Lattice multipolygons

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Abstract We discuss generalizations of some results on lattice polygons to certain piecewise linear loops which may have a self-intersection but have vertices in the lattice $\mathbb{Z}^2$. We first prove a formula on the rotation number of a unimodular sequence in $\mathbb{Z}^2$. This formula implies the generalized twelve-point theorem of Poonen and Rodriguez-Villegas. We then introduce the notion of lattice multipolygons, which is a generalization of lattice polygons, state the generalized Pick's formula, and discuss the classification of Ehrhart polynomials of lattice multipolygons and also of several natural subfamilies of lattice multipolygons.

Introduction

Lattice polygons are an elementary but fascinating object. Many interesting results such as Pick's formula are known for them. The results are interesting in themselves, but there are also a variety of proofs of the results that use advanced mathematics such as toric geometry, complex analysis, and modular forms (see, e.g., [4], [3], [9], [11]). These proofs are unexpected and make the study of lattice polygons more fruitful and intriguing.

Some of the results on lattice polygons are generalized to certain generalized polygons. For instance, Pick's [10] formula

$$A(P) = \sharp P^\circ + \frac{1}{2}B(P) - 1$$

for a lattice polygon P, where $A(P)$ is the area of P and $\sharp P^\circ$ (resp., $B(P)$) is the number of lattice points in the interior (resp., on the boundary) of P, is generalized in several directions. One of the generalizations is to certain piecewise linear loops which may have a self-intersection but have vertices in $\mathbb{Z}^2$ (see [5], [8]). As is well known, Pick's formula has an interpretation in toric geometry when P is convex (see [4], [9]), but the proof using toric geometry is not applicable when P is concave. However, once we develop toric geometry from the topological point of view, that is, *toric topology*, Pick's formula can be proved along the same line in full generality as is done in [8].

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Another such result on lattice polygons is the twelve-point theorem. It says that if P is a convex lattice polygon which contains the origin in its interior as a unique lattice point, then

$$B(P)+B(P^\vee)=12,$$

where $P^\vee$ is the lattice polygon dual to P. Several proofs are known to the theorem, and one of them again uses toric geometry. B. Poonen and F. Rodriguez-Villegas [11] provided a new proof by using modular forms. They also formulate a generalization of the twelve-point theorem and claim that their proof works in the general setting. It is mentioned in [11] that the proof using toric geometry is difficult to generalize, but a slight generalization of the proof of [8, Theorem 5.1], which uses toric topology and is along the same line of the proof using toric geometry, implies the generalized twelve-point theorem.

The generalized polygons considered in the generalization of the twelve-point theorem are called *legal loops*. A legal loop may have a self-intersection and is associated to a unimodular sequence of vectors $v_1,\dots,v_d$ in $\mathbb{Z}^2$. Here unimodular means that any consecutive two vectors v_i, v_{i+1} $(i=1,\dots,d)$ in the sequence form a basis of $\mathbb{Z}^2$, where $v_{d+1}=v_1$. Therefore, $\epsilon_i=\det(v_i,v_{i+1})$ is ±1. One sees that there is a unique integer a_i satisfying

$$\epsilon_{i-1}v_{i-1}+\epsilon_i v_{i+1}+a_i v_i=0$$

for each $i=1,\dots,d$. Note that $|a_i|$ is twice the area of the triangle with vertices v_{i-1}, v_{i+1}, and the origin. We prove that the rotation number of the unimodular sequence $v_1,\dots,v_d$ around the origin is given by (see Theorem 1.2)

$$\frac{1}{12}\Big(\sum_{i=1}^{d}a_i+3\sum_{i=1}^{d}\epsilon_i\Big).$$

The generalized twelve-point theorem easily follows from this formula. This formula was originally proved using toric topology, but after that, an elementary and combinatorial proof was found. We give it in Section 1 and the original proof in the Appendix. A different elementary proof of the above formula appeared in [14] while we were revising this article.

