(a,c)bd - \omega(b,c)ad - \omega(a,d)bc + \omega(b,d)ac\big)\end{aligned}$$

and

$$\pi(h_2 \otimes h_2) = 2\big((\varphi_3 \wedge \varphi_2) \otimes e_1'e_{-2}' + (\varphi_3 \wedge \varphi_1) \otimes e_1'e_2' - (\varphi_2 \wedge \varphi_1) \otimes (e_1')^2\big) \neq 0.$$

$(D_2, D_3, 2D_5)$. We have $V^*_{2D_5} = \mathsf{S}^2 W$, the equivariant map

$$\pi\colon (V^* \otimes \mathsf{\Lambda}^2_0 W) \otimes (V \otimes \mathsf{\Lambda}^2_0 W) \longrightarrow \mathsf{S}^2 W$$

given by

$$\begin{aligned}&\pi\big((\varphi \otimes a \wedge b) \otimes (v \otimes c \wedge d)\big)\\&\quad = \varphi(v)\big(\omega(a,c)bd - \omega(b,c)ad - \omega(a,d)bc + \omega(b,d)ac\big)\end{aligned}$$

and

$$\pi(h_2 \otimes h_3) = -2(e_1')^2 \neq 0.$$

$(D_3, D_4, D_1 + D_5)$. We have $V^*_{D_1+D_5} = \mathsf{\Lambda}^2 V \otimes W$, the equivariant map

$$\pi\colon (V \otimes \mathsf{\Lambda}^2_0 W) \otimes (V \otimes W) \longrightarrow \mathsf{\Lambda}^2 V \otimes W$$

given by

$$\pi\big((u \otimes a \wedge b) \otimes (v \otimes c)\big) = (u \wedge v) \otimes \big(\omega(a,c)b - \omega(b,c)a\big)$$

and

$$\pi(h_3 \otimes h_4) = -2(e_1 \wedge e_2) \otimes e_1' \neq 0.$$

(D_4, D_4, D_1). We have the equivariant map

$$\pi\colon (V \otimes W) \otimes (V \otimes W) \longrightarrow \mathsf{\Lambda}^2 V$$

given by

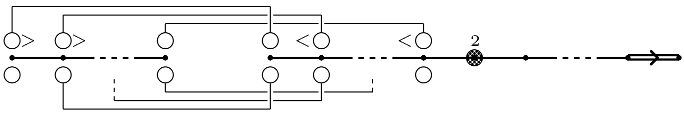
$$\pi\big((u \otimes a) \otimes (v \otimes b)\big) = \omega(a,b)(u \wedge v)$$

and

$$\pi(h_4 \otimes h_4) = -2(e_1 \wedge e_2) \neq 0. \qquad \square$$

2.4. Projective normality of $\mathsf{a}^{\mathsf{y}}(s,s) + \mathsf{b}'(t)$

Consider the wonderful variety X for a semisimple group G of type $\mathsf{A}_s \times \mathsf{B}_{s+t}$ with $s, t \geq 1$ defined by the following spherical system.



The spherical data and the Cartan pairing associated to this Luna diagram are described in Section 1.5.6. For convenience we number the spherical roots as

$$\sigma_{2i-1}=\alpha_i,\qquad \sigma_{2i}=\alpha'_i \quad \text{for } i=1,\dots,s,\qquad \sigma_{2s+1}=\sum_{i=s+1}^{s+t}2\alpha'_i.$$

There are $2s+2$ colors that we label as

$$D_{2i-1}=D^-_{\alpha_i},\quad \text{for } i=1,\dots,s,\qquad D_{2s+1}=D^-_{\alpha'_s},$$
$$D_{2i}=D^+_{\alpha_i},\quad \text{for } i=1,\dots,s,\qquad D_{2s+2}=D_{\alpha'_{s+1}}.$$

The weights of these colors are

$$\omega_{D_{2i-1}}=\omega_i+\omega'_{i-1}\quad \text{for } i=2,\dots,s,\qquad \omega_{D_1}=\omega_1,\qquad \omega_{D_{2s+1}}=\omega'_s,$$
$$\omega_{D_{2i}}=\omega_i+\omega'_i\quad \text{for } i=1,\dots,s,\qquad \omega_{D_{2s+2}}=\tilde\omega'_{s+1},$$

where $\tilde\omega'_{s+1}=\omega'_{s+1}$ if $t>1$ and $\tilde\omega'_{s+1}=2\omega'_{s+1}$ if $t=1$.

Note that X has the same Cartan matrix as that of the spherical nilpotent orbit studied in [4, Section 7.3]. It follows that the covering differences and the fundamental low triples are the same as those computed therein, since they only depend on the Cartan matrix. In particular, every covering difference γ satisfies $\mathrm{ht}(\gamma^+)=2$, and every fundamental triple is low. In order to prove the projective normality of X, in the following lemma we summarize some properties of its fundamental triples.

LEMMA 2.11

Let (D_p,D_q,F) be a fundamental triple, denote $\gamma=D_p+D_q-F$, and suppose that $\sigma_{2s+1}\in\mathrm{supp}_\Sigma(\gamma)$. Then p,q are even integers and $\sigma_1\notin\mathrm{supp}_\Sigma(\gamma)$. If moreover $\sigma_2\in\mathrm{supp}_\Sigma(\gamma)$, then $p+q-3\le 2s+1$ and $F=D_1+D_{p+q-3}$.

Proof

Take a sequence of coverings in $\mathbb{N}\Delta$

$$F=F_{n+1}<_\Sigma F_n<_\Sigma\cdots<_\Sigma F_1=D_p+D_q.$$

Denote $\gamma_i=F_i-F_{i+1}$. By [4, Propositions 3.2, 7.3] we have the following three possibilities:

- $\gamma_i=\sigma_{p_i}+\sigma_{p_i+2}+\cdots+\sigma_{q_i-1}=D_{p_i}+D_{q_i}-D_{p_i-1}-D_{q_i+1}$, for some integers p_i,q_i of different parity with $1\le p_i<q_i\le 2s+1$,
- $\gamma_i=\sigma_{p_i-1}+\sigma_{p_i}+\cdots+\sigma_{q_i}=D_{p_i}+D_{q_i}-D_{p_i-2}-D_{q_i+2}$, for some integers p_i,q_i of the same parity with $2\le p_i\le q_i\le 2s$,
- $\gamma_i=\sigma_{p_i}+\sigma_{p_i+2}+\cdots+\sigma_{q_i-2}+2(\sigma_{q_i}+\sigma_{q_i+2}+\cdots+\sigma_{2s})+\sigma_{2s+1}=D_{p_i}+D_{q_i}-D_{p_i-1}-D_{q_i-1}$, for some even integers p_i,q_i with $2\le p_i\le q_i\le 2s$.

Since $\sigma_{2s+1} \in \operatorname{supp}_\Sigma(\gamma)$, there is at least one γ_i of type 3. Let k be minimal with γ_k of type 3; because of the parity of p_k and q_k, the previous description implies that every γ_j with $j \neq k$ is of type 2. Moreover, it follows that $p_{i+1} = p_i - 2$ and $q_{i+1} = q_i + 2$ for all $i \neq k$ and that p_i, q_i are even (resp., odd) for all $i \leq k$ (resp., $i > k$).

Therefore, $p = p_1$ and $q = q_1$ are even integers and $2 \leq p \leq q \leq 2s+2$, and we get the equalities $p_{n+1} = p - 2n - 1$ and $q_{n+1} = q + 2n - 1$. Suppose that $k = n$. Then p_n and q_n are even and $2 \leq p_n \leq q_n \leq 2s+2$; hence, $1 \leq p_{n+1} \leq q_{n+1} \leq 2s+1$. Suppose instead $k < n$. Then p_n and q_n are odd and $2 \leq p_n \leq q_n \leq 2s$, and again we get $1 \leq p_{n+1} \leq q_{n+1} \leq 2s+1$.

To show the first claim, note that $\sigma_1 \in \operatorname{supp}_\Sigma(\gamma)$ if and only if $\sigma_1 \in \operatorname{supp}_\Sigma(\gamma_n)$. This is not the case if $k = n$. If $k < n$, then it also cannot happen, since then p_n and q_n would be odd. Similarly, $\sigma_2 \in \operatorname{supp}_\Sigma(\gamma)$ if and only if $\sigma_2 \in \operatorname{supp}_\Sigma(\gamma_n)$ if and only if $p_{n+1} = 1$. This means $n = \frac{p}{2} - 1$, which implies $q_{n+1} = p + q - 3$. $\square$

To prove the projective normality of X we will apply Lemma 2.6. First we describe the invariants. Let $V = \mathbb{C}^{s+1}$ with standard basis given by $e_1, \ldots, e_{s+1}$. Let $W = \mathbb{C}^{2n+1}$ where $n = s + t$. We choose a basis $e'_1, \ldots, e'_n, e'_0, e'_{-n}, \ldots, e'_{-1}$ and fix a bilinear symmetric form such that $\beta(e'_i, e'_j) = \delta_{i,-j}$ for all $i, j \geq 0$. Set $G = \mathrm{SL}(V^*) \times \mathrm{SO}(W, \beta)$, so that we can take H as the stabilizer of the line spanned by the vector $e = e_1 \otimes e'_0 + \sum_{i=2}^{s+1} e_i \otimes e'_{s-i+2}$. We have

$$V^*_{D_{2i-1}} = \Lambda^i V \otimes \Lambda^{i-1} W, \qquad V^*_{D_{2i}} = \Lambda^i V \otimes \Lambda^i W$$

for $i = 1, \ldots, s+1$. If we denote by h_i the vector $h_{D_i} \in V^*_{D_i}$, then in coordinates the vectors h_i are given as

$$h_{2i-1} = \sum_{2 \leq j_1 < \cdots < j_{i-1} \leq s+1} e_1 \wedge e_{j_1} \wedge \cdots \wedge e_{j_{i-1}} \otimes e'_{s-j_{i-1}+2} \wedge \cdots \wedge e'_{s-j_1+2},$$

$$\begin{aligned} h_{2i} = &\sum_{2 \leq j_1 < \cdots < j_{i-1} \leq s+1} e_1 \wedge e_{j_1} \wedge \cdots \wedge e_{j_{i-1}} \otimes e'_{s-j_{i-1}+2} \wedge \cdots \wedge e'_{s-j_1+2} \wedge e'_0 \\ &+ \sum_{2 \leq j_1 < \cdots < j_i \leq s+1} e_{j_1} \wedge \cdots \wedge e_{j_i} \otimes e'_{s-j_i+2} \wedge \cdots \wedge e'_{s-j_1+2}. \end{aligned}$$

PROPOSITION 2.12

The multiplication $m_{D,E}$ is surjective for all $D, E \in \mathbb{N}\Delta$.

Proof

As already noted, [4, Propositions 3.2, 7.3] show that every covering difference $\gamma \in \mathbb{N}\Sigma$ satisfies $\mathrm{ht}(\gamma^+) = 2$. It follows that every $D \in \Delta$ is minimal in $\mathbb{N}\Delta$ with respect to $\leq_\Sigma$; hence, every fundamental triple is low. Therefore, by Lemma 2.5 we have to check that $s^{D+E-F} V_F \subset V_D \cdot V_E$ for all fundamental triples (D, E, F). Let (D, E, F) be such a triple, and denote $\gamma = D + E - F$.

Suppose that $\sigma_{2s+1} \notin \operatorname{supp}_\Sigma(\gamma)$, and let X' be the G-stable divisor of X corresponding to the spherical root σ_{2s+1}. Then X' is a parabolic induction of a comodel wonderful variety of cotype A_{2s+1} (see [4, Section 5]). Hence, the inclusion $s^\gamma V_F \subset V_D \cdot V_E$ follows by Lemma 2.3 together with [4, Theorem 5.2].

Suppose that $\sigma_{2s+1} \in \mathrm{supp}_\Sigma(\gamma)$, and assume $D = D_p$ and $E = D_q$. Then $\sigma_1 \notin \mathrm{supp}_\Sigma(\gamma)$ by Lemma 2.11. We show that $s^\gamma V_F \subset V_D \cdot V_E$, proceeding by induction on s.

Suppose that $\sigma_2 \notin \mathrm{supp}_\Sigma(\gamma)$. Then $\mathrm{supp}_\Sigma(\gamma) \subset \{\sigma_3, \dots, \sigma_{2s+1}\}$. Let X'' be the G-stable subvariety of X obtained by intersecting the G-stable divisors corresponding to σ_1 and to σ_2. If $s > 1$, then X'' is a parabolic induction of the wonderful variety of type $\mathsf{a}^{\mathsf{y}}(s-1, s-1) + \mathsf{b}'(t)$. Therefore, the multiplication of sections of globally generated line bundles on X'' is always surjective by the inductive hypothesis thanks to Lemma 2.3. If instead $s = 1$, then X'' is a parabolic induction of a rank 1 wonderful symmetric variety Y, which is homogeneous under its automorphism group. Therefore, the multiplication of sections of globally generated line bundles on Y is always surjective, and the same holds for X'' by Lemma 2.3 again. In particular, since (D, E, F) is a low triple, it follows that the inclusion $s^\gamma V_F \subset V_D \cdot V_E$.

Suppose now that $\sigma_2 \in \mathrm{supp}_\Sigma(\gamma)$. Then by Lemma 2.11 it follows that $p = 2\ell$ and $q = 2m$ are even integers, and $F = D_1 + D_{2\ell+2m-3}$ with $2\ell + 2m - 3 \le 2s+1$. Hence, by Lemma 2.6 we need to find an equivariant map

$$\varphi\colon (\Lambda^\ell V \otimes \Lambda^\ell W) \otimes (\Lambda^m V \otimes \Lambda^m W) \longrightarrow V_{\omega_1 + \omega_{\ell+m-1}} \otimes \Lambda^{\ell+m-2} W$$

such that $\varphi(h_{2\ell} \otimes h_{2m}) \neq 0$. (The formula also makes sense when $\ell + m - 1 = s+1$, by setting $\omega_{s+1} = 0$.) Note that $V \otimes \Lambda^{\ell+m-1} V \simeq V_{\omega_1+\omega_{\ell+m-1}} \oplus \Lambda^{\ell+m} V$, and we denote by ρ_1 and ρ_2 the projection, respectively, onto the first factor and onto the second factor. In particular, the map ρ_2, up to a scalar factor, is just the wedge product. We will construct a map

$$\psi\colon (\Lambda^\ell V \otimes \Lambda^\ell W) \otimes (\Lambda^m V \otimes \Lambda^m W) \longrightarrow (V \otimes \Lambda^{\ell+m-1} V) \otimes \Lambda^{\ell+m-2} W$$

such that $\psi(h_{2\ell} \otimes h_{2m}) \neq 0$ and $(\rho_2 \otimes \mathrm{Id}) \circ \psi(h_{2\ell} \otimes h_{2m}) = 0$ so that the map $\varphi = (\rho_1 \otimes \mathrm{Id}) \circ \psi$ will have the desired properties.

Let $\pi_1\colon \Lambda^\ell W \otimes \Lambda^m W \longrightarrow \Lambda^{\ell+m-2} W$ be defined by

$$\begin{aligned}
&\pi_1(u_1 \wedge \dots \wedge u_\ell \otimes v_1 \wedge \dots \wedge v_m) \\
&\quad = \sum_{i,j} (-1)^{i+j} \beta(u_i, v_j) u_1 \wedge \dots \wedge \hat{u}_i \wedge \dots \wedge u_{\ell+1} \wedge v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_m.
\end{aligned}$$

Let $\pi_2\colon \Lambda^\ell V \otimes \Lambda^m V \longrightarrow V \otimes \Lambda^{\ell+m-1} V$ be defined by

$$\begin{aligned}
&\pi_2(u_1 \wedge \dots \wedge u_\ell \otimes v_1 \wedge \dots \wedge v_m) \\
&\quad = \sum_i (-1)^i u_i \otimes u_1 \wedge \dots \wedge \hat{u}_i \wedge \dots \wedge u_\ell \wedge v_1 \wedge \dots \wedge v_m.
\end{aligned}$$

Let $\pi_3\colon \Lambda^\ell V \otimes \Lambda^m V \longrightarrow \Lambda^{\ell+m} V$ be defined by $\pi_3(x \otimes y) = x \wedge y$. Finally, set $\psi = \pi_2 \otimes \pi_1$, so that $(\rho_2 \otimes \mathrm{Id}) \circ \psi = \pi_3 \otimes \pi_1$.

Note that the value of $\pi_2 \otimes \pi_1$ (resp., $\pi_3 \otimes \pi_1$) on $h_{2\ell} \otimes h_{2m}$ is the same as that on

$$\sum_{2\le j_1<\cdots<j_{\ell-1}\le s+1} e_1 \wedge e_{j_1} \wedge \cdots \wedge e_{j_{\ell-1}} \otimes e'_{s-j_{\ell-1}+2} \wedge \cdots \wedge e'_{s-j_1+2} \wedge e'_0$$

$$\otimes \sum_{2\le k_1<\cdots<k_{m-1}\le s+1} e_1 \wedge e_{k_1} \wedge \cdots \wedge e_{k_{m-1}} \otimes e'_{s-k_{m-1}+2} \wedge \cdots \wedge e'_{s-k_1+2} \wedge e'_0.$$

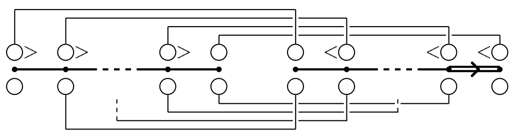
The first is equal to

$$\binom{\ell+m-2}{\ell-1} \sum (e_1 \otimes e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_{\ell+m-2}}) \otimes e'_{s-i_{\ell+m-2}+2} \wedge \cdots \wedge e'_{s-i_1+2}$$

(the sum being over $2 \le i_1 < \cdots < i_{\ell+m-2} \le s+1$). The second is equal to 0. □

2.5. Projective normality of $\mathsf{ab}^{\mathsf{y}}(s,s)$

Consider the wonderful variety X for a semisimple group G of type $\mathsf{A}_s \times \mathsf{B}_s$ with $s \ge 2$ defined by the following spherical system.



The spherical data and the Cartan pairing associated to this Luna diagram are described in Section 1.5.6. The spherical roots are simple roots. For convenience we enumerate them as

$$\sigma_{2i-1} = \alpha_i, \qquad \sigma_{2i} = \alpha'_i \quad \text{for } i = 1, \ldots, s.$$

There are $2s+1$ colors that we label as

$$D_{2i-1} = D^-_{\alpha_i}, \qquad D_{2i} = D^+_{\alpha_i} \quad \text{for } i = 1, \ldots, s, \qquad D_{2s+1} = D^-_{\alpha'_s}.$$

The weights of these colors are

$$\omega_{D_1} = \omega_1, \qquad \omega_{D_{2i}} = \omega_i + \omega'_i \quad \text{for } i = 1, \ldots, s,$$
$$\omega_{D_{2i-1}} = \omega_i + \omega'_{i-1} \quad \text{for } i = 2, \ldots, s, \qquad \omega_{D_{2s+1}} = \omega'_s.$$

Note that the G-stable divisor of X corresponding to σ_{2s} is a parabolic induction of a comodel wonderful variety of cotype A_{2s} (see [4, Section 5]). Therefore, we can restrict our study to the covering differences and the low triples of X which contain σ_{2s}.

LEMMA 2.13

Let $\gamma \in \mathbb{N}\Sigma$ be a covering difference in $\mathbb{N}\Delta$ with $\sigma_{2s} \in \operatorname{supp}_\Sigma \gamma$. Then either $\gamma = \sigma_{2s} = D_{2s} + D_{2s+1} - D_{2s-1}$, or $\gamma = \sigma_{2s-1} + \sigma_{2s} = -D_{2s-2} + 2D_{2s}$, or $\gamma = \sum_{i=\ell}^{s} \sigma_{2i} = D_{2\ell} - D_{2\ell-1}$ for some $1 \le \ell < s$. Every other covering difference satisfies $\operatorname{ht}(\gamma^+) = 2$.

Proof
Recall that the Cartan pairing is as follows (we also set $D_i = 0$ for all $i \leq 0$ and all $i \geq 2s+2$):

$$\sigma_i = -D_{i-1} + D_i + D_{i+1} - D_{i+2} \quad \text{for } i \neq 2s-2,$$
$$\sigma_{2s-2} = -D_{2s-3} + D_{2s-2} + D_{2s-1} - D_{2s} - D_{2s+1}.$$

Set $\gamma = \sum_{i=1}^{2s} a_i\sigma_i = \sum_{i=1}^{2s+1} c_i D_i$. We have the following identities (we set $a_i = 0$ for $i \leq 0$ and $i \geq 2s+1$):

$$c_i = -a_{i-2} + a_{i-1} + a_i - a_{i+1} \quad \text{for } i < 2s+1,$$
$$c_{2s+1} = -a_{2s-2} - a_{2s-1} + a_{2s}.$$

By hypothesis, we have $a_{2s} > 0$.

Let $k \geq 0$ be minimal with $c_{2s-2k} > 0$. Then $c_{2s-2i} \leq 0$ for all $0 \leq i < k$, and it follows that

$$a_{2s-2j} - a_{2s-2j+1} \geq a_{2s-2j+2} - a_{2s-2j+3}$$

for all $1 \leq j \leq k$, and $a_{2s-2j} \geq a_{2s} > 0$ for all $0 \leq j \leq k$.

If $k > 0$, set $\gamma_0 = \sum_{j=0}^{k} \sigma_{2s-2j} = -D_{2s-2k-1} + D_{2s-2k}$. Then $\gamma_0 \leq_\Sigma \gamma$ and $\gamma^+ - \gamma_0 \in \mathbb{N}\Delta$; hence, $\gamma = \gamma_0$ since γ is a covering difference.

We are left with the case $k = 0$. In particular, $c_{2s} > 0$. We claim that γ is necessarily equal to σ_{2s} or to $\sigma_{2s-1} + \sigma_{2s}$. Assume that $\gamma \neq \sigma_{2s}$ and $\gamma \neq \sigma_{2s-1} + \sigma_{2s}$. Since $a_{2s} > 0$ and $\sigma_{2s} = -D_{2s-1} + D_{2s} + D_{2s+1}$, it follows that $c_{2s+1} \leq 0$; hence, $a_{2s-1} > 0$. Since $\sigma_{2s-1} = -D_{2s-2} + D_{2s-1} + D_{2s} - D_{2s+1}$, it must be that $c_{2s-1} \leq 0$, and since $\sigma_{2s-1} + \sigma_{2s} = -D_{2s-2} + 2D_{2s}$, it must be that $c_{2s} = 1$. The latter implies

$$a_{2s-2} - a_{2s} = a_{2s-1} - 1 \geq 0.$$

Hence, $a_{2s-2} > 0$. Note that $c_{2s-2} \leq 0$. If indeed $c_{2s-2} > 0$, then $\gamma^+ - (\sigma_{2s-2} + \sigma_{2s}) = \gamma^+ - (-D_{2s-3} + D_{2s-2}) \in \mathbb{N}\Delta$; hence, $\gamma = -D_{2s-3} + D_{2s-2}$, contradicting $c_{2s} > 0$.

Let $j \geq 3$ be such that $a_{2s-j+1} - a_{2s-j+3} \geq 0$, and suppose that $a_{2s-i+1} > 0$ and $c_{2s-i+1} \leq 0$ for all i with $2 \leq i \leq j$. (Note that these conditions have just been proved for $j = 3$.) As $c_{2s-j+2} \leq 0$, it follows that

$$a_{2s-j} - a_{2s-j+2} \geq a_{2s-j+1} - a_{2s-j+3} \geq 0.$$

Hence, $a_{2s-j} > 0$. This implies $c_{2s-j} \leq 0$. If indeed $c_{2s-j} > 0$, set

$$\gamma_0 = \sum_{i=0}^{j/2} \sigma_{2s-2i} = -D_{2s-j-1} + D_{2s-j} \quad \text{if } j \text{ is even},$$

$$\gamma_0 = \sum_{i=1}^{(j+1)/2} \sigma_{2s-2i+1} = -D_{2s-j-1} + D_{2s-j} + D_{2s} - D_{2s+1} \quad \text{if } j \text{ is odd}.$$

Then $\gamma_0 \leq_\Sigma \gamma$ and $\gamma^+ - \gamma_0 \in \mathbb{N}\Delta$; hence, $\gamma = \gamma_0$, contradicting $c_{2s} > 0$ in the first case and $a_{2s} > 0$ in the second case.

By applying this argument recursively for $j=3,\dots,s-1$, it follows that $a_i>0$ for all $1\le i\le 2s$, $c_i\le 0$ for all $1\le i\le 2s-1$, and $a_1-a_3\ge 0$. In particular, $a_1+a_2-a_3=c_2\le 0$ and $a_1-a_3\ge 0$, which are in contradiction.

As already noted, the G-stable divisor of X corresponding to σ_{2s} is a parabolic induction of a comodel wonderful variety of cotype A_{2s}. Therefore, the covering differences γ with $\sigma_{2s}\notin \operatorname{supp}_\Sigma\gamma$ coincide with those studied in [4, Proposition 3.2], and they all satisfy $\operatorname{ht}(\gamma^+)=2$. $\square$

LEMMA 2.14

Let (D,E,F) be a low fundamental triple, denote $\gamma=D+E-F$, and suppose that $\sigma_{2s}\in\operatorname{supp}_\Sigma\gamma$. Then we have the following possibilities:

- $(D_{2m+1},D_{2s},D_{2m-1}+D_{2s+1})$ *for* $1\le m<s$, $\gamma=\sum_{i=2m}^{2s}\sigma_i$;
- (D_{2s},D_{2s},D_{2s-3}), $\gamma=\sigma_{2s-2}+\sigma_{2s-1}+2\sigma_{2s}$;
- (D_{2s},D_{2s},D_{2s-2}), $\gamma=\sigma_{2s-1}+\sigma_{2s}$;
- $(D_{2s},D_{2s+1},D_{2s-1})$, $\gamma=\sigma_{2s}$.

Proof

Set $D+E-F=\sum_{i=1}^{2s}a_i\sigma_i=\sum_{i=1}^{2s+1}c_iD_i$, and set also $D=D_{2s-p+1}$ and $E=D_{2s-q+1}$. By hypothesis, we have $a_{2s}>0$.

At least one of the two indices p and q must be odd. Indeed, if both p and q were even, then by taking a sequence

$$F=F_n<_\Sigma F_{n-1}<_\Sigma\cdots<_\Sigma F_0=D+E$$

of coverings in $\mathbb{N}\Delta$, $F_{i-1}-F_i$ would necessarily be a covering difference of a comodel spherical system of cotype A (see [4, Proposition 3.2.(2)]); hence, $\sigma_{2s}\notin\operatorname{supp}_\Sigma(D+E-F)$.

We claim that at least one of the two indices p and q must be equal to 1. Let us prove the claim. Assume both p and q are different from 1. We can assume that q is the minimal odd number between p and q. Since $c_{2s-2i}\le 0$ for every $0\le i<(q-1)/2$, as in the above proof, it follows that $a_{2s-2i}\ge a_{2s}>0$ for every $0\le i\le (q-1)/2$. Set

$$\gamma_0=\sum_{i=0}^{(q-1)/2}\sigma_{2s-2i}=-D_{2s-q}+D_{2s-q+1}$$

and $E'=D_{2s-q+1}-\gamma_0$. Then $E'\in\mathbb{N}\Delta$ and $F\le_\Sigma D+E'<_\Sigma D+E$; hence, (D,E,F) is not a low triple. Therefore, we can assume $q=1$.

Suppose that $p=0$. We have $D_{2s}+D_{2s+1}-\sigma_{2s}=D_{2s-1}$, but the latter is minuscule; therefore, we get only $(D_{2s},D_{2s+1},D_{2s-1})$.

Suppose that $p=1$. We have $-a_{2s-2}-a_{2s-1}+a_{2s}\le 0$ and $-a_{2s-2}+a_{2s-1}+a_{2s}=2$; hence, $a_{2s-1}>0$. Now, we have $2D_{2s}-(\sigma_{2s-1}+\sigma_{2s})=D_{2s-2}$, but in $\mathbb{N}\Delta$ the latter covers only D_{2s-3}, with $D_{2s-2}-(\sigma_{2s-2}+\sigma_{2s})=D_{2s-3}$. Therefore, if $a_{2s-2}=0$ or $a_{2s}=1$, we get only (D_{2s},D_{2s},D_{2s-2}). If $a_{2s-2}>0$ and $a_{2s}>1$, since D_{2s-3} is minuscule, we get only (D_{2s},D_{2s},D_{2s-3}).

Suppose that $p > 1$. We have $-a_{2s-2} - a_{2s-1} + a_{2s} \leq 0$ and $-a_{2s-2} + a_{2s-1} + a_{2s} = 1$; hence, $a_{2s-1} > 0$ and, since

$$a_{2s-2} - a_{2s} = a_{2s-1} - 1 \geq 0,$$

also $a_{2s-2} > 0$. For every $1 < i < p$, as $c_{2s-i+1} \leq 0$, we have

$$a_{2s-i-1} - a_{2s-i+1} \geq a_{2s-i} - a_{2s-i+2}.$$

Therefore, $a_{2s-j+1} > 0$ for all $1 \leq j \leq p+1$.

If p is odd, set

$$\gamma_0 = \sum_{i=0}^{(p-1)/2} \sigma_{2s-2i} = -D_{2s-p} + D_{2s-p+1}$$

and $D' = D_{2s-p+1} - \gamma_0$. Then $D' \in \mathbb{N}\Delta$ and $F \leq_\Sigma D' + E <_\Sigma D + E$; hence, (D, E, F) is not a low triple.

If p is even, we have

$$D_{2s-p+1} + D_{2s} - \sum_{i=1}^{p+1} \sigma_{2s-i+1} = D_{2s-p-1} + D_{2s+1},$$

but the latter is minuscule. We get $(D_{2s-p+1}, D_{2s}, D_{2s-p-1} + D_{2s+1})$. □

To prove the projective normality of X we will apply Lemma 2.6. This requires some computations, and we will need an explicit description of the invariants h_3 and h_{2s}.

Fix $V = \mathbb{C}^{s+1}$ with standard basis $e_1, \dots, e_{s+1}$, and fix $W = \mathbb{C}^{2s+1}$ with basis $e'_1, \dots, e'_s$, $e'_0, e'_{-s}, \dots, e'_{-1}$ and with a symmetric bilinear form β such that $\beta(e_i, e_j) = \delta_{i,-j}$. Set $G = \mathrm{SL}(V^*) \times \mathrm{Spin}(W, \beta)$, so that we can take H as the stabilizer of the line spanned by the vector $e = e_1 \otimes e'_0 + \sum_{i=2}^{s+1} e_i \otimes e'_{s-i+2}$.

We will need to use the spin module for the group $\mathrm{Spin}(W)$. Let us recall its construction. Let $W = U \oplus \mathbb{C}e'_0 \oplus U^*$, where U is the span of $e'_1, \dots, e'_s$ and U^* is the span of $e'_{-s}, \dots, e'_{-1}$ identified with the dual of U by the form β. Let $S = \Lambda U^*$, and rename the basis of U^* as $\psi_n = e'_{-n}, \dots, \psi_1 = e'_{-1}$. Define a map $\pi_S \colon W \otimes S \longrightarrow S$ by setting $\pi_S(e'_i \otimes \psi_{i_1} \wedge \cdots \wedge \psi_{i_k})$ equal to

$$\begin{aligned}
&\sum_{j=1}^{k} (-1)^{j-1} \beta(e'_i, e'_{-i_j}) \psi_{i_1} \wedge \cdots \wedge \hat{\psi}_{i_j} \wedge \cdots \wedge \psi_{i_k} \quad \text{if } i > 0, \\
&(-1)^k \psi_{i_1} \wedge \cdots \wedge \psi_{i_k} \quad \text{if } i = 0, \\
&\psi_{-i} \wedge \psi_{i_1} \wedge \cdots \wedge \psi_{i_k} \quad \text{if } i < 0.
\end{aligned}$$

Then π_S is G-equivariant, and its alternating square $\pi_S^2 \colon \Lambda^2 W \otimes S \longrightarrow S$ corresponds to the spin representation via the isomorphism $\Lambda^2 W \cong \mathfrak{so}(W, \beta)$. We have

$$V^*_{D_1} = V, \qquad V^*_{D_3} = \Lambda^2 V \otimes W, \qquad V^*_{D_{2s}} = \Lambda^s V \otimes S, \qquad V^*_{D_1 + D_{2s+1}} = V \otimes S.$$

The invariants, in coordinates, are

$$h_3 = \sum_{i=2}^{s+1} e_1 \wedge e_i \otimes e'_{s-i+2},$$

$$h_{2s} = e_2 \wedge \cdots \wedge e_{s+1} \otimes 1 + \sum_{i=2}^{s+1} (-1)^{i-1} e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{s+1} \otimes \psi_{s-i+2}.$$

PROPOSITION 2.15

The multiplication $m_{D,E}$ is surjective for all $D, E \in \mathbb{N}\Delta$.

Proof

By Lemma 2.13 every covering difference $\gamma \in \mathbb{N}\Sigma$ satisfies $\mathrm{ht}(\gamma^+) \leq 2$. Therefore, by Lemma 2.5 it is enough to check that $s^{D+E-F} V_F \subset V_D \cdot V_E$ for all low fundamental triples (D, E, F).

Suppose that $\sigma_{2s} \notin \mathrm{supp}_\Sigma(D+E-F)$, and let X' be the G-stable divisor of X corresponding to σ_{2s}. Then X' is a parabolic induction of a comodel wonderful variety of cotype A_{2s}; hence, the inclusion $s^\gamma V_F \subset V_D \cdot V_E$ follows by Lemma 2.3 together with [4, Theorem 5.2].

We are left to check that $s^{D+E-F} V_F \subset V_D V_E$ for all low fundamental triples (D, E, F) with $\sigma_{2s} \in \mathrm{supp}_\Sigma(D+E-F)$. Consider first the triple $(D_3, D_{2s}, D_1 + D_{2s+1})$. Then we have the projection $\pi\colon (\Lambda^2 V \otimes W) \otimes (\Lambda^s V \otimes S) \to V \otimes S$ given by

$$\pi\big((u_1 \wedge u_2 \otimes w) \otimes (v \otimes \psi)\big) = \big((u_2 \wedge v) u_1 - (u_1 \wedge v) u_2\big) \otimes \pi_s(w \otimes \psi),$$

where $\Lambda^{s+1} V \cong \mathbb{C}$ via $e_1 \wedge \cdots \wedge e_{s+1} \mapsto 1$, and we get $\pi(h_3 \otimes h_{2s}) = s(e_1 \otimes 1) \neq 0$.

We now proceed by induction on s. Assume $s = 2$. Then we are left to check the triples (D_4, D_4, D_1), (D_4, D_4, D_2), and (D_4, D_5, D_3).

(D_4, D_4, D_1). We have the projection $\pi\colon (\Lambda^2 V \otimes S) \otimes (\Lambda^2 V \otimes S) \to V$ given by $\pi(u_1 \wedge u_2 \otimes \varphi) \otimes (v_1 \wedge v_2 \otimes \psi)) = \pi'(\varphi \otimes \psi)((u_1 \wedge u_2 \wedge v_1) v_2 - (u_1 \wedge u_2 \wedge v_2) v_1)$, with $\Lambda^3 V \cong \mathbb{C}$ given by the identification $e_1 \wedge e_2 \wedge e_3 = 1$. Note that S is self-dual, and set $\pi'\colon S \otimes S \to \mathbb{C}$ given by $\varphi \otimes \psi \mapsto \varphi \wedge \psi$ followed by the identification $\psi_2 \wedge \psi_1 = 1$. We get $\pi(h_4 \otimes h_4) = -2e_1 \neq 0$.

(D_4, D_4, D_2) *and* (D_4, D_5, D_3). Since $\sigma_1, \sigma_2 \notin \mathrm{supp}_\Sigma(D+E-F)$, we can consider the intersection X' of the G-stable divisors of X corresponding to the spherical roots σ_1, σ_2. Then the sections in $V_D, V_E, s^{D+E-F} V_F$ do not vanish on X', so it is enough to prove that $s^{D+E-F} V_F \subset m'_{D,E}(V_D \otimes V_E)$, where m' denotes the multiplication of sections on X'. Consider in G the Levi subgroup L associated to the roots α_2, α'_2, which has semisimple factor of type $\mathsf{A}_1 \times \mathsf{A}_1$, and consider the comodel L-variety Y of cotype A_3. The wonderful G-variety X' is obtained by parabolic induction from Y. Hence, our claim follows by Lemma 2.3.

Assume $s > 2$. Then we are left to check the triples $(D_{2m+1}, D_{2s}, D_{2m-1} + D_{2s+1})$ with $1 < m < s$, $(D_{2s}, D_{2s}, D_{2s-3})$, $(D_{2s}, D_{2s}, D_{2s-2})$, and $(D_{2s}, D_{2s+1}, D_{2s-1})$. Let (D, E, F) be such a triple. Then $\sigma_1, \sigma_2 \notin \mathrm{supp}_\Sigma(D+E-F)$, and we can consider the intersection X' of the G-stable divisors of X corresponding to the spherical roots σ_1, σ_2. Take the Levi subgroup L of G, associated to the roots $\alpha_2, \ldots, \alpha_s, \alpha'_2, \ldots, \alpha'_s$, of semisimple type $\mathsf{A}_{s-1} \times \mathsf{B}_{s-1}$, and consider the comodel L-variety Y of type $\mathsf{ab}^{\mathsf{y}}(s-1, s-1)$. The wonderful G-variety X' is obtained by parabolic induction from Y. Hence, our claim follows by Lemma 2.3 and the induction hypothesis. □

3. Normality and semigroups

Recall that we have fixed a maximal torus T in K and Borel subgroup B of K containing T. We use $\mathcal{X}(T)$ for the weight lattice of T.

Let us denote by $\Gamma(Z)$ the weight semigroup of a K-spherical variety Z,

$$\Gamma(Z) = \{\lambda \in \mathcal{X}(T) : \mathrm{Hom}(\mathbb{C}[Z], V(\lambda)) \neq 0\}.$$

Let Ke be a spherical nilpotent orbit in $\mathfrak{p}$, and let Σ and Δ, respectively, be the set of spherical roots and the set of colors of the wonderful compactification of $K/K_{[e]}$. Let us denote by $D_{\mathfrak{p}}$ the element of $\mathbb{N}\Delta$ such that $\mathfrak{p} = V^*_{D_{\mathfrak{p}}}$. Provided that the multiplication of sections of globally generated line bundles on the wonderful compactification of $K/K_{[e]}$ is surjective, we have that $\overline{Ke} \subset \mathfrak{p}$ is normal if and only if $D_{\mathfrak{p}}$ is minuscule in $\mathbb{N}\Delta$ with respect to the partial order $\leq_\Sigma$ (see [4, Section 7]). If moreover $\widetilde{Ke}$ is the normalization of $\overline{Ke}$, then

$$\Gamma(\widetilde{Ke}) = \bigcup_{n \in \mathbb{N}} \{\omega_E : E \in \mathbb{N}\Delta, E \leq_\Sigma nD_{\mathfrak{p}}\};$$

that is, $\Gamma(\widetilde{Ke}) = \omega(\Gamma_{D_{\mathfrak{p}}})$, where $\Gamma_{D_{\mathfrak{p}}}$ is the subsemigroup of $\mathbb{N}\Delta$ given by

$$\Gamma_{D_{\mathfrak{p}}} = \bigcup_{n \in \mathbb{N}} \{E \in \mathbb{N}\Delta : E \leq_\Sigma nD_{\mathfrak{p}}\}.$$

In the present section we will study the normality of $\overline{Ke}$, and we will compute the weight semigroups $\Gamma(\widetilde{Ke})$ by computing the corresponding semigroups $\Gamma_{D_{\mathfrak{p}}}$. In particular, we will prove the following theorem.

THEOREM 3.1

Let $(\mathfrak{g}, \mathfrak{k})$ be a classical symmetric pair of non-Hermitian type. Then $\overline{Ke}$ is not normal if and only if $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}(m+n), \mathfrak{so}(m) + \mathfrak{so}(n))$ and the signed partition of Ke is $(+3, +2^{n-1}, +1^{m-n-1})$, with $n > 1$ odd, or $(-3, +2^{m-1}, -1^{n-m-1})$, with $m > 1$ odd.

In Appendix A these are cases 7.3 with $r = p$, 8.2 with $r = q$, and 8.3 with $r = p$. The normality or nonnormality of $\overline{Ke}$ as well as the generators of the weight semigroup $\Gamma(\widetilde{Ke})$ are given in Tables 2–10 in Appendix B.

In the tables we also provide the codimension of $\overline{Ke} \smallsetminus Ke$ in $\overline{Ke}$. Note that if $\overline{Ke}$ is normal and the codimension of $\overline{Ke} \smallsetminus Ke$ in $\overline{Ke}$ is greater than 1, then $\mathbb{C}[\overline{Ke}] = \mathbb{C}[Ke]$, so that the weight semigroup of Ke actually coincides with $\Gamma(\widetilde{Ke})$.

Note also that, in all cases where $\overline{Ke}$ is not normal, the normalization $\widetilde{Ke} \longrightarrow \overline{Ke}$ is not even bijective (see [15] for a general procedure to compute the K-orbits in $\widetilde{Ke}$ and in $\overline{Ke}$).