We also introduce the notion of lattice multipolygons. A *lattice multipolygon* is a piecewise linear loop with vertices in $\mathbb{Z}^2$ together with a sign function which assigns either $+$ or $-$ to each side and satisfies some mild condition. The piecewise linear loop may have a self-intersection, and we think of it as a sequence of points in $\mathbb{Z}^2$. A lattice polygon can naturally be regarded as a lattice multipolygon. The generalized Pick's formula holds for lattice multipolygons, so Ehrhart polynomials can be defined for them. The Ehrhart polynomial of a lattice multipolygon is of degree at most 2. The constant term is the rotation number of normal vectors to sides of the multipolygon and not necessarily 1 as it would be for ordinary Ehrhart polynomials. The other coefficients have geometric meanings similar to the ordinary ones, but they can be zero or negative unlike the ordinary ones. The family of lattice multipolygons has some natural subfamilies, for example, the family of all convex lattice polygons. We discuss the characterization of Ehrhart polynomials of not only all lattice multipolygons but also some natural subfamilies.

The structure of the present article is as follows. In Section 1, we give the elementary proof of the formula which describes the rotation number of a unimodular sequence of vectors in $\mathbb{Z}^2$ around the origin. Here the vectors in the sequence may go back and forth. The proof using toric topology is given in the Appendix. In Section 2, we observe that the formula implies the generalized twelve-point theorem. In Section 3, we introduce the notion of a lattice multipolygon and state the generalized Pick's formula for lattice multipolygons. In Section 4, we discuss the characterization of Ehrhart polynomials of lattice multipolygons and also of several natural subfamilies of lattice multipolygons.

1. Rotation number of a unimodular sequence

We say that a sequence of vectors $v_1, \dots, v_d$ in $\mathbb{Z}^2$ $(d \geq 2)$ is *unimodular* if each triangle with vertices $\mathbf{0}$, v_i, and v_{i+1} contains no lattice point except the vertices, where $\mathbf{0} = (0,0)$ and $v_{d+1} = v_1$. The vectors in the sequence are not necessarily counterclockwise or clockwise. They may go back and forth. We set

$$\epsilon_i = \det(v_i, v_{i+1}) \quad \text{for } i = 1, \dots, d. \tag{1.1}$$

In other words, $\epsilon_i = 1$ if the rotation from v_i to v_{i+1} (with angle less than π) is counterclockwise and $\epsilon_i = -1$ otherwise. Since each successive pair (v_j, v_{j+1}) is a basis of $\mathbb{Z}^2$ for $j = 1, \dots, d$, one has

$$(v_i, v_{i+1}) = (v_{i-1}, v_i) \begin{pmatrix} 0 & -\epsilon_{i-1}\epsilon_i \\ 1 & -\epsilon_i a_i \end{pmatrix}$$

with a unique integer a_i for each i. This is equivalent to

$$\epsilon_{i-1} v_{i-1} + \epsilon_i v_{i+1} + a_i v_i = 0. \tag{1.2}$$

Note that $|a_i|$ is twice the area of the triangle with vertices $\mathbf{0}$, v_{i-1}, and v_{i+1}.

EXAMPLE 1.1

(a) Take a unimodular sequence (see Figure 1 in Section 2)

$$\mathcal{P} = (v_1, \dots, v_5) = \big((1,0), (0,1), (-1,0), (0,-1), (-1,-1)\big).$$

Then

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_5 = 1, \qquad \epsilon_4 = -1, \qquad \text{and} \qquad a_1 = a_4 = a_5 = 1, \qquad a_2 = a_3 = 0,$$

and the rotation number of $\mathcal{P}$ around the origin is 1.

(b) Take another unimodular sequence (see Figure 2 in Section 2)

$$\mathcal{Q} = (v_1, \dots, v_6) = \big((1,0), (-1,1), (0,-1), (1,1), (-1,0), (1,-1)\big).$$

Then

$$\epsilon_1 = \cdots = \epsilon_6 = 1 \qquad \text{and} \qquad a_1 = a_6 = 0, \qquad a_2 = a_4 = 1, \qquad a_3 = a_5 = 2,$$

and the rotation number of $\mathcal{Q}$ around the origin is 2.

Our main result in this section is the following.

THEOREM 1.2

The rotation number of a unimodular sequence $v_1, \ldots, v_d$ $(d \geq 2)$ around the origin is given by

$$\frac{1}{12}\Big(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d} \epsilon_i\Big), \tag{1.3}$$

where the ϵ_i's and a_i's are the integers defined in (1.1) and (1.2).

For our proof of this theorem, we prepare the following lemma.

LEMMA 1.3

Let $v_1, \ldots, v_d$ be a unimodular sequence, and let v_j be a vector whose Euclidean norm is maximal among the vectors in the sequence, where $1 \leq j \leq d$. Then $a_j = 0$ or ± 1.