REMARK 3.2

The normality of $\overline{Ge}$ is well known and may be deduced from [22] (see also [28], when Ge is spherical under the action of G). In particular, if $(\mathfrak{g},\mathfrak{k})$ is a classical symmetric pair of non-Hermitian type, then $\overline{Ge}$ is normal in all but the following cases:

- $\mathfrak{g}=\mathfrak{sp}(2n)$ with $n>3$ and the partition of Ge is $(3^2,1^{2n-6})$ (cases 4.2 and 4.3, with $p+q>3$),
- $\mathfrak{g}=\mathfrak{sp}(2n)$ with $n>6$ and the partition of Ge is $(3^4,1^{2n-12})$ (cases 4.6 and 4.7, with $p+q>6$),
- $\mathfrak{g}=\mathfrak{so}(2n+1)$ and the partition of Ge is $(3,2^{n-1})$ (case 7.3, with $r=p=q-1$).

We now report the details of the computation of the semigroup $\Gamma_{D_\mathfrak{p}}$. We omit the cases where $K/K_{[e]}$ is a flag variety (cases 5.1 and 6.1 in Appendix A) or a parabolic induction of a symmetric variety (cases 1.1, 2.1, 3.1, 4.1, 7.1, 8.1, and 9.1, as well as cases 7.2, 7.3, 8.2, 8.3, 9.2, and 9.3 when $r=0$). In these cases the combinatorics of spherical systems is easier. By [20], the normality of $\overline{Ke}$ is already known in all these cases (see the discussion at the beginning of Appendix B), and the corresponding weight semigroups $\Gamma(\overline{Ke})$ were obtained in [2] by using different techniques.

3.1. Symplectic cases

Cases 4.2 $(q>1)$ *and 4.3* $(p>1)$. Let us deal with the case 4.2 $(q>1)$; the other one is similar. We have two spherical roots $\sigma_1=\alpha_1+\alpha_1'$ and $\sigma_2=\alpha_1'+2(\alpha_2'+\cdots+\alpha_{q-1}')+\alpha_q'$ and three colors $D_1=D_{\alpha_1}$, $D_2=D_{\alpha_2'}$, and $D_3=D_{\alpha_2}$.

We have $D_\mathfrak{p}=D_1$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal. Furthermore, $D_2+D_3=2D_1-\sigma_1$ and $D_3=2D_1-\sigma_1-\sigma_2$; therefore, $D_1, D_2+D_3, D_3\in\Gamma_{D_1}$.

Let us set $\sigma=\sum a_i\sigma_i\in\mathbb{N}\Sigma$ and $nD_1-\sigma=\sum c_iD_i\in\mathbb{N}\Delta$. We have

$$nD_1-\sigma=(n-2a_1)D_1+(a_1-a_2)D_2+a_1D_3,$$

and therefore, $c_3-c_2=a_2$. It follows that

$$\Gamma_{D_1}=\langle D_1, D_2+D_3, D_3\rangle_\mathbb{N}.$$

Cases 4.4 $(q>2)$ *and 4.5* $(p>2)$. Let us deal with case 4.4 $(q>2)$; the other one is similar. Let us keep the notation of Sections 1.5.7 and 2.3. Therefore, $D_1=D^-_{\alpha_2}$, $D_2=D^+_{\alpha_2}$, $D_3=D^-_{\alpha_1}$, $D_4=D^+_{\alpha_1}$, $D_5=D^-_{\alpha_1'}$, $D_6=D_{\alpha_3'}$, and $D_7=D_{\alpha_3}$.

Then we have $D_\mathfrak{p}=D_4$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal. Moreover, $D_2=2D_4-\sigma_3-\sigma_4$, $D_1=D_2-\sigma_2-\sigma_4-\sigma_5$; therefore, $D_1, D_2, D_4\in$

Γ_{D_4}. Moreover, $D_3+D_7=D_1+D_2-\sigma_1$, $D_5+D_7=D_2+D_4-\sigma_1-\sigma_2-\sigma_3-\sigma_4$, $D_6+D_7=D_2+D_4-\sigma_1-\sigma_2-\sigma_3-\sigma_4-\sigma_5$; therefore, $D_3+D_7, D_5+D_7, D_6+D_7 \in \Gamma_{D_4}$ as well.

Let us set $\sigma=\sum a_i\sigma_i \in \mathbb{N}\Sigma$ and $nD_4-\sigma=\sum c_iD_i \in \mathbb{N}\Delta$. We have

$$\begin{aligned} nD_4-\sigma &= (-a_1+a_2)D_1+(-a_1-a_2+a_3)D_2+(a_1-a_2-a_3+a_4)D_3 \\ &\quad +(n+a_2-a_3-a_4)D_4+(a_3-a_4+a_5)D_5+(a_2-a_5)D_6+a_1D_7, \end{aligned}$$

and therefore, $c_3+c_5+c_6=c_7$. It follows that

$$\Gamma_{D_4}=\langle D_1, D_2, D_4, D_3+D_7, D_5+D_7, D_6+D_7\rangle_{\mathbb{N}}.$$

Cases 4.4 $(q=2)$ *and 4.5* $(p=2)$. Let us deal with case 4.4 ($q=2$); the other one is similar. Label the spherical roots $\sigma_1=\alpha_2$, $\sigma_2=\alpha_2'$, $\sigma_3=\alpha_1$, $\sigma_4=\alpha_1'$, and label the colors $D_1=D^-_{\alpha_2}$, $D_2=D^+_{\alpha_2}$, $D_3=D^-_{\alpha_1}$, $D_4=D^+_{\alpha_1}$, $D_5=D^-_{\alpha_1'}$, $D_6=D_{\alpha_3}$.

Then we have $D_{\mathfrak{p}}=D_4$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal. Moreover, $D_2=2D_4-\sigma_3-\sigma_4$, $D_1=D_2-\sigma_2-\sigma_4$; therefore, $D_1, D_2, D_4 \in \Gamma_{D_4}$. Similarly $D_3+D_6=D_1+D_2-\sigma_1$ and $D_5+D_6=D_2+D_4-\sigma_1-\sigma_2-\sigma_3-\sigma_4$; therefore, $D_3+D_6, D_5+D_6 \in \Gamma_{D_4}$ as well.

Let us set $\sigma=\sum a_i\sigma_i \in \mathbb{N}\Sigma$ and $nD_4-\sigma=\sum c_iD_i \in \mathbb{N}\Delta$. We have

$$\begin{aligned} nD_4-\sigma &= (-a_1+a_2)D_1+(-a_1-a_2+a_3)D_2+(a_1-a_2-a_3+a_4)D_3 \\ &\quad +(n+a_2-a_3-a_4)D_4+(a_2+a_3-a_4)D_5+a_1D_6, \end{aligned}$$

and therefore, $c_3+c_5=c_6$. It follows that

$$\Gamma_{D_4}=\langle D_1, D_2, D_4, D_3+D_6, D_5+D_6\rangle_{\mathbb{N}}.$$

Cases 4.6 and 4.7. Let us deal with case 4.6; the other one is similar. In this case $D_{\mathfrak{p}}$ is minuscule in $\mathbb{N}\Delta$. By following Example 2.7, this does not imply that the ring $\bigoplus_{n\in\mathbb{N}}\Gamma(X,\mathcal{L}_{nD_{\mathfrak{p}}})$ is generated by its degree 1 component $V_{D_{\mathfrak{p}}}=\Gamma(X,\mathcal{L}_{D_{\mathfrak{p}}})$; indeed, the multiplication of sections of globally generated line bundles on X is not necessarily surjective. However, using our methods we are still able to compute the normality and the weight semigroups of $\overline{Ke}$.

Enumerate the spherical roots and the colors of X as in Example 2.7. Then $D_{\mathfrak{p}}=D_4$, and by definition

$$\Gamma(\overline{Ke})=\bigcup_{n\in\mathbb{N}}\{\omega_E : E\in\mathbb{N}\Delta, V_E\subset V^n_{D_4}\}.$$

LEMMA 3.3

The following inclusions hold:

(1) $V_{D_1}\subset V^2_{D_4}$ *(where* $D_1=2D_4-\sigma_2-\sigma_3-2\sigma_4$*),*
(2) $V_{D_2}\subset V^2_{D_4}$ *(where* $D_2=2D_4-\sigma_3-\sigma_4$*),*
(3) $V_{D_3}\subset V_{D_1}V_{D_2}$ *(where* $D_3=D_1+D_2-\sigma_1$*),*
(4) $V_{D_5}\subset V_{D_1}V_{D_4}$ *(where* $D_5=D_1+D_4-\sigma_1-\sigma_3$*),*
(5) $V_{D_6}\subset V^2_{D_1}$ *(where* $D_6=2D_1-2\sigma_1-\sigma_3-\sigma_5$*).*

Proof
Consider the G-stable divisor $X' \subset X$ corresponding to σ_5: it is a parabolic induction of a wonderful variety of type $\mathsf{ab}^{\mathsf{y}}(2,2)$. By Proposition 2.15 together with Lemma 2.3 it follows that the multiplication of sections is surjective for all pairs of globally generated line bundles on X'. Denote by $\rho' : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X')$ the restriction of line bundles, and for $D \in \mathbb{N}\Delta$ set $D' = \rho'(D)$. By Lemma 2.14, (D'_4, D'_4, D'_1) and (D'_4, D'_4, D'_2) are low triples for X', and since $D'_5 \leq D'_1 + D'_4$ and $D'_3 \leq D'_1 + D'_2$ are coverings in $\mathrm{Pic}(X')$, it follows that (D'_1, D'_4, D'_5) and (D'_1, D'_2, D'_3) are low triples for X' as well. On the other hand, for all $D \in \mathbb{N}\Delta$ the G-modules V_D and $V_{D'}$ are canonically identified, and since the restriction $\Gamma(X, \mathcal{L}_D) \longrightarrow \Gamma(X', \mathcal{L}_{D'})$ is surjective, we get the inclusions (1), (2), (3), and (4).

We are left with the inclusion (5). Consider the distinguished subset of colors $\Delta_0 = \{D_2, D_3, D_4, D_5\}$, and denote by Y the quotient of X by Δ_0. Then Y is a rank 1 wonderful variety with spherical root $2\sigma_1 + \sigma_3 + \sigma_5$ whose set of colors is identified with $\{D_1, D_6\}$. By Lemma 2.2 we have that $\Gamma(X, \mathcal{L}_{nD_1}) = \Gamma(Y, \mathcal{L}_{nD_1})$ for all $n \in \mathbb{N}$. Since $D_6 \leq 2D_1$ is a covering in $\mathrm{Pic}(Y)$, the triple (D_1, D_1, D_6) is low in $\mathrm{Pic}(Y)$. On the other hand Y is a parabolic induction of a rank 1 symmetric variety, and for such a variety, the multiplication of sections of globally generated line bundles is known to be always surjective. By Lemma 2.3 the same holds for Y, and since it corresponds to a low triple we get the inclusion $V_{D_6} \subset V^2_{D_1}$. $\square$

PROPOSITION 3.4
We have that $\overline{Ke}$ is normal, and $\Gamma(\overline{Ke})$ is generated by the weights

$$\omega_2, \omega_4, \omega_1 + \omega'_1, \omega_2 + \omega'_2, \omega_1 + \omega_3 + \omega'_2, \omega_3 + \omega'_1.$$

Proof
Clearly, $\Gamma(\overline{Ke}) \subset \omega(\mathbb{N}\Delta)$. On the other hand by the previous lemma we have that $\omega(D) \in \Gamma(\overline{Ke})$ for all $D \in \Delta$; therefore, $\Gamma(\overline{Ke}) = \omega(\mathbb{N}\Delta)$ and the description of the generators follows by the description of the map ω.

Note that the weights $\omega(D_1), \ldots, \omega(D_6)$ are linearly independent. Therefore, $\Gamma(\overline{Ke})$ is a saturated semigroup of weights. (That is, if $\Gamma_{\mathbb{Z}} \subset \mathcal{X}(T)$ is the sublattice generated by $\Gamma(\overline{Ke})$ and if $\Gamma_{\mathbb{Q}^+} \subset \mathcal{X}(T) \otimes \mathbb{Q}$ is the cone generated by $\Gamma(\overline{Ke})$, then $\Gamma(\overline{Ke}) = \Gamma_{\mathbb{Z}} \cap \Gamma_{\mathbb{Q}^+}$.) It follows that $\overline{Ke}$ is normal. $\square$

3.2. Orthogonal cases

3.2.1. Tail cases

Cases without Roman numerals. Let us deal with case 7.2 ($r < q-1$). Cases 7.3 ($r < \min\{p, q-1\}$), 8.2 ($r < q$), 8.3 ($r < p$), 9.2 ($r < \min\{p-1, q-1\}$), and 9.3 ($r < \min\{p-1, q-1\}$) are similar. Suppose that $r > 0$; otherwise, we have a symmetric case. Let us keep the notation of Section 1.5.8. Therefore, for all $i = 1, \ldots, r$, $D_{2i-1} = D^-_{\alpha_i}$ and $D_{2i} = D^+_{\alpha_i}$; furthermore, $D_{2r+1} = D^-_{\alpha'_r}$, $D_{2r+2} =$

$D_{\alpha'_{r+1}}$, and $D_{2r+3} = D_{\alpha_{r+1}}$. To have a uniform notation, we denote

$$\tilde{D}_{2r+3} = \begin{cases} D_{2r+3} & \text{if } r < p-1, \\ 2D_{2r+3} & \text{if } r = p-1. \end{cases}$$

We have $D_{\mathfrak{p}} = D_2$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal.

Set $\tilde{D}_1 = D_1$, $\tilde{D}_2 = D_2$, and for all $k = 3, \dots, 2r+2$,

$$\tilde{D}_k = D_2 + D_{k-2} - (\sigma_1 + \cdots + \sigma_{k-2}).$$

Note that

$$\tilde{D}_k = \begin{cases} D_k & \text{if } k \leq 2r, \\ D_k + \tilde{D}_{2r+3} & \text{if } k = 2r+1, 2r+2. \end{cases}$$

PROPOSITION 3.5

The semigroup Γ_{D_2} is generated by $\tilde{D}_{2i}$ and $\tilde{D}_{2i-1} + \tilde{D}_{2j-1}$ for all $i, j = 1, \dots, r+1$.

Proof

Since $D_2 \in \Gamma_{D_2}$, by induction on the even indices, it follows that $\tilde{D}_{2i} \in \Gamma_{D_2}$ for all $i \leq r+1$. On the other hand,

$$D_1 = D_2 - \left(\sigma_2 + \sigma_4 + \cdots + \sigma_{2r-2} + \sigma_{2r} + \frac{\sigma_{2r+1}}{2}\right).$$

Therefore, for the odd indices, we get $\tilde{D}_{2i-1} + \tilde{D}_{2j-1} \in \Gamma_{D_2}$ for all $i, j \leq r+1$.

Let $\sigma \in \mathbb{N}\Sigma$, and suppose that $nD_2 - \sigma \in \mathbb{N}\Delta$. Let $\sigma = \sum a_i \sigma_i$ and $nD_2 - \sigma = \sum c_i D_i$. Note that if $r = p-1$, then c_{2r+3} is even. Therefore, $nD_2 - \sigma \in \langle D_1, \dots, D_{2r+2}, \tilde{D}_{2r+3} \rangle_{\mathbb{N}}$, and we write

$$nD_2 - \sigma = b_1 D_1 + \cdots + b_{2r+2} D_{2r+2} + b_{2r+3} \tilde{D}_{2r+3}.$$

Expressing the coefficients $b_1, \dots, b_{2r+3}$ with respect to $a_1, \dots, a_{2r+1}$ we get that $\sum_{i=1}^{r+1} b_{2i-1} = 2a_{2r+1}$ and $b_{2r+1} + b_{2r+2} = b_{2r+3}$. The claim follows. ☐

Cases with Roman numerals. Let us deal with case 9.2 ($r = p-1 < q-1$), (I). Case (II) and cases 7.3 ($r = q-1 < p$) and 9.3 ($r = q-1 < p-1$) are similar. First, let us suppose $r > 1$. Let us keep the notation of Section 1.5.8. Therefore, for all $i = 1, \dots, r$, $D_{2i-1} = D^-_{\alpha_i}$ and $D_{2i} = D^+_{\alpha_i}$; furthermore, $D_{2r+1} = D^-_{\alpha'_r}$, $D_{2r+2} = D_{\alpha'_{r+1}}$, and $D_{2r+3} = D_{\alpha_p}$. We have $D_{\mathfrak{p}} = D_2$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal.

Set $\tilde{D}_1 = D_1$ and $\tilde{D}_2 = D_2$, and define inductively, for all $k = 3, \dots, 2r+2$,

$$\tilde{D}_k = D_2 + \tilde{D}_{k-2} - (\sigma_1 + \cdots + \sigma_{k-2}).$$

Note that

$$\tilde{D}_k = \begin{cases} D_k & \text{if } k \leq 2r-2, \\ D_k + D_{2r+3} & \text{if } k = 2r-1, 2r, \\ D_k + 2D_{2r+3} & \text{if } k = 2r+1, 2r+2. \end{cases}$$

PROPOSITION 3.6

The semigroup Γ_{D_2} is generated by $\tilde{D}_{2i}$ and $\tilde{D}_{2i-1}+\tilde{D}_{2j-1}$ for all $i,j=1,\ldots,r+1$.

Proof

Since $D_2 \in \Gamma_{D_2}$, for the even indices, it follows that $\tilde{D}_{2i} \in \Gamma_{D_2}$ for all $i \le r+1$. On the other hand

$$D_1 = D_2 - \Big(\sigma_2+\sigma_4+\cdots+\sigma_{2r-2}+\sigma_{2r}+\frac{\sigma_{2r+1}}{2}\Big).$$

Therefore, for the odd indices, we get $\tilde{D}_{2i-1}+\tilde{D}_{2j-1} \in \Gamma_{D_2}$ for all $i,j \le r+1$.

Let $\sigma \in \mathbb{N}\Sigma$, and suppose that $nD_2-\sigma \in \mathbb{N}\Delta$. Denote $\sigma=\sum a_i\sigma_i$ and $nD_2-\sigma=\sum c_iD_i$. Expressing the coefficients $c_1,\ldots,c_{2r+3}$ with respect to $a_1,\ldots,a_{2r+1}$ we get that $\sum_{i=1}^{r+1} c_{2i-1}=2a_{2r+1}$ and $c_{2r-1}+c_{2r}+2c_{2r+1}+2c_{2r+2}=c_{2r+3}$. The claim follows. □

We are left with the case $r=1$, the case $r=0$ being symmetric. Here we have $D_{\mathfrak{p}}=D_2+D_5$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal. Proceeding as above we get the same semigroup

$$\Gamma_{D_2+D_5}=\langle D_2+D_5, D_4+2D_5, 2D_1+2D_5, D_1+D_3+3D_5, 2D_3+4D_5\rangle_{\mathbb{N}}.$$

3.2.2. Collapsed tails of type B

Cases without Roman numerals. Let us deal with case 8.2 ($r=q$). Cases 7.3 ($r=p<q-1$) and 8.3 ($r=p$) are similar. Let us keep the notation of Sections 1.5.6 and 2.5. Therefore, $D_{2i-1}=D^-_{\alpha_i}$ and $D_{2i}=D^+_{\alpha_i}$ for all $i=1,\ldots,r$, $D_{2r+1}=D^-_{\alpha'_q}$, and $D_{2r+2}=D_{\alpha_{r+1}}$.

We have $D_{\mathfrak{p}}=D_2$. Note that D_2 is not minuscule; indeed, $D_1=D_2-\sum_{i=1}^r \sigma_{2i}$. Therefore, $\overline{Ke}$ is not normal.

To have a uniform notation set

$$\tilde{D}_{2r+2}=\begin{cases} D_{2r+2} & \text{if } r<p-1,\\ 2D_{2r+2} & \text{if } r=p-1.\end{cases}$$

Set $\tilde{D}_1=D_1$ and $\tilde{D}_2=D_2$, and define inductively for all $i=3,\ldots,2r+1$

$$\tilde{D}_i=D_2+\tilde{D}_{i-2}-(\sigma_1+\cdots+\sigma_{i-2}).$$

Note that

$$\tilde{D}_i=\begin{cases} D_i & \text{if } i\le 2r-1,\\ D_{2r}+D_{2r+1} & \text{if } i=2r,\\ 2D_{2r+1}+\tilde{D}_{2r+2} & \text{if } i=2r+1.\end{cases}$$

PROPOSITION 3.7

The semigroup Γ_{D_2} is generated by $\tilde{D}_1,\ldots,\tilde{D}_{2r+1}$.

Proof
Let $\sigma \in \mathbb{N}\Sigma$, and suppose that $nD_2 - \sigma \in \mathbb{N}\Delta$. Denote $\sigma = \sum a_i\sigma_i$ and $nD_2 - \sigma = \sum c_i D_i$. Note that if $r = p-1$, then c_{2r+3} is even. Therefore, $nD_2 - \sigma \in \langle D_1, \dots, D_{2r+2}, \tilde{D}_{2r+3}\rangle_{\mathbb{N}}$, and we write

$$nD_2 - \sigma = b_1 D_1 + \cdots + b_{2r+2}D_{2r+2} + b_{2r+3}\tilde{D}_{2r+3}.$$

By expressing the spherical roots in terms of colors it follows that $b_{2r+1} = b_{2r} + 2b_{2r+2}$. The claim follows. $\square$

Case with Roman numerals. We deal here with case 7.3 ($r = p = q-1$), (I); case (II) is similar. The notation will be slightly different than before: let us enumerate the spherical roots $\Sigma = \{\sigma_1, \dots, \sigma_{2r}\}$ as

$$\sigma_{2i-1} = \alpha_i', \qquad \sigma_{2i} = \alpha_i \quad \text{for all } i = 1, \dots, r.$$

Accordingly, we enumerate the colors as $D_{2i-1} = D^-_{\alpha_i'}$ and $D_{2i} = D^+_{\alpha_i'}$ for all $i = 1, \dots, r$, $D_{2r+1} = D^-_{\alpha_p}$, and $D_{2r+2} = D_{\alpha_q'}$.

We have $D_{\mathfrak{p}} = D_2$. Note that D_2 is not minuscule; indeed, $D_1 = D_2 - \sum_{i=1}^r \sigma_{2i}$. Therefore, $\overline{Ke}$ is not normal.

Set $\tilde{D}_1 = D_1$ and $\tilde{D}_2 = D_2$, and define inductively for all $i = 3, \dots, 2r+1$

$$\tilde{D}_i = D_2 + \tilde{D}_{i-2} - (\sigma_1 + \cdots + \sigma_{i-2}).$$

Note that

$$\tilde{D}_i = \begin{cases} D_i & \text{if } i \le 2r-2, \\ D_{2r-1} + D_{2r+2} & \text{if } i = 2r-1, \\ D_{2r} + D_{2r+1} + D_{2r+2} & \text{if } i = 2r, \\ 2D_{2r+1} + 2D_{2r+2} & \text{if } i = 2r+1. \end{cases}$$

PROPOSITION 3.8
The semigroup Γ_{D_2} is generated by $\tilde{D}_1, \dots, \tilde{D}_{2r+1}$.

Proof
Let $\sigma \in \mathbb{N}\Sigma$, and suppose that $nD_2 - \sigma \in \mathbb{N}\Delta$. Let $\sigma = \sum a_i\sigma_i$ and $nD_2 - \sigma = \sum c_i D_i$. By expressing the spherical roots in terms of colors it follows that $c_{2r+1} - c_{2r} = 2a_{2r-1}$ and $c_{2r-1} + c_{2r+1} = c_{2r+2}$. The claim follows. $\square$

3.2.3. Collapsed tails of type D
Cases without Roman numerals. Let us deal with case 7.2 ($r = q-1$). Cases 9.2 ($r = q-1 < p-1$) and 9.3 ($r = p-1 < q-1$) are similar. Let us keep the notation of Section 1.5.8. Therefore, for all $i = 1, \dots, r$, $D_{2i-1} = D^-_{\alpha_i}$ and $D_{2i} = D^+_{\alpha_i}$; furthermore, $D_{2r+1} = D^-_{\alpha_r'}$, $D_{2r+2} = D^-_{\alpha_{r+1}'}$, and $D_{2r+3} = D_{\alpha_{r+1}}$. To have a uniform notation, we denote

$$\tilde{D}_{2r+3} = \begin{cases} D_{2r+3} & \text{if } r < p-1, \\ 2D_{2r+3} & \text{if } r = p-1. \end{cases}$$

We have $D_{\mathfrak{p}} = D_2$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal.

Set $\tilde{D}_1 = D_1$, $\tilde{D}_2 = D_2$, and for all $k = 3, \ldots, 2r+2$,

$$\tilde{D}_k = D_2 + D_{k-2} - (\sigma_1 + \cdots + \sigma_{k-2}).$$

Note that

$$\tilde{D}_k = \begin{cases} D_k & \text{if } k \leq 2r, \\ D_k + D_{2r+2} + \tilde{D}_{2r+3} & \text{if } k = 2r+1, 2r+2. \end{cases}$$

Furthermore, set

$$\tilde{D}'_{2r+2} = D_2 + D_{2r} - (\sigma_1 + \cdots + \sigma_{2r-1} + \sigma_{2r+1}) = 2D_{2r+1} + \tilde{D}_{2r+3}.$$

PROPOSITION 3.9

The semigroup Γ_{D_2} is generated by $\tilde{D}_{2i}$, $\tilde{D}_{2i-1} + \tilde{D}_{2j-1}$ for all $i, j = 1, \ldots, r+1$ with $i + j \leq 2r+1$, and $\tilde{D}'_{2r+2}$.

Proof

Since $D_2 \in \Gamma_{D_2}$, for the even indices, it follows that $\tilde{D}_{2i} \in \Gamma_{D_2}$ for all $i \leq r+1$, and $\tilde{D}'_{2r+2} \in \Gamma_{D_2}$ as well. On the other hand,

$$D_1 = D_2 - \Big(\sigma_2 + \sigma_4 + \cdots + \sigma_{2r-2} + \frac{\sigma_{2r} + \sigma_{2r+1}}{2}\Big).$$

Therefore, for the odd indices, we get $\tilde{D}_{2i-1} + \tilde{D}_{2j-1} \in \Gamma_{D_2}$ for all $i, j \leq r+1$.

Let $\sigma \in \mathbb{N}\Sigma$ and suppose that $nD_2 - \sigma \in \mathbb{N}\Delta$. Denote $\sigma = \sum a_i \sigma_i$ and $nD_2 - \sigma = \sum c_i D_i$. Note that if $r = p-1$, then c_{2r+3} is even. Therefore, $nD_2 - \sigma \in \langle D_1, \ldots, D_{2r+2}, \tilde{D}_{2r+3} \rangle_{\mathbb{N}}$, and we write

$$nD_2 - \sigma = b_1 D_1 + \cdots + b_{2r+2} D_{2r+2} + b_{2r+3} \tilde{D}_{2r+3}.$$

Expressing the coefficients $b_1, \ldots, b_{2r+3}$ with respect to $a_1, \ldots, a_{2r+1}$ we get that $\sum_{i=1}^{r+1} b_{2i-1} = 2a_{2r+1}$ and $b_{2r+1} + b_{2r+2} = 2b_{2r+3}$. The claim follows. □

Cases with Roman numerals. Let us deal with case 9.2 ($r = p-1 = q-1$), (I). Case (II) and case 9.3 ($r = q-1 = p-1$) are similar. First, let us suppose $r > 1$. Let us keep the notation of Section 1.5.8. Therefore, for all $i = 1, \ldots, r$, $D_{2i-1} = D^-_{\alpha_i}$ and $D_{2i} = D^+_{\alpha_i}$; furthermore, $D_{2r+1} = D^-_{\alpha'_{q-1}}$, $D_{2r+2} = D^-_{\alpha'_q}$, and $D_{2r+3} = D_{\alpha_p}$. We have $D_{\mathfrak{p}} = D_2$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal.

Set $\tilde{D}_1 = D_1$ and $\tilde{D}_2 = D_2$, and define inductively, for all $k = 3, \ldots, 2r+2$,

$$\tilde{D}_k = D_2 + \tilde{D}_{k-2} - (\sigma_1 + \cdots + \sigma_{k-2}).$$

Note that

$$\tilde{D}_k = \begin{cases} D_k & \text{if } k \leq 2r-2, \\ D_k + D_{2r+3} & \text{if } k = 2r-1, 2r, \\ D_k + D_{2r+2} + 2D_{2r+3} & \text{if } k = 2r+1, 2r+2. \end{cases}$$

Furthermore, set

$$\tilde{D}'_{2r+2} = D_2 + \tilde{D}_{2r} - (\sigma_1 + \cdots + \sigma_{2r-1} + \sigma_{2r+1}) = 2D_{2r+1} + 2D_{2r+3}.$$

PROPOSITION 3.10

The semigroup Γ_{D_2} is generated by $\tilde{D}_{2i}$, $\tilde{D}_{2i-1} + \tilde{D}_{2j-1}$ for all $i, j = 1, \ldots, r+1$ with $i + j \leq 2r+1$, and $\tilde{D}'_{2r+2}$.

Proof

Since $D_2 \in \Gamma_{D_2}$, for the even indices, it follows that $\tilde{D}_{2i} \in \Gamma_{D_2}$ for all $i \leq r+1$, and $\tilde{D}'_{2r+2} \in \Gamma_{D_2}$ as well. On the other hand

$$D_1 = D_2 - \Big(\sigma_2 + \sigma_4 + \cdots + \sigma_{2r-2} + \frac{\sigma_{2r} + \sigma_{2r+1}}{2}\Big).$$

Therefore, for the odd indices, we get $\tilde{D}_{2i-1} + \tilde{D}_{2j-1} \in \Gamma_{D_2}$ for all $i, j \leq r+1$.

Let $\sigma \in \mathbb{N}\Sigma$, and suppose that $nD_2 - \sigma \in \mathbb{N}\Delta$. Let $\sigma = \sum a_i \sigma_i$ and $nD_2 - \sigma = \sum c_i D_i$. Expressing the coefficients $c_1, \ldots, c_{2r+3}$ with respect to $a_1, \ldots, a_{2r+1}$ we get that $\sum_{i=1}^{r+1} c_{2i-1} = 2a_{2r+1}$ and $c_{2r-1} + c_{2r} + c_{2r+1} + c_{2r+2} = c_{2r+3}$. The claim follows. □

We are left with the case $r = 1$, the case $r = 0$ being symmetric. Here we have $D_{\mathfrak{p}} = D_2 + D_5$, which is minuscule in $\mathbb{N}\Delta$; therefore, $\overline{Ke}$ is normal. Proceeding as above we get the same semigroup; that is, $\Gamma_{D_2+D_5}$ is generated by $D_2 + D_5$, $2D_4 + 2D_5$, $2D_3 + 2D_5$, $2D_1 + 2D_5$, $D_1 + D_3 + D_4 + 3D_5$.

Appendix A: List of spherical nilpotent K-orbits in $\mathfrak{p}$ in the classical non-Hermitian cases

Here we report the list of the spherical nilpotent K-orbits in $\mathfrak{p}$ for all symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of classical non-Hermitian type. Every K-orbit in $\mathfrak{p}$ is labeled with the signed partition of the corresponding real nilpotent orbit, via the Kostant–Sekiguchi–Đoković bijection. For every orbit we provide a representative e and a normal triple containing it $\{h, e, f\}$.

For all $i \in \mathbb{Z}$, let $\mathfrak{k}(i)$ be the ad$h$-eigenspace in $\mathfrak{k}$ of eigenvalue i. We denote by Q the parabolic subgroup of K whose Lie algebra is equal to

$$\operatorname{Lie} Q = \bigoplus_{i \geq 0} \mathfrak{k}(i).$$

In each case we describe the centralizer of h, which we denote by K_h or by L, which is a Levi subgroup of Q. We denote by Q^{u} the unipotent radical of Q. Then we describe the centralizer of e, which we denote by K_e. A Levi subgroup of K_e is always given by L_e, the centralizer of e in L. The unipotent radical of K_e is either equal to Q^{u} or equal to a cosimple L_e-submodule of Q^{u}. In the latter case, there always exist some simple L_e-submodules in $\mathfrak{k}(1)$, say, $W_0, \ldots, W_d$, which we determine, with the following properties. They are isomorphic as L_e-modules but lie in pairwise distinct isotypical L-components. By denoting by V

the L_e-complement of $W_0 \oplus \cdots \oplus W_d$ in $\operatorname{Lie} Q^{\mathrm{u}}$, as an L_e-module,

$$\operatorname{Lie} K_e^{\mathrm{u}} = W \oplus V,$$

where W is a cosimple L_e-submodule of $W_0 \oplus \cdots \oplus W_d$ which projects nontrivially on every summand $W_0, \ldots, W_d$. Actually, the integer $d+1$, the number of the above simple L_e-modules $W_0, \ldots, W_d$, will only be equal to 2 or 3.

REMARK A.1
As already mentioned, the list of spherical nilpotent K-orbits in $\mathfrak{p}$ is in [19], and all the data in our list, such as a representative and its centralizer, can be directly computed using the information contained therein, with one exception. There is one missing case in [19], corresponding to the signed partition $(+3^4, +1^{2n-8})$ for the symmetric pair $(\mathfrak{sp}(2n+4), \mathfrak{sp}(2n) + \mathfrak{sp}(4))$ with $n \geq 4$ (cases 4.6 and 4.7 in Appendix A). The lack comes from a small mistake in [19, Lemma 7.2]; we have checked that there is no further missing case arising from that lemma. The smallest case of this family, which is for $n = 4$, was already present in [27, Example 5.8].

A.1 $\mathfrak{sl}(2n)/\mathfrak{sp}(2n)$
We set $K = \mathrm{Sp}(2n)$, $n \geq 2$, $\mathfrak{p} = V(\omega_2)$. Let us fix a basis $e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}$ of $\mathbb{C}^{2n}$, a skew-symmetric bilinear form ω such that $\omega(e_i, e_j) = \delta_{i,-j}$ for $1 \leq i \leq n$, and $K = \mathrm{Sp}(\mathbb{C}^{2n}, \omega)$. Then ω can be seen as a linear form on $\Lambda^2 \mathbb{C}^{2n}$, $\omega(e_i \wedge e_j) = \omega(e_i, e_j)$, and

$$\mathfrak{p} = \ker \omega \subset \Lambda^2 \mathbb{C}^{2n}.$$

A.1.1 $(2^r, 1^{n-2r})$ $r \geq 1$
We take

$$e = \sum_{i=1}^{r} e_i \wedge e_{2r-i+1}, \qquad h(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq 2r, \\ -e_i & \text{if } -2r \leq i \leq -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$f = \sum_{i=1}^{r} e_{-2r+i-1} \wedge e_{-i}.$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(2r) \times \mathrm{Sp}(2n-4r)$. The centralizer of e is $K_e = L_e Q^{\mathrm{u}}$, where $L_e \cong \mathrm{Sp}(2r) \times \mathrm{Sp}(2n-4r)$.

A.2 $\mathfrak{sl}(2n+1)/\mathfrak{so}(2n+1)$
We set $K = \mathrm{SO}(2n+1)$, $n \geq 2$, $\mathfrak{p} = V(2\omega_1)$. If $n = 1$, then $\mathfrak{p} = V(4\omega)$. Let us fix a basis $e_1, \ldots, e_n, e_0, e_{-n}, \ldots, e_{-1}$ of $\mathbb{C}^{2n+1}$, a symmetric bilinear form β such that $\beta(e_i, e_j) = \delta_{i,-j}$ for all i, j, and $K = \mathrm{SO}(\mathbb{C}^{2n+1}, \beta)$. Then β can be seen as a linear form on $\mathsf{S}^2 \mathbb{C}^{2n+1}$, $\beta(e_i e_j) = \beta(e_i, e_j)$, and

$$\mathfrak{p} = \ker \beta \subset \mathsf{S}^2 \mathbb{C}^{2n+1}.$$

A.2.1 $(2^r, 1^{2n-2r+1})$, $r \geq 1$
We take

$$e = \sum_{i=1}^{r} e_i e_{r-i+1}, \qquad h(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq r, \\ -e_i & \text{if } -r \leq i \leq -1, \\ 0 & \text{otherwise,} \end{cases} \qquad f = \sum_{i=1}^{r} e_{-r+i-1} e_{-i}.$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(r) \times \mathrm{SO}(2n-2r+1)$. The centralizer of e is $K_e = L_e Q^{\mathrm{u}}$, where $L_e \cong \mathrm{O}(r) \times \mathrm{SO}(2n-2r+1)$.

A.3 $\mathfrak{sl}(2n)/\mathfrak{so}(2n)$
We set $K = \mathrm{SO}(2n)$, $n \geq 3$, $\mathfrak{p} = V(2\omega_1)$. If $n = 2$, then $\mathfrak{p} = V(2\omega + 2\omega')$. Let us fix a basis $e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}$ of $\mathbb{C}^{2n}$, a symmetric bilinear form β such that $\beta(e_i, e_j) = \delta_{i,-j}$ for all i, j, and $K = \mathrm{SO}(\mathbb{C}^{2n}, \beta)$. Then β can be seen as a linear form on $\mathsf{S}^2\mathbb{C}^{2n}$, $\beta(e_i e_j) = \beta(e_i, e_j)$, and

$$\mathfrak{p} = \ker \beta \subset \mathsf{S}^2\mathbb{C}^{2n}.$$

Let us denote by τ the linear endomorphism of $\mathbb{C}^{2n}$ switching e_n and e_{-n} and fixing all the other basis vectors. The conjugation by τ is an involutive internal automorphism of $\mathfrak{g}$, leaving $\mathfrak{k}$ and $\mathfrak{p}$ stable, and inducing the nontrivial involution of the Dynkin diagram of $\mathfrak{k}$.

A.3.1 $(2^r, 1^{2n-2r})$, $r \geq 1$
If $r < n$, we take

$$e = \sum_{i=1}^{r} e_i e_{r-i+1}, \qquad h(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq r, \\ -e_i & \text{if } -r \leq i \leq -1, \\ 0 & \text{otherwise,} \end{cases} \qquad f = \sum_{i=1}^{r} e_{-r+i-1} e_{-i}.$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(r) \times \mathrm{SO}(2n-2r)$. The centralizer of e is $K_e = L_e Q^{\mathrm{u}}$, where $L_e \cong \mathrm{O}(r) \times \mathrm{SO}(2n-2r)$.

If $r = n$, then there exist two orbits labeled I and II. Case (I) can be described as above by specializing r equal to n. Case (II) can be obtained from case (I) by conjugating by τ.

A.4 $\mathfrak{sp}(2p+2q)/\mathfrak{sp}(2p) + \mathfrak{sp}(2q)$
We set $K = \mathrm{Sp}(2p) \times \mathrm{Sp}(2q)$, $p, q \geq 1$, $\mathfrak{p} = V(\omega_1 + \omega_1')$. Let us fix a basis $e_1, \ldots, e_p, e_{-p}, \ldots, e_{-1}$ of $\mathbb{C}^{2p}$ and a skew-symmetric bilinear form ω such that $\omega(e_i, e_j) = \delta_{i,-j}$ for $1 \leq i \leq p$. Similarly, let us fix a basis $e_1', \ldots, e_q', e_{-q}', \ldots, e_{-1}'$ of $\mathbb{C}^{2q}$ and a skew-symmetric bilinear form ω' such that $\omega'(e_i', e_j') = \delta_{i,-j}$ for $1 \leq i \leq q$. Then $K = \mathrm{Sp}(\mathbb{C}^{2p}, \omega) \times \mathrm{Sp}(\mathbb{C}^{2q}, \omega')$ and

$$\mathfrak{p} = \mathbb{C}^{2p} \otimes \mathbb{C}^{2q}.$$

A.4.1 $(+2^{2r}, +1^{2p-2r}, -1^{2q-2r})$, $r \geq 1$
We take

$$e = \sum_{i=1}^{r} e_i \otimes e'_{r-i+1}, \qquad f = -\sum_{i=1}^{r} e_{-r+i-1} \otimes e'_{-i},$$

$$h(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq r, \\ -e_i & \text{if } -r \leq i \leq -1, \\ 0 & \text{otherwise,} \end{cases} \qquad h(e'_i) = \begin{cases} e'_i & \text{if } 1 \leq i \leq r, \\ -e'_i & \text{if } -r \leq i \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(r) \times \mathrm{Sp}(2p-2r) \times \mathrm{GL}(r) \times \mathrm{Sp}(2q-2r)$. The centralizer of e is $K_e = L_e Q^{\mathrm{u}}$, where $L_e \cong \mathrm{GL}(r) \times \mathrm{Sp}(2p-2r) \times \mathrm{Sp}(2q-2r)$, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally, $A \mapsto (A, A^{-1})$, into the $\mathrm{GL}(r) \times \mathrm{GL}(r)$ factor of L.

A.4.2 $(+3^2, +1^{2p-4}, -1^{2q-2})$
We take

$$e = e_1 \otimes e'_{-1} - e_2 \otimes e'_1, \qquad f = 2e_{-2} \otimes e'_{-1} + 2e_{-1} \otimes e'_1,$$

$$h(e_i) = \begin{cases} 2e_i & \text{if } 1 \leq i \leq 2, \\ -2e_i & \text{if } -2 \leq i \leq -1, \\ 0 & \text{otherwise,} \end{cases} \qquad h(e'_i) = 0 \ \forall i.$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(2) \times \mathrm{Sp}(2p-4) \times \mathrm{Sp}(2q)$. The centralizer of e is $K_e = L_e Q^{\mathrm{u}}$, where $L_e \cong \mathrm{SL}(2) \times \mathrm{Sp}(2p-4) \times \mathrm{Sp}(2q-2)$, and the $\mathrm{SL}(2) \times \mathrm{Sp}(2q-2)$ factor of L_e is embedded as

$$(A, B) \mapsto (A, A, B)$$

into $\mathrm{SL}(2) \times \mathrm{Sp}(2) \times \mathrm{Sp}(2q-2)$, where the $\mathrm{SL}(2)$ factor is included in the $\mathrm{GL}(2)$ factor of L and the $\mathrm{Sp}(2) \times \mathrm{Sp}(2q-2)$ factor is included in the $\mathrm{Sp}(2q)$ factor of L.