Proof

It follows from (1.2) and the maximality of the Euclidean norm of v_j that we have

$$\|a_j v_j\| = \| -\epsilon_{j-1} v_{j-1} - \epsilon_j v_{j+1}\| \leq \|v_{j-1}\| + \|v_{j+1}\| \leq \|v_j\| + \|v_j\|, \tag{1.4}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. Therefore, $|a_j| \leq 1$ or $|a_j| = 2$ and the equality holds in (1.4). However, the latter case does not occur because the vectors v_{j-1}, v_j, v_{j+1} are not parallel, proving the lemma. □

Proof of Theorem 1.2

We give a proof by induction on d.

When $d = 2$, the rotation number of v_1, v_2 is zero while $a_1 = a_2 = 0$ and $\epsilon_1 + \epsilon_2 = 0$. Therefore, the theorem holds in this case.

When $d = 3$, we may assume that $(v_1, v_2) = ((1,0),(0,1))$ or $(v_1, v_2) = ((0,1),(1,0))$ through an (orientation-preserving) unimodular transformation on $\mathbb{R}^2$, and then v_3 is one of $(1,1)$, $(-1,1)$, $(1,-1)$, and $(-1,-1)$. Now, it is immediate to check that the rotation number of each unimodular sequence coincides with (1.3).

Let $d \geq 4$, and assume that the theorem holds for any unimodular sequence with at most $d-1$ vectors. Let v_j be a vector in the unimodular sequence $v_1, \ldots, v_d$ whose Euclidean norm is maximal among the vectors in the sequence. Then Lemma 1.3 says that $a_j = 0$ or ± 1.

The case where $a_j = 0$, *that is,*

$$\epsilon_{j-1}v_{j-1} + \epsilon_j v_{j+1} = 0. \tag{1.5}$$

In this case, we consider a subsequence $v_1, \dots, v_{j-2}, v_{j+1}, \dots, v_d$ obtained by removing two vectors v_{j-1} and v_j from the given unimodular sequence. Since

$$\left|\det(v_{j-2}, v_{j+1})\right| = \left|\det(v_{j-2}, -\epsilon_{j-1}\epsilon_j v_{j-1})\right| = 1,$$

the subsequence is also unimodular. Set

$$v'_i = \begin{cases} v_i & \text{for } 1 \leq i \leq j-2, \\ v_{i+2} & \text{for } j-1 \leq i \leq d-2, \end{cases} \tag{1.6}$$

and define ϵ'_i and a'_i for the unimodular sequence $v'_1, \dots, v'_{d-2}$ similarly to (1.1) and (1.2); that is,

$$\epsilon'_i = \det(v'_i, v'_{i+1}), \qquad \epsilon'_{i-1}v'_{i-1} + \epsilon'_i v'_{i+1} + a'_i v'_i = 0. \tag{1.7}$$

Then, it follows from (1.5)–(1.7) and (1.1) that

$$\epsilon'_i = \begin{cases} \epsilon_i & \text{for } 1 \leq i \leq j-3, \\ -\epsilon_{j-2}\epsilon_{j-1}\epsilon_j & \text{for } i = j-2, \\ \epsilon_{i+2} & \text{for } j-1 \leq i \leq d-2. \end{cases} \tag{1.8}$$

It also follows from (1.5)–(1.8) and (1.2) that

$$\begin{aligned} a'_{j-2}v_{j-2} = a'_{j-2}v'_{j-2} &= -\epsilon'_{j-3}v'_{j-3} - \epsilon'_{j-2}v'_{j-1} \\ &= -\epsilon_{j-3}v_{j-3} - (-\epsilon_{j-2}\epsilon_{j-1}\epsilon_j)(-\epsilon_{j-1}\epsilon_j v_{j-1}) \\ &= -\epsilon_{j-3}v_{j-3} - \epsilon_{j-2}v_{j-1} = a_{j-2}v_{j-2} \end{aligned}$$

and

$$\begin{aligned} a'_{j-1}v_{j+1} = a'_{j-1}v'_{j-1} &= -\epsilon'_{j-2}v'_{j-2} - \epsilon'_{j-1}v'_j \\ &= \epsilon_{j-2}\epsilon_{j-1}\epsilon_j v_{j-2} - \epsilon_{j+1}v_{j+2} \\ &= -\epsilon_{j-1}\epsilon_j(-\epsilon_{j-2}v_{j-2} - \epsilon_{j-1}v_j) - \epsilon_j v_j - \epsilon_{j+1}v_{j+2} \\ &= -\epsilon_{j-1}\epsilon_j a_{j-1}v_{j-1} + a_{j+1}v_{j+1} \\ &= a_{j-1}v_{j+1} + a_{j+1}v_{j+1} = (a_{j-1} + a_{j+1})v_{j+1}. \end{aligned}$$