A.4.3 $(-3^2, +1^{2p-2}, -1^{2q-4})$
This case can be obtained from case 4.2 by switching the roles of p and q.

A.4.4 $(+3^2, +2^2, +1^{2p-6}, -1^{2q-4})$
We take

$$e = e_1 \otimes e'_{-2} - e_2 \otimes e'_2 - e_3 \otimes e'_1,$$

$$f = e_{-3} \otimes e'_{-1} + 2e_{-2} \otimes e'_{-2} + 2e_{-1} \otimes e'_2,$$

$$h(e_i) = \begin{cases} 2e_i & \text{if } 1 \le i \le 2, \\ e_i & \text{if } i = 3, \\ -e_i & \text{if } i = -3, \\ -2e_i & \text{if } -2 \le i \le -1, \\ 0 & \text{otherwise,} \end{cases} \qquad h(e'_i) = \begin{cases} e'_i & \text{if } i = 1, \\ -e'_i & \text{if } i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(2) \times \mathrm{GL}(1) \times \mathrm{Sp}(2p-6) \times \mathrm{GL}(1) \times \mathrm{Sp}(2q-2)$. The centralizer of e is $K_e = L_e K_e^{\mathrm{u}}$, where $L_e \cong \mathrm{SL}(2) \times \mathrm{GL}(1) \times \mathrm{Sp}(2p-3) \times \mathrm{Sp}(2q-4)$, the $\mathrm{SL}(2) \times \mathrm{Sp}(2q-4)$ factor of L_e is embedded as

$$(A, B) \mapsto (A, A, B)$$

into $\mathrm{SL}(2) \times \mathrm{Sp}(2) \times \mathrm{Sp}(2q-4)$, where the $\mathrm{SL}(2)$ factor is included in the $\mathrm{GL}(2)$ factor of L and the $\mathrm{Sp}(2) \times \mathrm{Sp}(2q-4)$ factor is included in the $\mathrm{Sp}(2q-2)$ factor of L, and the $\mathrm{GL}(1)$ factor of L_e is embedded skew-diagonally

$$z \mapsto (z, z^{-1})$$

into the $\mathrm{GL}(1) \times \mathrm{GL}(1)$ factor of L. The quotient $\mathrm{Lie}\, Q^{\mathrm{u}} / \mathrm{Lie}\, K_e^{\mathrm{u}}$ is a simple L_e-module of dimension 2 as follows. In $\mathfrak{k}(1)$ there are exactly two simple L_e-submodules, W_0, W_1, of highest weight ω_1 with respect to the $\mathrm{SL}(2)$ factor, isomorphic as L_e-modules but lying in two distinct isotypical L-components. Let V be the L_e-complement of $W_0 \oplus W_1$ in $\mathrm{Lie}\, Q^{\mathrm{u}}$. As an L_e-module, $\mathrm{Lie}\, K_e^{\mathrm{u}}$ is the direct sum of V and a simple L_e-submodule of $W_0 \oplus W_1$ which projects nontrivially on both summands W_0 and W_1.

A.4.5 $(-3^2, +2^2, +1^{2p-4}, -1^{2q-6})$

This case can be obtained from case 4.4 by switching the roles of p and q.

A.4.6 $(+3^4, +1^{2p-8})$, $q = 2$

We take

$$\begin{aligned} e &= e_1 \otimes e'_{-1} + e_2 \otimes e'_{-2} - e_3 \otimes e'_2 - e_4 \otimes e'_1, \\ f &= 2(e_{-4} \otimes e'_{-1} + e_{-3} \otimes e'_{-2} + e_{-2} \otimes e'_2 + e_{-1} \otimes e'_1), \end{aligned}$$

$$h(e_i) = \begin{cases} 2e_i & \text{if } 1 \le i \le 4, \\ -2e_i & \text{if } -4 \le i \le -1, \\ 0 & \text{otherwise,} \end{cases} \qquad h(e'_i) = 0 \ \forall i.$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(4) \times \mathrm{Sp}(2p-8) \times \mathrm{Sp}(4)$. The centralizer of e is $K_e = L_e Q^{\mathrm{u}}$, where $L_e \cong \mathrm{Sp}(4) \times \mathrm{Sp}(2p-8)$, and the $\mathrm{Sp}(4)$ factor of L_e is embedded diagonally, $A \mapsto (A, A)$, into the $\mathrm{GL}(4) \times \mathrm{Sp}(4)$ factor of L.

A.4.7 $(-3^4, -1^{2q-8})$, $p = 2$

This case can be obtained from case 4.6 by switching the roles of p and q.

A.5 $\mathfrak{so}(2n+1)/\mathfrak{so}(2n)$

We set $K=\mathrm{SO}(2n)$, $n\geq 3$, $\mathfrak{p}=V(\omega_1)$. If $n=2$, then $\mathfrak{p}=V(\omega+\omega')$. Let us fix a basis $e_1,\dots,e_n,e_{-n},\dots,e_{-1}$ of $\mathbb{C}^{2n}$, a symmetric bilinear form β such that $\beta(e_i,e_j)=\delta_{i,-j}$ for all i,j, and $K=\mathrm{SO}(\mathbb{C}^{2n},\beta)$. Then

$$\mathfrak{p}=\mathbb{C}^{2n}.$$

A.5.1 $(+3,+1^{2n-2})$

We take

$$e=e_1,\qquad h(e_i)=\begin{cases}2e_i & \text{if } i=1,\\ -2e_i & \text{if } i=-1,\\ 0 & \text{otherwise,}\end{cases}\qquad f=-2e_{-1}.$$

Let $Q=LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L=K_h\cong \mathrm{GL}(1)\times\mathrm{SO}(2n-2)$. The centralizer of e is $K_e=L_eQ^{\mathrm{u}}$, where $L_e\cong\mathrm{SO}(2n-2)$.

A.6 $\mathfrak{so}(2n+2)/\mathfrak{so}(2n+1)$

We set $K=\mathrm{SO}(2n+1)$, $n\geq 2$, $\mathfrak{p}=V(\omega_1)$. If $n=1$, then $\mathfrak{p}=V(2\omega)$. Let us fix a basis $e_1,\dots,e_n,e_0,e_{-n},\dots,e_{-1}$ of $\mathbb{C}^{2n+1}$, a symmetric bilinear form β such that $\beta(e_i,e_j)=\delta_{i,-j}$ for all i,j, and $K=\mathrm{SO}(\mathbb{C}^{2n+1},\beta)$. Then

$$\mathfrak{p}=\mathbb{C}^{2n+1}.$$

A.6.1 $(+3,+1^{2n-1})$

We take

$$e=e_1,\qquad h(e_i)=\begin{cases}2e_i & \text{if } i=1,\\ -2e_i & \text{if } i=-1,\\ 0 & \text{otherwise,}\end{cases}\qquad f=-2e_{-1}.$$

Let $Q=LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L=K_h\cong \mathrm{GL}(1)\times\mathrm{SO}(2n-1)$. The centralizer of e is $K_e=L_eQ^{\mathrm{u}}$, where $L_e\cong\mathrm{SO}(2n-1)$.

A.7 $\mathfrak{so}(2p+2q+1)/\mathfrak{so}(2p+1)+\mathfrak{so}(2q)$

We set $K=\mathrm{SO}(2p+1)\times\mathrm{SO}(2q)$, $p\geq 2$, $q\geq 3$, $\mathfrak{p}=V(\omega_1+\omega_1')$. If $p=1$ and $q\geq 3$, then $\mathfrak{p}=V(2\omega+\omega_1')$. If $p\geq 2$ and $q=2$, then $\mathfrak{p}=V(\omega_1+\omega'+\omega'')$. If $p=1$ and $q=2$, then $\mathfrak{p}=V(2\omega+\omega'+\omega'')$. Let us fix a basis $e_1,\dots,e_p,e_0,e_{-p},\dots,e_{-1}$ of $\mathbb{C}^{2p+1}$ and a symmetric bilinear form β such that $\beta(e_i,e_j)=\delta_{i,-j}$ for all i,j. Similarly, let us fix a basis $e_1',\dots,e_q',e_{-q}',\dots,e_{-1}'$ of $\mathbb{C}^{2q}$ and a symmetric bilinear form β' such that $\beta'(e_i',e_j')=\delta_{i,-j}$ for all i,j. Then $K=\mathrm{SO}(\mathbb{C}^{2p+1},\beta)\times\mathrm{SO}(\mathbb{C}^{2q},\beta')$ and

$$\mathfrak{p}=\mathbb{C}^{2p+1}\otimes\mathbb{C}^{2q}.$$

Let us denote by τ the linear endomorphism of $\mathbb{C}^{2p+2q+1}$ switching e_q' and e_{-q}' and fixing all the other basis vectors. The conjugation by τ is an involutive internal automorphism of $\mathfrak{g}$, leaving $\mathfrak{k}$ and $\mathfrak{p}$ stable, and inducing the nontrivial involution of the Dynkin diagram of $\mathfrak{k}$.

A.7.1 $(+2^{2r}, +1^{2p+1-2r}, -1^{2q-2r})$, $r \geq 1$
If $r < q$, we take

$$e = \sum_{i=1}^{r} e_i \otimes e'_{r-i+1}, \qquad f = -\sum_{i=1}^{r} e_{-r+i-1} \otimes e'_{-i},$$

$$h(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq r, \\ -e_i & \text{if } -r \leq i \leq -1, \\ 0 & \text{otherwise,} \end{cases} \qquad h(e'_i) = \begin{cases} e'_i & \text{if } 1 \leq i \leq r, \\ -e'_i & \text{if } -r \leq i \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(r) \times \mathrm{SO}(2p-2r+1) \times \mathrm{GL}(r) \times \mathrm{SO}(2q-2r)$. The centralizer of e is $K_e = L_e Q^{\mathrm{u}}$, where $L_e \cong \mathrm{GL}(r) \times \mathrm{SO}(2p-2r+1) \times \mathrm{SO}(2q-2r)$, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally, $A \mapsto (A, A^{-1})$, into the $\mathrm{GL}(r) \times \mathrm{GL}(r)$ factor of L.

If $r = q$, then there exist two orbits labeled I and II. Case (I) can be described as above by specializing r equal to q. Case (II) can be obtained from case (I) by conjugating by τ.

A.7.2 $(+3, +2^{2r}, +1^{2p-1-2r}, -1^{2q-1-2r})$
If $r \leq q-2$, we take

$$e = e_1 \otimes (e'_q + e'_{-q}) + \sum_{i=1}^{r} e_{i+1} \otimes e'_{r-i+1},$$

$$f = -\Big(\sum_{i=1}^{r} e_{-r+i-2} \otimes e'_{-i}\Big) - e_{-1} \otimes (e'_q + e'_{-q}),$$

$$h(e_i) = \begin{cases} 2e_i & \text{if } i = 1, \\ e_i & \text{if } 2 \leq i \leq r+1, \\ -e_i & \text{if } -r-1 \leq i \leq -2, \\ -2e_i & \text{if } i = -1, \\ 0 & \text{otherwise,} \end{cases} \qquad h(e'_i) = \begin{cases} e'_i & \text{if } 1 \leq i \leq r, \\ -e'_i & \text{if } -r \leq i \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(1) \times \mathrm{GL}(r) \times \mathrm{SO}(2p-2r-1) \times \mathrm{GL}(r) \times \mathrm{SO}(2q-2r)$. The centralizer of e is $K_e = L_e K_e^{\mathrm{u}}$, where $L_e \cong \mathrm{GL}(r) \times \mathrm{SO}(2p-2r-1) \times \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2q-2r-1))$, the $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2p-2r-1))$ factor of L_e is embedded as

$$(z, A) \mapsto (z, z, A)$$

into $\mathrm{GL}(1) \times \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2q-2r-1))$, where the $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2q-2r-1))$ factor is included in the $\mathrm{SO}(2q-2r)$ factor of L, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally

$$B \mapsto (B, B^{-1})$$

into the $\mathrm{GL}(r) \times \mathrm{GL}(r)$ factor of L. The quotient $\operatorname{Lie} Q^{\mathrm{u}} / \operatorname{Lie} K_e^{\mathrm{u}}$ is a simple L_e-module of dimension r as follows. In $\mathfrak{k}(1)$ there are exactly two simple L_e-submodules, W_0, W_1, of highest weight ω_{r-1} with respect to the $\mathrm{GL}(r)$ factor, isomorphic as L_e-modules but lying in two distinct isotypical L-components. Let V be the L_e-complement of $W_0 \oplus W_1$ in $\operatorname{Lie} Q^{\mathrm{u}}$. As an L_e-module, $\operatorname{Lie} K_e^{\mathrm{u}}$ is the direct sum of V and a simple L_e-submodule of $W_0 \oplus W_1$ which projects nontrivially on both summands W_0 and W_1.

If $r = q-1$, then the normal triple h, e, f, the parabolic subgroup $Q = LQ^{\mathrm{u}}$, and L_e have the same description, with $K_e = L_e K_e^{\mathrm{u}}$. The quotient $\operatorname{Lie} Q^{\mathrm{u}} / \operatorname{Lie} K_e^{\mathrm{u}}$ remains a simple L_e-module of dimension $q-1$, but here in $\mathfrak{k}(1)$ there are exactly three simple L_e-submodules, W_0, W_1, W_2, of highest weight ω_{q-2} with respect to the $\mathrm{GL}(q-1)$ factor, isomorphic as L_e-modules but lying in three distinct isotypical L-components. Let V be the L_e-complement of $W_0 \oplus W_1 \oplus W_2$ in $\operatorname{Lie} Q^{\mathrm{u}}$. As an L_e-module, $\operatorname{Lie} K_e^{\mathrm{u}}$ is the direct sum of V and a cosimple L_e-submodule of $W_0 \oplus W_1 \oplus W_2$ which projects nontrivially on every summand W_0, W_1, and W_2.

A.7.3 $(-3, +2^{2r}, +1^{2p-2r}, -1^{2q-2-2r})$
If $r \le q-2$, we take

$$e = \Big(\sum_{i=1}^{r} e_i \otimes e'_{r-i+2} \Big) + e_0 \otimes e'_1, \qquad f = -2e_0 \otimes e'_{-1} - \sum_{i=1}^{r} e_{-r+i-1} \otimes e'_{-i-1},$$

$$h(e_i) = \begin{cases} e_i & \text{if } 1 \le i \le r, \\ -e_i & \text{if } -r \le i \le -1, \\ 0 & \text{otherwise,} \end{cases} \qquad h(e'_i) = \begin{cases} 2e'_i & \text{if } i = 1, \\ e'_i & \text{if } 2 \le i \le r+1, \\ -e'_i & \text{if } -r-1 \le i \le -2, \\ -2e'_i & \text{if } i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(r) \times \mathrm{SO}(2p-2r+1) \times \mathrm{GL}(1) \times \mathrm{GL}(r) \times \mathrm{SO}(2q-2r-2)$. The centralizer of e is $K_e = L_e K_e^{\mathrm{u}}$, where $L_e \cong \mathrm{GL}(r) \times \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2p-2r)) \times \mathrm{SO}(2q-2r-2)$, the $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2p-2r))$ factor of L_e is embedded as

$$(z, A) \mapsto (z, A, z)$$

into $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2p-2r)) \times \mathrm{GL}(1)$, where the $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2p-2r))$ factor is included in the $\mathrm{SO}(2p-2r+1)$ factor of L, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally

$$B \mapsto (B, B^{-1})$$

into the $\mathrm{GL}(r) \times \mathrm{GL}(r)$ factor of L. The quotient $\operatorname{Lie} Q^{\mathrm{u}} / \operatorname{Lie} K_e^{\mathrm{u}}$ is a simple L_e-module of dimension r as follows. In $\mathfrak{k}(1)$ there are exactly two simple L_e-submodules, W_0, W_1, of highest weight ω_1 with respect to the $\mathrm{GL}(r)$ factor, isomorphic as L_e-modules but lying in two distinct isotypical L-components. Let V be the L_e-complement of $W_0 \oplus W_1$ in $\operatorname{Lie} Q^{\mathrm{u}}$. As an L_e-module, $\operatorname{Lie} K_e^{\mathrm{u}}$ is the direct sum of V and a simple L_e-submodule of $W_0 \oplus W_1$ which projects nontrivially on both summands W_0 and W_1.

If $r = q-1$, then there exist two orbits labeled I and II. Case (I) can be described as above by specializing r equal to $q-1$. Case (II) can be obtained from case (I) by conjugating by τ.

A.8 $\mathfrak{so}(2p+2q+2)/\mathfrak{so}(2p+1)+\mathfrak{so}(2q+1)$

We set $K = \mathrm{SO}(2p+1)\times\mathrm{SO}(2q+1)$, $p,q \geq 2$, $\mathfrak{p} = V(\omega_1+\omega_1')$. If $p=1$ and $q \geq 2$, then $\mathfrak{p} = V(2\omega+\omega_1')$. If $p \geq 2$ and $q=1$, then $\mathfrak{p} = V(\omega_1+2\omega')$. If $p=q=1$, then $\mathfrak{p} = V(2\omega+2\omega')$.

Let us fix a basis $e_1,\dots,e_p,e_0,e_{-p},\dots,e_{-1}$ of $\mathbb{C}^{2p+1}$ and a symmetric bilinear form β such that $\beta(e_i,e_j) = \delta_{i,-j}$ for all i,j. Similarly, let us fix a basis $e_1',\dots,e_q',e_0',\ e_{-q}',\dots,e_{-1}'$ of $\mathbb{C}^{2q+1}$ and a symmetric bilinear form β' such that $\beta'(e_i',e_j') = \delta_{i,-j}$ for all i,j. Then $K = \mathrm{SO}(\mathbb{C}^{2p+1},\beta)\times\mathrm{SO}(\mathbb{C}^{2q+1},\beta')$ and

$$\mathfrak{p} = \mathbb{C}^{2p+1}\otimes\mathbb{C}^{2q+1}.$$

A.8.1 $(+2^{2r},+1^{2p+1-2r},-1^{2q+1-2r})$, $r \geq 1$

We take

$$e = \sum_{i=1}^{r} e_i \otimes e_{r-i+1}', \qquad f = -\sum_{i=1}^{r} e_{-r+i-1}\otimes e_{-i}',$$

$$h(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq r,\\ -e_i & \text{if } -r \leq i \leq -1,\\ 0 & \text{otherwise,}\end{cases} \qquad h(e_i') = \begin{cases} e_i' & \text{if } 1 \leq i \leq r,\\ -e_i' & \text{if } -r \leq i \leq -1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(r)\times\mathrm{SO}(2p-2r+1)\times\mathrm{GL}(r)\times\mathrm{SO}(2q-2r+1)$. The centralizer of e is $K_e = L_eQ^{\mathrm{u}}$, where $L_e \cong \mathrm{GL}(r)\times\mathrm{SO}(2p-2r+1)\times\mathrm{SO}(2q-2r+1)$, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally, $A \mapsto (A, A^{-1})$, into the $\mathrm{GL}(r)\times\mathrm{GL}(r)$ factor of L.

A.8.2 $(+3,+2^{2r},+1^{2p-1-2r},-1^{2q-2r})$

We take

$$e = e_1\otimes e_0' + \sum_{i=1}^{r} e_{i+1}\otimes e_{r-i+1}', \qquad f = -\Big(\sum_{i=1}^{r} e_{-r+i-2}\otimes e_{-i}'\Big) - 2e_{-1}\otimes e_0',$$

$$h(e_i) = \begin{cases} 2e_i & \text{if } i=1,\\ e_i & \text{if } 2 \leq i \leq r+1,\\ -e_i & \text{if } -r-1 \leq i \leq -2,\\ -2e_i & \text{if } i=-1,\\ 0 & \text{otherwise,}\end{cases} \qquad h(e_i') = \begin{cases} e_i' & \text{if } 1 \leq i \leq r,\\ -e_i' & \text{if } -r \leq i \leq -1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $Q = LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L = K_h \cong \mathrm{GL}(1)\times\mathrm{GL}(r)\times\mathrm{SO}(2p-2r-1)\times\mathrm{GL}(r)\times\mathrm{SO}(2q-2r+1)$. The centralizer of e

is $K_e = L_e K_e^{\mathrm{u}}$, where $L_e \cong \mathrm{GL}(r) \times \mathrm{SO}(2p-2r-1) \times \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2q-2r))$, the $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2q-2r))$ factor of L_e is embedded as

$$(z, A) \mapsto (z, z, A)$$

into $\mathrm{GL}(1) \times \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2q-2r))$, where the $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2q-2r))$ factor is included in the $\mathrm{SO}(2q-2r+1)$ factor of L, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally

$$B \mapsto (B, B^{-1})$$

into the $\mathrm{GL}(r) \times \mathrm{GL}(r)$ factor of L. The quotient $\mathrm{Lie}\, Q^{\mathrm{u}} / \mathrm{Lie}\, K_e^{\mathrm{u}}$ is a simple L_e-module of dimension r as follows. In $\mathfrak{k}(1)$ there are exactly two simple L_e-submodules, W_0, W_1, of highest weight ω_{r-1} with respect to the $\mathrm{GL}(r)$ factor, isomorphic as L_e-modules but lying in two distinct isotypical L-components. Let V be the L_e-complement of $W_0 \oplus W_1$ in $\mathrm{Lie}\, Q^{\mathrm{u}}$. As an L_e-module, $\mathrm{Lie}\, K_e^{\mathrm{u}}$ is the direct sum of V and a simple L_e-submodule of $W_0 \oplus W_1$ which projects nontrivially on both summands W_0 and W_1.

A.8.3 $(-3, +2^{2r}, +1^{2p-2r}, -1^{2q-1-2r})$
This case can be obtained from case 8.2 by switching the roles of p and q.

A.9 $\mathfrak{so}(2p+2q)/\mathfrak{so}(2p)+\mathfrak{so}(2q)$
We set $K = \mathrm{SO}(2p) \times \mathrm{SO}(2q)$, $p, q \geq 3$, $\mathfrak{p} = V(\omega_1 + \omega_1')$. If $p = 2$ and $q \geq 3$, then $\mathfrak{p} = V(\omega + \omega' + \omega_1'')$. If $p \geq 3$ and $q = 2$, then $\mathfrak{p} = V(\omega_1 + \omega' + \omega'')$. If $p = 2$ and $q = 2$, then $\mathfrak{p} = V(\omega + \omega' + \omega'' + \omega''')$. Let us fix a basis $e_1, \ldots, e_p, e_{-p}, \ldots, e_{-1}$ of $\mathbb{C}^{2p}$ and a symmetric bilinear form β such that $\beta(e_i, e_j) = \delta_{i,-j}$ for all i, j. Similarly, let us fix a basis $e_1', \ldots, e_q', e_{-q}', \ldots, e_{-1}'$ of $\mathbb{C}^{2q}$ and a symmetric bilinear form β' such that $\beta'(e_i', e_j') = \delta_{i,-j}$ for all i, j. Then $K = \mathrm{SO}(\mathbb{C}^{2p}, \beta) \times \mathrm{SO}(\mathbb{C}^{2q}, \beta')$ and

$$\mathfrak{p} = \mathbb{C}^{2p} \otimes \mathbb{C}^{2q}.$$

Let us denote by τ the linear endomorphism of $\mathbb{C}^{2p+2q}$ switching e_p and e_{-p} and fixing all the other basis vectors. Similarly, let us denote by τ' the linear endomorphism of $\mathbb{C}^{2p+2q}$ switching e_q' and e_{-q}' and fixing all the other basis vectors. The conjugation by τ (and by τ', resp.) is an involutive external automorphism of $\mathfrak{g}$, leaving $\mathfrak{k}$ and $\mathfrak{p}$ stable, and inducing the nontrivial involution of the first (the second, resp.) connected component of the Dynkin diagram of $\mathfrak{k}$.

A.9.1 $(+2^{2r}, +1^{2p-2r}, -1^{2q-2r})$, $r \geq 1$
If $r < p$ and $r < q$, we take

$$e = \sum_{i=1}^{r} e_i \otimes e_{r-i+1}', \qquad f = -\sum_{i=1}^{r} e_{-r+i-1} \otimes e_{-i}',$$

$$h(e_i)=\begin{cases} e_i & \text{if } 1\le i\le r,\\ -e_i & \text{if } -r\le i\le -1,\\ 0 & \text{otherwise,}\end{cases}\qquad h(e'_i)=\begin{cases} e'_i & \text{if } 1\le i\le r,\\ -e'_i & \text{if } -r\le i\le -1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $Q=LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L=K_h\cong \mathrm{GL}(r)\times\mathrm{SO}(2p-2r)\times\mathrm{GL}(r)\times\mathrm{SO}(2q-2r)$. The centralizer of e is $K_e=L_eQ^{\mathrm{u}}$, where $L_e\cong\mathrm{GL}(r)\times\mathrm{SO}(2p-2r)\times\mathrm{SO}(2q-2r)$, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally, $A\mapsto(A,A^{-1})$, into the $\mathrm{GL}(r)\times\mathrm{GL}(r)$ factor of L.

If $r=p$ and $r<q$, then there exist two orbits labeled I and II. Case (I) can be described as above by specializing r equal to p. Case (II) can be obtained from case (I) by conjugating by τ.

If $r<p$ and $r=q$, then there exist two orbits labeled I and II. Case (I) can be described as above by specializing r equal to q. Case (II) can be obtained from case (I) by conjugating by τ'.

If $r=p=q$, then there exist four orbits with a double label I or II. Case (I, I) can be described as above by specializing r equal to $p=q$. Case (I, II) can be obtained from case (I, I) by conjugating by τ'. Case (II, I) can be obtained from case (I, I) by conjugating by τ. Case (II, II) can be obtained from case (I, I) by conjugating by τ and τ'.

A.9.2 $(+3,+2^{2r},+1^{2p-2-2r},-1^{2q-1-2r})$
If $r\le p-2$ and $r\le q-2$, we take

$$e=e_1\otimes(e'_q+e'_{-q})+\sum_{i=1}^{r}e_{i+1}\otimes e'_{r-i+1},$$

$$f=-\Big(\sum_{i=1}^{r}e_{-r+i-2}\otimes e'_{-i}\Big)-e_{-1}\otimes(e'_q+e'_{-q}),$$

$$h(e_i)=\begin{cases} 2e_i & \text{if } i=1,\\ e_i & \text{if } 2\le i\le r+1,\\ -e_i & \text{if } -r-1\le i\le -2,\\ -2e_i & \text{if } i=-1,\\ 0 & \text{otherwise,}\end{cases}\qquad h(e'_i)=\begin{cases} e'_i & \text{if } 1\le i\le r,\\ -e'_i & \text{if } -r\le i\le -1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $Q=LQ^{\mathrm{u}}$ be the corresponding parabolic subgroup of K, so that $L=K_h\cong \mathrm{GL}(1)\times\mathrm{GL}(r)\times\mathrm{SO}(2p-2r-2)\times\mathrm{GL}(r)\times\mathrm{SO}(2q-2r)$. The centralizer of e is $K_e=L_eK_e^{\mathrm{u}}$, where $L_e\cong\mathrm{GL}(r)\times\mathrm{SO}(2p-2r-2)\times\mathrm{S}(\mathrm{O}(1)\times\mathrm{O}(2q-2r-1))$, the $\mathrm{S}(\mathrm{O}(1)\times\mathrm{O}(2p-2r-1))$ factor of L_e is embedded as

$$(z,A)\mapsto(z,z,A)$$

into $\mathrm{GL}(1)\times\mathrm{S}(\mathrm{O}(1)\times\mathrm{O}(2q-2r-1))$, where the $\mathrm{S}(\mathrm{O}(1)\times\mathrm{O}(2q-2r-1))$ factor is included in the $\mathrm{SO}(2q-2r)$ factor of L, and the $\mathrm{GL}(r)$ factor of L_e is embedded skew-diagonally

$$B\mapsto(B,B^{-1})$$

into the $\mathrm{GL}(r)\times\mathrm{GL}(r)$ factor of L. The quotient $\mathrm{Lie}\,Q^{\mathrm{u}}/\mathrm{Lie}\,K_e^{\mathrm{u}}$ is a simple L_e-module of dimension r as follows. In $\mathfrak{k}(1)$ there are exactly two simple L_e-submodules, W_0, W_1, of highest weight ω_{r-1} with respect to the $\mathrm{GL}(r)$ factor, isomorphic as L_e-modules but lying in two distinct isotypical L-components. Let V be the L_e-complement of $W_0 \oplus W_1$ in $\mathrm{Lie}\,Q^{\mathrm{u}}$. As an L_e-module, $\mathrm{Lie}\,K_e^{\mathrm{u}}$ is the direct sum of V and a simple L_e-submodule of $W_0 \oplus W_1$ which projects nontrivially on both summands W_0 and W_1.

If $r \le p-2$ and $r = q-1$, then the normal triple h, e, f, the parabolic subgroup $Q = LQ^{\mathrm{u}}$, and L_e have the same description, with $K_e = L_e K_e^{\mathrm{u}}$. The quotient $\mathrm{Lie}\,Q^{\mathrm{u}}/\mathrm{Lie}\,K_e^{\mathrm{u}}$ remains a simple L_e-module of dimension $q-1$, but here in $\mathfrak{k}(1)$ there are exactly three simple L_e-submodules, W_0, W_1, W_2, of highest weight ω_{q-2} with respect to the $\mathrm{GL}(q-1)$ factor, isomorphic as L_e-modules but lying in three distinct isotypical L-components. Let V be the L_e-complement of $W_0 \oplus W_1 \oplus W_2$ in $\mathrm{Lie}\,Q^{\mathrm{u}}$. As an L_e-module, $\mathrm{Lie}\,K_e^{\mathrm{u}}$ is the direct sum of V and a cosimple L_e-submodule of $W_0 \oplus W_1 \oplus W_2$ which projects nontrivially on every summand W_0, W_1, and W_2.

If $r = p-1$, then there exist two orbits labeled I and II. Case (I) can be described as above by specializing r equal to $p-1$. Case (II) can be obtained from case (I) by conjugating by τ.

A.9.3 $(-3, +2^{2r}, +1^{2p-1-2r}, -1^{2q-2-2r})$
This case can be obtained from case 9.2 by switching the roles of p and q.

Appendix B: Tables of spherical nilpotent K-orbits in $\mathfrak{p}$ in the classical non-Hermitian cases

Let $e \in \mathcal{N}_{\mathfrak{p}}$, and let $\{h, e, f\}$ be a normal triple containing it. The action of the semisimple element h on $\mathfrak{g}$ induces a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$, where we denote $\mathfrak{g}(i) = \{x \in \mathfrak{g} : [h,x] = ix\}$. This defines the *height* of e (which actually depends only on Ge), defined as $\mathrm{ht}(e) = \max\{i : \mathfrak{g}(i) \ne 0\}$. By [27, Theorem 2.6], the orbit Ge is spherical if and only if $\mathrm{ht}(e) \le 3$.

Similarly, one may consider the action of h on $\mathfrak{p}$, and the corresponding $\mathbb{Z}$-grading $\mathfrak{p} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{p}(i)$, where $\mathfrak{p}(i) = \mathfrak{p} \cap \mathfrak{g}(i)$. This defines the $\mathfrak{p}$-*height* of e (which actually depends only on Ke), defined as $\mathrm{ht}_{\mathfrak{p}}(e) = \max\{i : \mathfrak{p}(i) \ne 0\}$. By [27, Theorems 5.1, 5.6], the orbit Ke is spherical if $\mathrm{ht}(e) \le 3$, whereas if Ke is spherical, then $\mathrm{ht}(e) \le 4$ and $\mathrm{ht}_{\mathfrak{p}}(e) \le 3$. Similarly to the adjoint case, $\overline{Ke}$ is normal if $\mathrm{ht}_{\mathfrak{p}}(e) = 2$, in which case Hesselink's [16] proof of the normality of the closure of a nilpotent adjoint G-orbit of height 2 essentially applies (see [20, Proposition 2.1]).

In Tables 2–10, for every spherical orbit $Ke \subset \mathcal{N}_{\mathfrak{p}}$, we report its signed partition (column 2), the Kostant–Dynkin diagram and the height of Ge (columns 3, 4), the Kostant–Dynkin diagram and the $\mathfrak{p}$-height of Ke (columns 5 and 6), the normality of $\overline{Ke}$ (column 7), the codimension of $\overline{Ke} \smallsetminus Ke$ in $\overline{Ke}$ (column 8), and the weight semigroup of $\widetilde{Ke}$ (column 9).

In the orthogonal cases, the generators of the weight semigroups given in the tables are expressed in terms of the following variation of the fundamental weights of an irreducible root system R

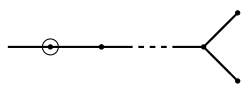
$$\varpi_i = \begin{cases} 2\omega_i & \text{if } i = n \text{ and } R = \mathsf{B}_n, \\ \omega_{n-1} + \omega_n & \text{if } i = n-1, n \text{ and } R = \mathsf{D}_n, \\ \omega_i & \text{otherwise,} \end{cases}$$

and we set $\varpi_0 = 0$.

In all cases with a Roman numeral, (I) or (II), one K-orbit is obtained from the other one by applying an involutive automorphism of a factor of K of type D. In some of these cases, the generators of $\Gamma(\widetilde{Ke})$ are given just for the K-orbit labeled with (I); the generators for the other one are obtained by switching ω_{p-1} and ω_p (resp., ω'_{q-1} and ω'_q) if the first (resp., the second) component is the one involved by the above-mentioned automorphism. Which component is involved by the automorphism is clear from the Kostant–Dynkin diagrams of the two orbits.

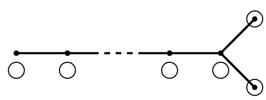
In Tables 11–19, for every spherical nilpotent orbit Ke in $\mathfrak{p}$, we report the Luna diagram and the set of spherical roots of the spherical system of $K[e]$. For every family of K-orbits, we draw the Luna diagram for values of the parameters n, p, q big enough with respect to r. When r becomes close to n, p, or q, the diagram may change. Let us explain how it changes.

Whenever K has a factor of type D_t, where the diagram ends with

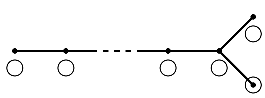


(the corresponding simple root α_s moving a color of type b, with $\alpha_{s+1}, \ldots, \alpha_t$ belonging to S^{p}) as in case 3.1, the given diagram is for $s < t-1$. If $s = t-1$, then both the simple roots α_{t-1} and α_t move colors of type b. If $s = t$, with Roman numeral (I), then $\alpha_{t-1} \in \operatorname{supp} \Sigma$ and α_t moves a color of type b. If $s = t$, with Roman numeral (II), then $\alpha_t \in \operatorname{supp} \Sigma$ and α_{t-1} moves a color of type b. For example, the diagram of case 3.1 becomes as follows.

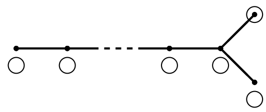
$r = n-1$



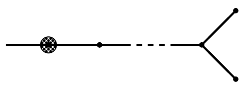
$r = n$ (I)



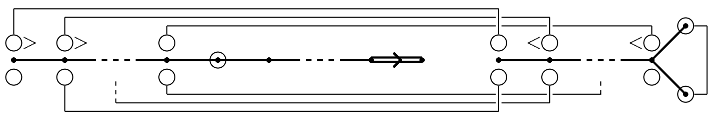
$r = n$ (II)



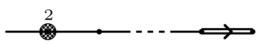
Whenever K has a factor of type D_t, where the diagram ends with a tail



(the corresponding simple root α_s moving a color of type b, with $\alpha_{s+1}, \dots, \alpha_t$ belonging to S^{p} and $2(\alpha_s + \cdots + \alpha_{t-2}) + \alpha_{t-1} + \alpha_t$ belonging to Σ) as in case 7.2, the given diagram is for $s < t-1$. If $s = t-1$, then the simple roots α_{t-1} and α_t move the same color of type b ($\alpha_{t-1} + \alpha_t$ is a spherical root). For example, the diagram of case 7.2 for $r = q-2$ becomes as follows.



Whenever K has a factor of type B_t, where the diagram ends with a tail



(the corresponding simple root α_s moving a color of type b, with $\alpha_{s+1}, \dots, \alpha_t$ belonging to S^{p} and $2(\alpha_s + \cdots + \alpha_t)$ belonging to Σ) as in case 7.3, the given diagram is for $s < t$. If $s = t$, then the simple root α_t moves a color of type 2a ($2\alpha_t$ is a spherical root). For example, the diagram of the case 7.3 for $r = p-1$ becomes as follows.

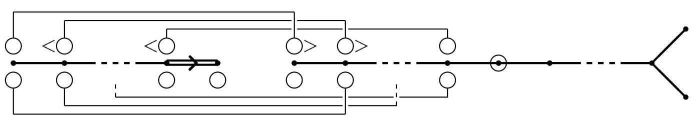


Table 2. $G = \mathsf{A}_{2n-1},\ K = \mathsf{C}_n\ (n \geq 2)$

	Signed partition	Diagram of Ge	$\mathrm{ht}(e)$	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
1.1	$(2^r, 1^{n-2r}),\ r \geq 1$	$(\underbrace{0\ldots0}_{2r-1}10\ldots01\underbrace{0\ldots0}_{2r-1})$ if $2r < n$ $(\underbrace{0\ldots0}_{n-1}20\ldots0)$ if $2r = n$	2	$(\underbrace{0\ldots0}_{2r-1}10\ldots0)$ if $2r < n$ $(0\ldots02)$ if $2r = n$	2	$+$	$4(n-2r+1)$	$\omega_2, \omega_4, \ldots, \omega_{2r}$

Table 3. $G = \mathsf{A}_{2n},\ K = \mathsf{B}_n\ (n \geq 1)$

	Signed partition	Diagram of Ge	$\mathrm{ht}(e)$	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
2.1	$(2^r, 1^{2n-2r+1}),\ r \geq 1$	$(\underbrace{0\ldots0}_{r-1}10\ldots01\underbrace{0\ldots0}_{r-1})$	2	$(\underbrace{0\ldots0}_{r-1}10\ldots0)$	2	$+$	$2(n-r+1)$	$2\varpi_1, \ldots, 2\varpi_r$

Table 4. $G = \mathsf{A}_{2n-1},\ K = \mathsf{D}_n\ (n \geq 2)$

	Signed partition	Diagram of Ge	$\mathrm{ht}(e)$	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
3.1 $r < n$	$(2^r, 1^{2n-2r}),\ r \geq 1$	$(\underbrace{0\ldots0}_{2r-1}10\ldots01\underbrace{0\ldots0}_{2r-1})$	2	$(\underbrace{0\ldots0}_{r-1}10\ldots0)$ if $r < n-1$ $(0\ldots011)$ if $r = n-1$	2	$+$	$2(n-r)+1$	$2\varpi_1, \ldots, 2\varpi_r$
3.1 $r = n$	(2^n) (I) or (II)	$(\underbrace{0\ldots0}_{n-1}20\ldots0)$	2	(I) $(0\ldots002)$ (II) $(0\ldots020)$	2	$+$	1	(I) $2\varpi_1, \ldots, 2\varpi_{n-1}, 4\omega_n$ (II) $2\varpi_1, \ldots, 2\varpi_{n-1}, 4\omega_{n-1}$

Table 5. $G = \mathsf{C}_{p+q},\ K = \mathsf{C}_p \times \mathsf{C}_q\ (p, q \geq 1)$

	Signed partition	Diagram of Ge	ht(e)	Diagram of Ke	ht$_{\mathfrak{p}}(e)$	$\overline{Ke}$	codim$(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
4.1	$(+2^{2r}, +1^{2p-2r}, -1^{2q-2r}),\ r \geq 1$	$(\underbrace{0\ldots0}_{2r-1}10\ldots0)$ if $2r < p+q$ $(0\ldots02)$ if $r = p = q$	2	$(\underbrace{0\ldots0}_{r-1}10\ldots0,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$ if $r < p, r < q$ $(0\ldots02,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$ if $r = p < q$ $(\underbrace{0\ldots0}_{r-1}10\ldots0,\ 0\ldots02)$ if $r = q < p$ $(0\ldots02,\ 0\ldots02)$ if $r = p = q$	2	+	$2(p+q-2r)+3$	$\omega_1 + \omega_1', \ldots, \omega_r + \omega_r'$
4.2	$(+3^2, +1^{2p-4}, -1^{2q-2})$	$(020\ldots0)$	4	$(020\ldots0,\ 0\ldots0)$ if $p > 2$ $(04,\ 0\ldots0)$ if $p = 2$	2	+	1	$\omega_1 + \omega_1', \omega_2$ if $q = 1$ $\omega_1 + \omega_1', \omega_2, \omega_2 + \omega_2'$ if $q > 1$
4.3	$(-3^2, +1^{2p-2}, -1^{2q-4})$	$(020\ldots0)$	4	$(0\ldots0,\ 020\ldots0)$ if $q > 2$ $(0\ldots0,\ 04)$ if $q = 2$	2	+	1	$\omega_1 + \omega_1', \omega_2'$ if $p = 1$ $\omega_1 + \omega_1', \omega_2 + \omega_2', \omega_2'$ if $p > 1$
4.4 $q > 2$	$(+3^2, +2^2, +1^{2p-6}, -1^{2q-4})$	$(01010\ldots0)$	4	$(0110\ldots0,\ 10\ldots0)$ if $p > 3$ $(012,\ 10\ldots0)$ if $p = 3$	3	+	3	$\omega_1 + \omega_3 + \omega_2', \omega_1 + \omega_1', \omega_2,$ $\omega_2 + \omega_2', \omega_3 + \omega_1', \omega_3 + \omega_3'$
4.4 $q = 2$	$(+3^2, +2^2, +1^{2p-6})$	$(01010\ldots0)$	4	$(0110\ldots0,\ 10)$ if $p > 3$ $(012,\ 10)$ if $p = 3$	3	+	$2p-3$	$\omega_1 + \omega_3 + \omega_2', \omega_1 + \omega_1',$ $\omega_2, \omega_2 + \omega_2', \omega_3 + \omega_1'$
4.5 $p > 2$	$(-3^2, +2^2, +1^{2p-4}, -1^{2q-6})$	$(01010\ldots0)$	4	$(10\ldots0,\ 0110\ldots0)$ if $q > 3$ $(10\ldots0,\ 012)$ if $q = 3$	3	+	3	$\omega_1 + \omega_1', \omega_1 + \omega_3', \omega_2 + \omega_1' + \omega_3',$ $\omega_2 + \omega_2', \omega_3 + \omega_3', \omega_2'$
4.5 $p = 2$	$(-3^2, +2^2, -1^{2q-6})$	$(01010\ldots0)$	4	$(10,\ 0110\ldots0)$ if $q > 3$ $(10,\ 012)$ if $q = 3$	3	+	$2q-3$	$\omega_1 + \omega_1', \omega_1 + \omega_3', \omega_2 + \omega_1' + \omega_3',$ $\omega_2 + \omega_2', \omega_2'$
4.6	$(+3^4, +1^{2p-8}),\ q = 2$	$(00020\ldots0)$	4	$(00020\ldots0, 00)$ if $p > 4$ $(0004, 00)$ if $p = 4$	2	+	$2p-6$	$\omega_2, \omega_4, \omega_1 + \omega_1', \omega_2 + \omega_2',$ $\omega_1 + \omega_3 + \omega_2', \omega_3 + \omega_1'$
4.7	$(-3^4, -1^{2q-8}),\ p = 2$	$(00020\ldots0)$	4	$(00, 00020\ldots0)$ if $q > 4$ $(00, 0004)$ if $q = 4$	2	+	$2q-6$	$\omega_2', \omega_4', \omega_1 + \omega_1', \omega_2 + \omega_2',$ $\omega_2 + \omega_1' + \omega_3', \omega_1 + \omega_3'$