Therefore,

$$a'_i = \begin{cases} a_i & \text{for } 1 \leq i \leq j-2, \\ a_{j-1} + a_{j+1} & \text{for } i = j-1, \\ a_{i+2} & \text{for } j \leq i \leq d-2. \end{cases} \tag{1.9}$$

Since $a_j = 0$, it follows from (1.8) and (1.9) that

$$\begin{aligned}&\frac{1}{12}\Big(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d}\epsilon_i\Big) - \frac{1}{12}\Big(\sum_{i=1}^{d-2} a'_i + 3\sum_{i=1}^{d-2}\epsilon'_i\Big)\\ &\quad = \frac{1}{4}(\epsilon_{j-2}+\epsilon_{j-1}+\epsilon_j-\epsilon'_{j-2}) = \frac{1}{4}(\epsilon_{j-2}+\epsilon_{j-1}+\epsilon_j+\epsilon_{j-2}\epsilon_{j-1}\epsilon_j),\end{aligned}\tag{1.10}$$

which is $+1$ (resp., -1) if ϵ_{j-2}, ϵ_{j-1}, and ϵ_j are all $+1$ (resp., -1), and 0 otherwise. On the other hand, one can see that if the rotation number of $v_1, \dots, v_d$ is r, then that of $v'_1, \dots, v'_{d-2}$ is equal to $r-1$ (resp., $r+1$) if ϵ_{j-2}, ϵ_{j-1}, and ϵ_j are all $+1$ (resp., -1), and r otherwise. This result, the hypothesis of induction, and (1.10) show that $\frac{1}{12}(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d}\epsilon_i)$ is the rotation number of $v_1, \dots, v_d$.

The case where $a_j = \pm 1$, we have

$$\epsilon_j v_{j+1} + \epsilon_{j-1} v_{j-1} + a_j v_j = 0.\tag{1.11}$$

In this case, we consider a subsequence $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_d$ obtained by removing the v_j from the given unimodular sequence. Since

$$\big|\det(v_{j-1}, v_{j+1})\big| = \big|\det(v_{j-1}, -\epsilon_j\epsilon_{j-1}v_{j-1} - \epsilon_j a_j v_j)\big| = \big|\det(v_{j-1}, v_j)\big| = 1,$$

the subsequence is also unimodular. Set

$$v'_i = \begin{cases} v_i & \text{for } 1 \le i \le j-1,\\ v_{i+1} & \text{for } j \le i \le d-1,\end{cases}\tag{1.12}$$

and define ϵ'_i and a'_i for the unimodular sequence $v'_1, \dots, v'_{d-1}$ as before by (1.7). Then, it follows from (1.7), (1.11), (1.12), and (1.1) that

$$\epsilon'_i = \begin{cases} \epsilon_i & \text{for } 1 \le i \le j-2,\\ -\epsilon_{j-1}\epsilon_j a_j & \text{for } i = j-1,\\ \epsilon_{i+1} & \text{for } j \le i \le d-1.\end{cases}\tag{1.13}$$

It also follows from (1.11)–(1.13), (1.7), and (1.2) that

$$\begin{aligned} a'_{j-1}v_{j-1} = a'_{j-1}v'_{j-1} &= -\epsilon'_{j-2}v'_{j-2} - \epsilon'_{j-1}v'_j\\ &= -\epsilon_{j-2}v_{j-2} + \epsilon_{j-1}\epsilon_j a_j v_{j+1}\\ &= -\epsilon_{j-2}v_{j-2} + \epsilon_{j-1}a_j(-a_j v_j - \epsilon_{j-1}v_{j-1})\\ &= -\epsilon_{j-2}v_{j-2} - \epsilon_{j-1}v_j - a_j v_{j-1}\\ &= a_{j-1}v_{j-1} - a_j v_{j-1} = (a_{j-1} - a_j)v_{j-1}\end{aligned}$$