Table 6. $G = \mathsf{B}_n,\ K = \mathsf{D}_n\ (n \geq 2)$

	Signed partition	Diagram of Ge	$\mathrm{ht}(e)$	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
5.1	$(+3, +1^{2n-2})$	$(2, 0, \ldots, 0)$	2	$(10\ldots0)$ if $n > 2$ (11) if $n = 2$	2	$+$	$2n - 1$	ω_1 if $n > 2$ $\omega + \omega'$ if $n = 2$

Table 7. $G = \mathsf{D}_{n+1},\ K = \mathsf{B}_n\ (n \geq 1)$

	Signed partition	Diagram of Ge	$\mathrm{ht}(e)$	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
6.1	$(+3, +1^{2n-1})$	$(2, 0, \ldots, 0)$	2	$(10\ldots0)$	2	$+$	$2n$	ω_1 if $n > 1$ 2ω if $n = 1$

Table 8. $G = \mathsf{B}_{p+q},\ K = \mathsf{B}_p \times \mathsf{D}_q\ (p \geq 1,\ q \geq 2)$

	Signed partition	Diagram of Ge	ht(e)	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
7.1 $r < q$	$(+2^{2r}, +1^{2p+1-2r}, -1^{2q-2r}),\ r \geq 1$	$(\underbrace{0\ldots0}_{2r-1}10\ldots0)$	2	$(\underbrace{0\ldots0}_{r-1}10\ldots0,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$ if $r < q-1$ $(\underbrace{0\ldots0}_{q-2}10\ldots0,\ 0\ldots011)$ if $r = q-1$	2	$+$	$2(p+q-2r+1)$	$\varpi_1 + \varpi'_1, \ldots, \varpi_r + \varpi'_r$
7.1 $r = q$	$(+2^{2q}, +1^{2p+1-2q})$ (I) or (II)	$(\underbrace{0\ldots0}_{2q-1}10\ldots0)$	2	(I) $(\underbrace{0\ldots0}_{q-1}10\ldots0,\ 0\ldots002)$ (II) $(\underbrace{0\ldots0}_{q-1}10\ldots0,\ 0\ldots020)$	2	$+$	$2(p-r+1)$	(I) $\varpi_1 + \varpi'_1, \ldots, \varpi_{q-1} + \varpi'_{q-1}$, $\varpi_q + 2\omega'_q$ (II) $\varpi_1 + \varpi'_1, \ldots, \varpi_{q-1} + \varpi'_{q-1}$, $\varpi_q + 2\omega'_{q-1}$
7.2 $r = 0$	$(+3, +1^{2p-1}, -1^{2q-1})$	$(20\ldots0)$	2	$(20\ldots0,\ 0\ldots0)$	2	$+$	1	$\varpi_1 + \varpi'_1,\ 2\varpi_1$
7.2 $0 < r < q-1$	$(+3, +2^{2r}, +1^{2p-1-2r}, -1^{2q-1-2r})$	$(1\underbrace{0\ldots0}_{2r-1}10\ldots0)$	3	$(1\underbrace{0\ldots0}_{r-1}10\ldots0,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$	3	$+$	$\min\{r+1,$ $2(p+q-2r)-1\}$	$\varpi_i + \varpi'_i$ $(i = 1, \ldots, r+1)$, $\varpi_i + \varpi_j + \varpi'_{i-1} + \varpi'_{j-1}$ $(i, j = 1, \ldots, r+1)$
7.2 $r = q-1$	$(+3, +2^{2q-2}, +1^{2p+1-2q}, -1)$	$(1\underbrace{0\ldots0}_{2q-3}10\ldots0)$	3	$(1\underbrace{0\ldots0}_{q-2}10\ldots0,\ 0\ldots011)$	3	$+$	$\min\{q,$ $2(p-q)+3\}$	$\varpi_i + \varpi'_i$ $(i = 1, \ldots, q-1)$, $\varpi_q + 2\omega'_{q-1},\ \varpi_q + 2\omega'_q$, $\varpi_i + \varpi_j + \varpi'_{i-1} + \varpi'_{j-1}$ $(i, j = 1, \ldots, q,\ i+j < 2q)$
7.3 $r = 0$	$(-3, +1^{2p}, -1^{2q-2})$	$(20\ldots0)$	2	$(0\ldots0,\ 20\ldots0)$	2	$+$	1	$\varpi_1 + \varpi'_1,\ 2\varpi'_1$
7.3 $0 < r < p, q-1$	$(-3, +2^{2r}, +1^{2p-2r}, -1^{2q-2-2r})$	$(1\underbrace{0\ldots0}_{2r-1}10\ldots0)$	3	$(\underbrace{0\ldots0}_{r-1}10\ldots0,\ 1\underbrace{0\ldots0}_{r-1}10\ldots0)$ if $r < q-2$ $(\underbrace{0\ldots0}_{q-3}10\ldots0,\ 10\ldots011)$ if $r = q-2$	3	$+$	$\min\{r+1,$ $2(p+q-2r)-1\}$	$\varpi_i + \varpi'_i$ $(i = 1, \ldots, r+1)$, $\varpi_{i-1} + \varpi_{j-1} + \varpi'_i + \varpi'_j$ $(i, j = 1, \ldots, r+1)$
7.3 $r = q-1 < p$	$(-3, +2^{2q-2}, +1^{2p+2-2q})$ (I) or (II)	$(1\underbrace{0\ldots0}_{2q-3}10\ldots0)$	3	(I) $(\underbrace{0\ldots0}_{q-2}10\ldots0,\ 10\ldots002)$ (II) $(\underbrace{0\ldots0}_{q-2}10\ldots0,\ 10\ldots020)$	3	$+$	$\min\{q,$ $2(p-q)+3\}$	(I) $\varpi_i + \varpi'_i$ $(i = 1, \ldots, q-1)$, $\varpi_q + 2\omega'_q,\ 2\varpi_{q-1} + 4\omega'_q$, $\varpi_{i-1} + \varpi_{j-1} + \varpi'_i + \varpi'_j$ $(i, j = 1, \ldots, q-1)$, $\varpi_{i-1} + \varpi_{q-1} + \varpi'_i + 2\omega'_q$ $(i = 1, \ldots, q-1)$
7.3 $r = p < q-1$	$(-3, +2^{2p}, -1^{2q-2-2p})$	$(1\underbrace{0\ldots0}_{2p-1}10\ldots0)$	3	$(\underbrace{0\ldots0}_{p-1}10\ldots0,\ 1\underbrace{0\ldots0}_{p-1}10\ldots0)$ if $p < q-2$ $(\underbrace{0\ldots0}_{p-1}10\ldots0,\ 10\ldots011)$ if $p = q-2$	3	$-$	$2(q-p)-1$	$\varpi_i + \varpi'_i$ $(i = 1, \ldots, p)$, $\varpi_{i-1} + \varpi'_i$ $(i = 1, \ldots, p+1)$
7.3 $r = p = q-1$	$(-3, +2^{2p})$ (I) or (II)	$(10\ldots01)$	3	(I) $(0\ldots01,\ 10\ldots002)$ (II) $(0\ldots01,\ 10\ldots020)$	3	$-$	1	(I) $\varpi_i + \varpi'_i$ $(i = 1, \ldots, p)$, $\varpi_{i-1} + \varpi'_i$ $(i = 1, \ldots, p)$, $\varpi_p + 2\omega'_{p+1}$

Table 9. $G = \mathsf{D}_{p+q+1},\ K = \mathsf{B}_p \times \mathsf{B}_q\ (p, q \geq 1)$

	Signed partition	Diagram of Ge	$\mathrm{ht}(e)$	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
8.1	$(+2^{2r}, +1^{2p+1-2r}, -1^{2q+1-2r}),\ r \geq 1$	$(\underbrace{0\ldots0}_{2r-1}10\ldots0)$ if $2r < p+q$ $(0\ldots011)$ if $r = p = q$	2	$(\underbrace{0\ldots0}_{r-1}10\ldots0,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$	2	+	$2(p+q-2r)+3$	$\varpi_i + \varpi_i'\ (i = 1, \ldots, r)$
8.2 $r = 0$	$(+3, +1^{2p-1}, -1^{2q})$	$(20\ldots0)$	2	$(20\ldots0,\ 0\ldots0)$	2	+	1	$\varpi_1 + \varpi_1',\ 2\varpi_1$
8.2 $0 < r < q$	$(+3, +2^{2r}, +1^{2p-1-2r}, -1^{2q-2r})$	$(1\underbrace{0\ldots0}_{2r-1}10\ldots0)$	3	$(1\underbrace{0\ldots0}_{r-1}10\ldots0,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$	3	+	$\min\{r+1,$ $2(p+q-2r)\}$	$\varpi_i + \varpi_i'\ (i = 1, \ldots, r+1)$, $\varpi_i + \varpi_j + \varpi_{i-1}' + \varpi_{j-1}'$ $(i, j = 1, \ldots, r+1)$
8.2 $r = q$	$(+3, +2^{2q}, +1^{2p-1-2q})$	$(1\underbrace{0\ldots0}_{2q-1}10\ldots0)$ if $q < p-1$ $(10\ldots011)$ if $q = p-1$	3	$(1\underbrace{0\ldots0}_{q-1}10\ldots0,\ \underbrace{0\ldots0}_{q-1}10\ldots0)$	3	−	$2(p-q)$	$\varpi_i + \varpi_i'\ (i = 1, \ldots, q)$, $\varpi_i + \varpi_{i-1}'\ (i = 1, \ldots, q+1)$
8.3 $r = 0$	$(-3, +1^{2p}, -1^{2q-1})$	$(20\ldots0)$	2	$(0\ldots0,\ 20\ldots0)$	2	+	1	$\varpi_1 + \varpi_1',\ 2\varpi_1'$
8.3 $0 < r < p$	$(-3, +2^{2r}, +1^{2p-2r}, -1^{2q-1-2r})$	$(1\underbrace{0\ldots0}_{2r-1}10\ldots0)$	3	$(\underbrace{0\ldots0}_{r-1}10\ldots0,\ 1\underbrace{0\ldots0}_{r-1}10\ldots0)$	3	+	$\min\{r+1,$ $2(p+q-2r)\}$	$\varpi_i + \varpi_i'\ (i = 1, \ldots, r+1)$, $\varpi_{i-1} + \varpi_{j-1} + \varpi_i' + \varpi_j'$ $(i, j = 1, \ldots, r+1)$
8.3 $r = p$	$(-3, +2^{2p}, -1^{2q-1-2p})$	$(1\underbrace{0\ldots0}_{2p-1}10\ldots0)$ if $p < q-1$ $(10\ldots011)$ if $p = q-1$	3	$(\underbrace{0\ldots0}_{p-1}10\ldots0,\ 1\underbrace{0\ldots0}_{p-1}10\ldots0)$	3	−	$2(q-p)$	$\varpi_i + \varpi_i'\ (i = 1, \ldots, p)$, $\varpi_{i-1} + \varpi_i'\ (i = 1, \ldots, p+1)$

Table 10. $G = \mathsf{D}_{p+q},\ K = \mathsf{D}_p \times \mathsf{D}_q\ (p, q \geq 2)$

	Signed partition	Diagram of Ge	$\mathrm{ht}(e)$	Diagram of Ke	$\mathrm{ht}_{\mathfrak{p}}(e)$	$\overline{Ke}$	$\mathrm{codim}(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
9.1 $r < p,\ q$	$(+2^{2r}, +1^{2p-2r}, -1^{2q-2r}),\ r \geq 1$	$(\underbrace{0\ldots0}_{2r-1}10\ldots0)$ if $2r < p+q-1$ $(0\ldots011)$ if $2r = p+q-1$	2	$(\underbrace{0\ldots0}_{r-1}10\ldots0,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$ if $r < p-1,\ q-1$ $(0\ldots011,\ \underbrace{0\ldots0}_{r-1}10\ldots0)$ if $r = p-1 < q-1$ $(\underbrace{0\ldots0}_{r-1}10\ldots0,\ 0\ldots011)$ if $r = q-1 < p-1$ $(0\ldots011,\ 0\ldots011)$ if $r = p-1 = q-1$	2	+	$2(p+q-2r)+1$	$\varpi_i + \varpi'_i\ (i = 1, \ldots, r)$
9.1 $r = p < q$	$(+2^{2p}, -1^{2q-2p})$ (I) or (II)	$(\underbrace{0\ldots0}_{2p-1}10\ldots0)$ if $p < q-1$ $(0\ldots011)$ if $p = q-1$	2	(I) $(0\ldots002,\ \underbrace{0\ldots0}_{p-1}10\ldots0)$ if $p < q-1$ $(0\ldots002,\ 0\ldots011)$ if $p = q-1$ (II) $(0\ldots020,\ \underbrace{0\ldots0}_{p-1}10\ldots0)$ if $p < q-1$ $(0\ldots020,\ 0\ldots011)$ if $p = q-1$	2	+	$2(q-r)+1$	$\varpi_i + \varpi'_i\ (i = 1, \ldots, p-1)$ and $2\omega_p + 2\varpi'_p$ if (I) $2\omega_{p-1} + 2\varpi'_p$ if (II)
9.1 $r = q < p$	$(+2^{2q}, +1^{2p-2q})$ (I) or (II)	$(\underbrace{0\ldots0}_{2q-1}10\ldots0)$ if $q < p-1$ $(0\ldots011)$ if $q = p-1$	2	(I) $(\underbrace{0\ldots0}_{q-1}10\ldots0,\ 0\ldots002)$ if $q < p-1$ $(0\ldots011, 0\ldots002)$ if $q = p-1$ (II) $(\underbrace{0\ldots0}_{q-1}10\ldots0,\ 0\ldots020)$ if $q < p-1$ $(0\ldots011, 0\ldots020)$ if $q = p-1$	2	+	$2(p-r)+1$	$\varpi_i + \varpi'_i\ (i = 1, \ldots, q-1)$ and $\varpi_q + 2\omega'_q$ if (I) $\varpi_q + 2\omega'_{q-1}$ if (II)
9.1 $r = p = q$	$(+2^{2p})$ (I, I), (I, II), (II, I) or (II, II)	$(0\ldots002)$ if (I, I) or (II, II) $(0\ldots020)$ if (II, I) or (I, II)	2	(I, I) $(0\ldots002,\ 0\ldots002)$ (I, II) $(0\ldots002,\ 0\ldots020)$ (II, I) $(0\ldots020,\ 0\ldots002)$ (II, II) $(0\ldots020,\ 0\ldots020)$	2	+	1	$\varpi_i + \varpi'_i\ (i = 1, \ldots, p-1)$ and $2\omega_p + 2\omega'_p$ if (I, I) $2\omega_p + 2\omega'_{p-1}$ if (I, II) $2\omega_{p-1} + 2\omega'_p$ if (II, I) $2\omega_{p-1} + 2\omega'_{p-1}$ if (II, II)

Table 10. (continued)

	Signed partition	Diagram of Ge	ht(e)	Diagram of Ke	ht$_{\mathfrak{p}}(e)$	$\overline{Ke}$	codim$(\overline{Ke} \smallsetminus Ke)$	Generators of $\Gamma(\widetilde{Ke})$
9.2 $r=0$	$(+3,+1^{2p-2},-1^{2q-1})$	$(20\dots0)$	2	$(20\dots0,\ 0\dots0)$	2	+	1	$\varpi_1+\varpi_1',\ 2\varpi_1$
9.2 $0<r<p-1,q-1$	$(+3,+2^{2r},+1^{2p-2-2r},-1^{2q-1-2r})$	$(1\underbrace{0\dots0}_{2r-1}10\dots0)$	3	$(1\underbrace{0\dots0}_{r-1}10\dots0,\ \underbrace{0\dots0}_{r-1}10\dots0)$ if $r<p-2$ $(10\dots011,\ \underbrace{0\dots0}_{r-1}10\dots0)$ if $r=p-2$	3	+	$\min\{r+1,$ $2(p+q-2r)-2\}$	$\varpi_i+\varpi_i'\ (i=1,\dots,r+1)$, $\varpi_i+\varpi_j+\varpi_{i-1}'+\varpi_{j-1}'$ $(i,j=1,\dots,r+1)$
9.2 $r=p-1<q-1$	$(+3,+2^{2p-2},-1^{2q+1-2p})$ (I) or (II)	$(1\underbrace{0\dots0}_{2p-3}10\dots0)$	3	(I) $(10\dots002,\ \underbrace{0\dots0}_{p-2}10\dots0)$ (II) $(10\dots020,\ \underbrace{0\dots0}_{p-2}10\dots0)$	3	+	$\min\{p,$ $2(q-p)+2\}$	(I) $\varpi_i+\varpi_i'\ (i=1,\dots,p-1)$, $2\omega_p+\varpi_p',\ 4\omega_p+2\varpi_{p-1}'$, $\varpi_i+\varpi_j+\varpi_{i-1}'+\varpi_{j-1}'$ $(i,j=1,\dots,p-1)$, $\varpi_i+2\omega_p+\varpi_{i-1}'+\varpi_{p-1}'$ $(i=1,\dots,p-1)$
9.2 $r=q-1<p-1$	$(+3,+2^{2q-2},+1^{2p-2q},-1)$	$(1\underbrace{0\dots0}_{2q-3}10\dots0)$	3	$(1\underbrace{0\dots0}_{q-2}10\dots0,\ 0\dots011)$ if $q<p-1$ $(10\dots011,\ 0\dots011)$ if $q=p-1$	3	+	$\min\{q,$ $2(p-q)+2\}$	$\varpi_i+\varpi_i'\ (i=1,\dots,q-1)$, $\varpi_q+2\omega_{q-1}',\ \varpi_q+2\omega_q'$, $\varpi_i+\varpi_j+\varpi_{i-1}'+\varpi_{j-1}'$ $(i,j=1,\dots,q,\ i+j<2q)$
9.2 $r=p-1=q-1$	$(+3,+2^{2p-2},-1)$ (I) or (II)	$(10\dots011)$	3	(I) $(10\dots002,\ 0\dots011)$ (II) $(10\dots020,\ 0\dots011)$	3	+	2	(I) $\varpi_i+\varpi_i'\ (i=1,\dots,p-1)$, $2\omega_p+2\omega_{p-1}',\ 2\omega_p+2\omega_p'$, $\varpi_i+\varpi_j+\varpi_{i-1}'+\varpi_{j-1}'$ $(i,j=1,\dots,p-1)$, $\varpi_i+2\omega_p+\varpi_{i-1}'+\varpi_{p-1}'$ $(i=1,\dots,p-1)$
9.3 $r=0$	$(-3,+1^{2p-1},-1^{2q-2})$	$(20\dots0)$	2	$(0\dots0,\ 20\dots0)$	2	+	1	$\varpi_1+\varpi_1',2\varpi_1'$
9.3 $0<r<p-1,q-1$	$(-3,+2^{2r},+1^{2p-1-2r},-1^{2q-2-2r})$	$(1\underbrace{0\dots0}_{2r-1}10\dots0)$	3	$(\underbrace{0\dots0}_{r-1}10\dots0,\ 1\underbrace{0\dots0}_{r-1}10\dots0)$ if $r<q-2$ $(\underbrace{0\dots0}_{r-1}10\dots0,\ 10\dots011)$ if $r=q-2$	3	+	$\min\{r+1,$ $2(p+q-2r)-2\}$	$\varpi_i+\varpi_i'\ (i=1,\dots,r+1)$, $\varpi_{i-1}+\varpi_{j-1}+\varpi_i'+\varpi_j'$ $(i,j=1,\dots,r+1)$
9.3 $r=q-1<p-1$	$(-3,+2^{2q-2},+1^{2p+1-2q})$ (I) or (II)	$(1\underbrace{0\dots0}_{2q-3}10\dots0)$	3	(I) $(\underbrace{0\dots0}_{q-2}10\dots0,\ 10\dots002)$ (II) $(\underbrace{0\dots0}_{q-2}10\dots0,\ 10\dots020)$	3	+	$\min\{q,$ $2(p-q)+2\}$	(I) $\varpi_i+\varpi_i'\ (i=1,\dots,q-1)$, $\varpi_q+2\omega_q',\ 2\varpi_{q-1}+4\omega_q'$, $\varpi_{i-1}+\varpi_{j-1}+\varpi_i'+\varpi_j'$ $(i,j=1,\dots,q-1)$, $\varpi_{i-1}+\varpi_{q-1}+\varpi_i'+2\omega_q'$ $(i=1,\dots,q-1)$
9.3 $r=p-1<q-1$	$(-3,+2^{2p-2},+1,-1^{2q-2p})$	$(1\underbrace{0\dots0}_{2p-3}10\dots0)$	3	$(0\dots011,\ 1\underbrace{0\dots0}_{p-2}10\dots0)$ if $p<q-1$ $(0\dots011,\ 10\dots011)$ if $p=q-1$	3	+	$\min\{p,$ $2(q-p)+2\}$	$\varpi_i+\varpi_i'\ (i=1,\dots,p-1)$, $2\omega_{p-1}+\varpi_p',\ 2\omega_p+\varpi_p'$, $\varpi_{i-1}+\varpi_{j-1}+\varpi_i'+\varpi_j'$ $(i,j=1,\dots,p,\ i+j<2p)$
9.3 $r=p-1=q-1$	$(-3,+2^{2p-2},+1)$ (I) or (II)	$(10\dots011)$	3	(I) $(0\dots011,\ 10\dots002)$ (II) $(0\dots011,\ 10\dots020)$	3	+	2	(I) $\varpi_i+\varpi_i'\ (i=1,\dots,q-1)$, $2\omega_{q-1}+2\omega_q',\ 2\omega_q+2\omega_q'$, $\varpi_{i-1}+\varpi_{j-1}+\varpi_i'+\varpi_j'$ $(i,j=1,\dots,q-1)$, $\varpi_{i-1}+\varpi_{q-1}+\varpi_i'+2\omega_q'$ $(i=1,\dots,q-1)$

Table 11. $G = \mathsf{A}_{2n-1},\ K = \mathsf{C}_n\ (n \geq 2)$

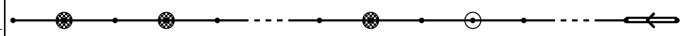
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
1.1	$(2^r, 1^{n-2r}),\ r \geq 1$		$\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}$ $(i = 1, \ldots, r-1)$

Table 12. $G = \mathsf{A}_{2n},\ K = \mathsf{B}_n\ (n \geq 1)$

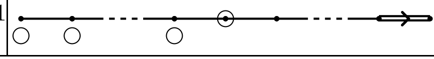
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
2.1	$(2^r, 1^{2n-2r+1}),\ r \geq 1$		$2\alpha_1, \ldots, 2\alpha_{r-1}$

Table 13. $G = \mathsf{A}_{2n-1},\ K = \mathsf{D}_n\ (n \geq 2)$

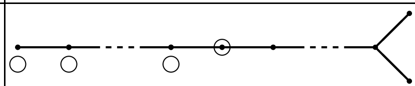
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
3.1	$(2^r, 1^{2n-2r}),\ r \geq 1$ (I) or (II) if $r = n$		$2\alpha_1, \ldots, 2\alpha_{r-1}$ if $r < n$ $2\alpha_1, \ldots, 2\alpha_{n-1}$ if $r = n$ (I) $2\alpha_1, \ldots, 2\alpha_{n-2},\ 2\alpha_n$ if $r = n$ (II)

Table 14. $G = \mathsf{C}_{p+q},\ K = \mathsf{C}_p \times \mathsf{C}_q\ (p, q \geq 1)$

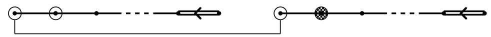
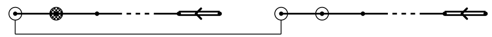
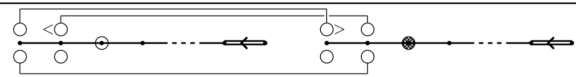
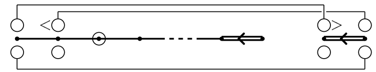
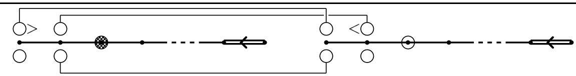
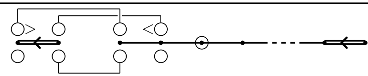
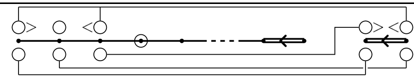
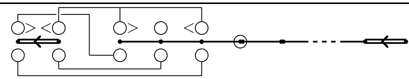
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
4.1	$(+2^{2r}, +1^{2p-2r}, -1^{2q-2r}),\ r \geq 1$		$\alpha_1 + \alpha'_1, \ldots, \alpha_{r-1} + \alpha'_{r-1}$
4.2	$(+3^2, +1^{2p-4}, -1^{2q-2})$		$\alpha_1 + \alpha'$ if $q = 1$ $\alpha_1 + \alpha'_1,\ \alpha'_1 + 2(\alpha'_2 + \cdots + \alpha'_{q-1}) + \alpha'_q$ if $q > 1$
4.3	$(-3^2, +1^{2p-2}, -1^{2q-4})$		$\alpha + \alpha'_1$ if $p = 1$ $\alpha_1 + \alpha'_1,\ \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{p-1}) + \alpha_p$ if $p > 1$
4.4 $q > 2$	$(+3^2, +2^2, +1^{2p-6}, -1^{2q-4})$		$\alpha_1,\ \alpha_2,\ \alpha'_1,\ \alpha'_2,\ \alpha'_2 + 2(\alpha'_3 + \cdots + \alpha'_{q-1}) + \alpha'_q$
4.4 $q = 2$	$(+3^2, +2^2, +1^{2p-6})$		$\alpha_1,\ \alpha_2,\ \alpha'_1,\ \alpha'_2$
4.5 $p > 2$	$(-3^2, +2^2, +1^{2p-4}, -1^{2q-6})$		$\alpha_1,\ \alpha_2,\ \alpha'_1,\ \alpha'_2,\ \alpha_2 + 2(\alpha_3 + \cdots + \alpha_{p-1}) + \alpha_p$
4.5 $p = 2$	$(-3^2, +2^2, -1^{2q-6})$		$\alpha_1,\ \alpha_2,\ \alpha'_1,\ \alpha'_2$
4.6	$(+3^4, +1^{2p-8}),\ q = 2$		$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha'_1,\ \alpha'_2$
4.7	$(-3^4, -1^{2q-8}),\ p = 2$		$\alpha_1,\ \alpha_2,\ \alpha'_1,\ \alpha'_2,\ \alpha'_3$

Table 15. $G = \mathsf{B}_n,\ K = \mathsf{D}_n\ (n \geq 2)$

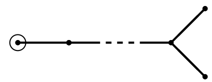
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
5.1	$(+3, +1^{2n-2})$		none

Table 16. $G = \mathsf{D}_{n+1},\ K = \mathsf{B}_n\ (n \geq 1)$

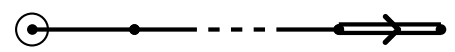
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
6.1	$(+3, +1^{2n-1})$		none

Table 17. $G = \mathsf{B}_{p+q},\ K = \mathsf{B}_p \times \mathsf{D}_q\ (p \geq 1, q \geq 2)$

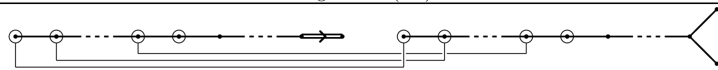
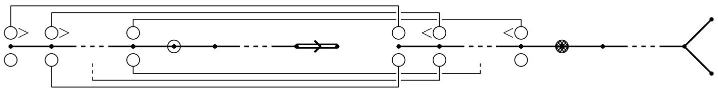
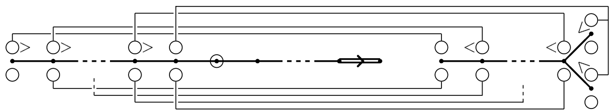
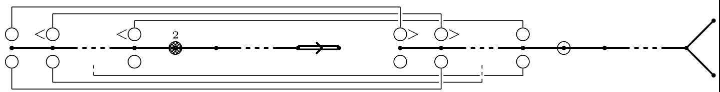
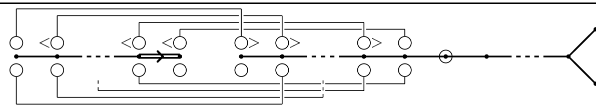
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
7.1	$(+2^{2r}, +1^{2p+1-2r}, -1^{2q-2r}),\ r \geq 1$		$\alpha_1 + \alpha'_1, \ldots, \alpha_{r-1} + \alpha'_{r-1}$
7.2 $r < q-1$	$(+3, +2^{2r}, +1^{2p-1-2r}, -1^{2q-1-2r})$		$\alpha_1, \ldots, \alpha_r,\ \alpha'_1, \ldots, \alpha'_r,$ $2(\alpha'_{r+1} + \cdots + \alpha'_{q-2}) + \alpha'_{q-1} + \alpha'_q$
7.2 $r = q-1$	$(+3, +2^{2q-2}, +1^{2p+1-2q}, -1)$		$\alpha_1, \ldots, \alpha_{q-1},\ \alpha'_1, \ldots, \alpha'_q$
7.3 $r < p$	$(-3, +2^{2r}, +1^{2p-2r}, -1^{2q-2-2r})$ (I) or (II) if $r = q-1$		$\left.\begin{array}{r}\alpha_1, \ldots, \alpha_r, \\ \alpha'_1, \ldots, \alpha'_r, \\ 2(\alpha_{r+1} + \cdots + \alpha_p)\end{array}\right\}$ if $r < q-1$ $\left.\begin{array}{r}\alpha_1, \ldots, \alpha_{q-1}, \\ \alpha'_1, \ldots, \alpha'_{q-1}, \\ 2(\alpha_q + \cdots + \alpha_p)\end{array}\right\}$ if $r = q-1$ (I) $\left.\begin{array}{r}\alpha_1, \ldots, \alpha_{q-1}, \\ \alpha'_1, \ldots, \alpha'_{q-2},\ \alpha'_q, \\ 2(\alpha_q + \cdots + \alpha_p)\end{array}\right\}$ if $r = q-1$ (II)
7.3 $r = p$	$(-3, +2^{2p}, -1^{2q-2-2p})$ (I) or (II) if $p = q-1$		$\left.\begin{array}{r}\alpha_1, \ldots, \alpha_p, \\ \alpha'_1, \ldots, \alpha'_p\end{array}\right\}$ if $p < q-1$ $\left.\begin{array}{r}\alpha_1, \ldots, \alpha_p, \\ \alpha'_1, \ldots, \alpha'_p\end{array}\right\}$ if $p = q-1$ (I) $\left.\begin{array}{r}\alpha_1, \ldots, \alpha_p, \\ \alpha'_1, \ldots, \alpha'_{p-1},\ \alpha'_{p+1}\end{array}\right\}$ if $p = q-1$ (II)

Table 18. $G = \mathsf{D}_{p+q+1}$, $K = \mathsf{B}_p \times \mathsf{B}_q$ $(p \geq 1,\ q \geq 1)$

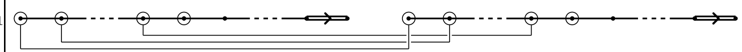
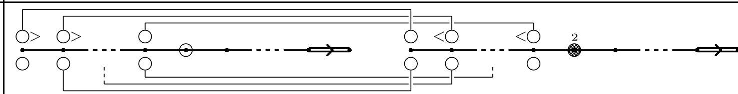
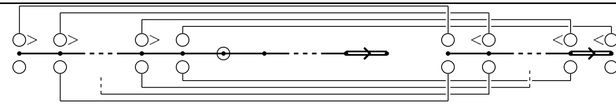
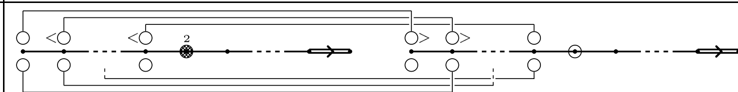
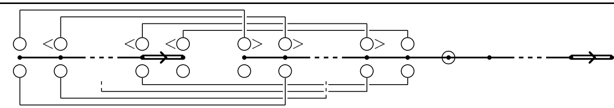
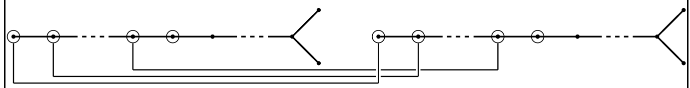
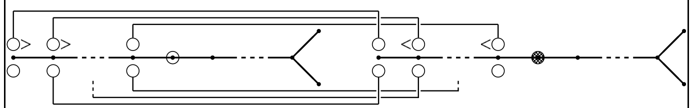
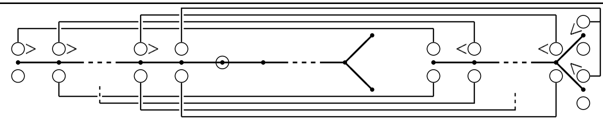
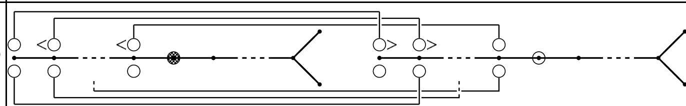
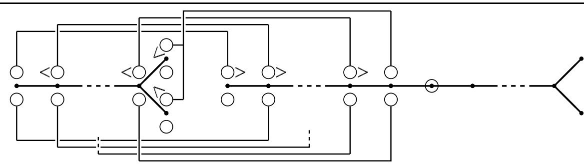
	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
8.1	$(+2^{2r}, +1^{2p+1-2r}, -1^{2q+1-2r})$, $r \geq 1$		$\alpha_1 + \alpha'_1, \ldots, \alpha_{r-1} + \alpha'_{r-1}$
8.2 $r < q$	$(+3, +2^{2r}, +1^{2p-1-2r}, -1^{2q-2r})$		$\alpha_1, \ldots, \alpha_r,\ \alpha'_1, \ldots, \alpha'_r,$ $2(\alpha'_{r+1} + \cdots + \alpha'_q)$
8.2 $r = q$	$(+3, +2^{2q}, +1^{2p-1-2q})$		$\alpha_1, \ldots, \alpha_q,\ \alpha'_1, \ldots, \alpha'_q$
8.3 $r < p$	$(-3, +2^{2r}, +1^{2p-2r}, -1^{2q-1-2r})$		$\alpha_1, \ldots, \alpha_r,\ \alpha'_1, \ldots, \alpha'_r,$ $2(\alpha_{r+1} + \cdots + \alpha_p)$
8.3 $r = p$	$(-3, +2^{2p}, -1^{2q-1-2p})$		$\alpha_1, \ldots, \alpha_p,\ \alpha'_1, \ldots, \alpha'_p$

Table 19. $G = \mathsf{D}_{p+q},\ K = \mathsf{D}_p \times \mathsf{D}_q\ (p, q \geq 2)$

	Signed partition	Diagram of $\mathbb{P}(Ke)$	Spherical roots
9.1	$(+2^{2r}, +1^{2p-2r}, -1^{2q-2r})$, $r \geq 1$ (I) or (II) if $r = p < q$ or $r = q < p$ (I, I), (I, II), (II, I) or (II, II) if $r = p = q$		$\alpha_1 + \alpha'_1, \dots, \alpha_{r-1} + \alpha'_{r-1}$ if $r < p,\ q$ $\alpha_1 + \alpha'_1, \dots, \alpha_{p-1} + \alpha'_{p-1}$ if $r = p < q$ (I) $\alpha_1 + \alpha'_1, \dots, \alpha_{p-2} + \alpha'_{p-2}, \alpha_p + \alpha'_{p-1}$ if $r = p < q$ (II) $\alpha_1 + \alpha'_1, \dots, \alpha_{q-1} + \alpha'_{q-1}$ if $r = q < p$ (I) $\alpha_1 + \alpha'_1, \dots, \alpha_{q-2} + \alpha'_{q-2}, \alpha_{q-1} + \alpha'_q$ if $r = q < p$ (II) $\alpha_1 + \alpha'_1, \dots, \alpha_{p-1} + \alpha'_{p-1}$ if $r = p = q$ (I, I) $\alpha_1 + \alpha'_1, \dots, \alpha_{p-2} + \alpha'_{p-2}, \alpha_{p-1} + \alpha'_p$ if $r = p = q$ (I, II) $\alpha_1 + \alpha'_1, \dots, \alpha_{p-2} + \alpha'_{p-2}, \alpha_p + \alpha'_{p-1}$ if $r = p = q$ (II, I) $\alpha_1 + \alpha'_1, \dots, \alpha_{p-2} + \alpha'_{p-2}, \alpha_p + \alpha'_p$ if $r = p = q$ (II, II)
9.2 $r < q - 1$	$(+3, +2^{2r}, +1^{2p-2-2r}, -1^{2q-1-2r})$ (I) or (II) if $r = p - 1$		$\alpha_1, \dots, \alpha_r,\ \alpha'_1, \dots, \alpha'_r,\ 2(\alpha'_{r+1} + \dots + \alpha'_{q-2}) + \alpha'_{q-1} + \alpha'_q$ if $r < p - 1$ $\alpha_1, \dots, \alpha_{p-1},\ \alpha'_1, \dots, \alpha'_{p-1},\ 2(\alpha'_p + \dots + \alpha'_{q-2}) + \alpha'_{q-1} + \alpha'_q$ if $r = p - 1$ (I) $\alpha_1, \dots, \alpha_{p-2},\ \alpha_p,\ \alpha'_1, \dots, \alpha'_{p-1},\ 2(\alpha'_p + \dots + \alpha'_{q-2}) + \alpha'_{q-1} + \alpha'_q$ if $r = p - 1$ (II)
9.2 $r = q - 1$	$(+3, +2^{2q-2}, +1^{2p-2q}, -1)$ (I) or (II) if $p = q$		$\alpha_1, \dots, \alpha_{q-1},\ \alpha'_1, \dots, \alpha'_q$ if $p > q$ $\alpha_1, \dots, \alpha_{q-1},\ \alpha'_1, \dots, \alpha'_q$ if $p = q$ (I) $\alpha_1, \dots, \alpha_{q-2},\ \alpha_q,\ \alpha'_1, \dots, \alpha'_q$ if $p = q$ (II)
9.3 $r < p - 1$	$(-3, +2^{2r}, +1^{2p-1-2r}, -1^{2q-2-2r})$ (I) or (II) if $r = q - 1$		$\alpha_1, \dots, \alpha_r,\ \alpha'_1, \dots, \alpha'_r,\ 2(\alpha_{r+1} + \dots + \alpha_{p-2}) + \alpha_{p-1} + \alpha_p$ if $r < q - 1$ $\alpha_1, \dots, \alpha_{q-1},\ \alpha'_1, \dots, \alpha'_{q-1},\ 2(\alpha_q + \dots + \alpha_{p-2}) + \alpha_{p-1} + \alpha_p$ if $r = q - 1$ (I) $\alpha_1, \dots, \alpha_{q-1},\ \alpha'_1, \dots, \alpha'_{q-2},\ \alpha'_q,\ 2(\alpha_q + \dots + \alpha_{p-2}) + \alpha_{p-1} + \alpha_p$ if $r = q - 1$ (II)
9.3 $r = p - 1$	$(-3, +2^{2p-2}, +1, -1^{2q-2p})$ (I) or (II) if $p = q$		$\alpha_1, \dots, \alpha_p,\ \alpha'_1, \dots, \alpha'_{p-1}$ if $p < q$ $\alpha_1, \dots, \alpha_p,\ \alpha'_1, \dots, \alpha'_{p-1}$ if $p = q$ (I) $\alpha_1, \dots, \alpha_p,\ \alpha'_1, \dots, \alpha'_{p-2},\ \alpha'_p$ if $p = q$ (II)

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