and

$$\begin{aligned} a'_j v_{j+1} = a'_j v'_j &= -\epsilon'_{j-1}v'_{j-1} - \epsilon'_j v'_{j+1}\\ &= \epsilon_{j-1}\epsilon_j a_j v_{j-1} - \epsilon_{j+1}v_{j+2}\\ &= \epsilon_j a_j(-a_j v_j - \epsilon_j v_{j+1}) - \epsilon_{j+1}v_{j+2}\end{aligned}$$

$$= -a_j v_{j+1} - \epsilon_j v_j - \epsilon_{j+1} v_{j+2}$$
$$= -a_j v_{j+1} + a_{j+1} v_{j+1} = (a_{j+1} - a_j) v_{j+1}.$$

Therefore,

$$a'_i = \begin{cases} a_i & \text{for } 1 \le i \le j-2, \\ a_{j-1} - a_j & \text{for } i = j-1, \\ a_{j+1} - a_j & \text{for } i = j, \\ a_{i+1} & \text{for } j+1 \le i \le d-1. \end{cases} \tag{1.14}$$

It follows from (1.13) and (1.14) that

$$\begin{aligned} &\frac{1}{12}\Big(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d} \epsilon_i\Big) - \frac{1}{12}\Big(\sum_{i=1}^{d-1} a'_i + 3\sum_{i=1}^{d-1} \epsilon'_i\Big) \\ &\quad = \frac{1}{4}(a_j + \epsilon_{j-1} + \epsilon_j - \epsilon'_{j-1}) = \frac{1}{4}\big((1 + \epsilon_{j-1}\epsilon_j)a_j + \epsilon_{j-1} + \epsilon_j\big), \end{aligned} \tag{1.15}$$

which is a_j if both ϵ_{j-1} and ϵ_j are a_j and 0 otherwise. On the other hand, one can see that if the rotation number of $v_1, \dots, v_d$ is r, then that of $v'_1, \dots, v'_{d-1}$ is equal to $r - a_j$ if both ϵ_{j-1} and ϵ_j are a_j and r otherwise. This result, the hypothesis of induction, and (1.15) show that $\frac{1}{12}(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d} \epsilon_i)$ is the rotation number of $v_1, \dots, v_d$. This completes the proof of the theorem. □

REMARK

A different elementary proof of Theorem 1.2 is given in [14].

2. Generalized twelve-point theorem

Let P be a convex lattice polygon whose only interior lattice point is the origin. Then the dual $P^\vee$ to P is also a convex lattice polygon whose only interior lattice point is the origin. Let $B(P)$ denote the total number of lattice points on the boundary of P. The following fact is well known.

THEOREM 2.1 (TWELVE-POINT THEOREM)

We have that $B(P) + B(P^\vee) = 12$.

Several proofs are known for this theorem (see [2], [12], [11]). Poonen and Rodriguez-Villegas [11] gave a proof using modular forms. They also formulated a generalization of the twelve-point theorem and claimed that their proof works in the general setting. In this section, we will explain the generalized twelve-point theorem and observe that it follows from Theorem 1.2.

If P is a convex lattice polygon whose only interior lattice point is the origin and $v_1, \dots, v_d$ are the vertices of P arranged counterclockwise, then every v_i is primitive and the triangle with the vertices $\mathbf{0}$, v_i, and v_{i+1} has no lattice point in the interior for each i, where $v_{d+1} = v_1$ as usual. This observation motivates the following definition (see [11], [2]).

DEFINITION

A sequence of vectors $\mathcal{P} = (v_1, \dots, v_d)$, where $v_1, \dots, v_d$ are in $\mathbb{Z}^2$ and $d \geq 2$, is called a *legal loop* if every v_i is primitive and, whenever $v_i \neq v_{i+1}$, v_i and v_{i+1} are linearly independent (i.e., $v_i \neq -v_{i+1}$) and the triangle with the vertices $\mathbf{0}$, v_i, and v_{i+1} has no lattice point in the interior. We say that a legal loop is *reduced* if $v_i \neq v_{i+1}$ for any i. A (nonreduced) legal loop $\mathcal{P}$ naturally determines a reduced legal loop, denoted $\mathcal{P}_{\mathrm{red}}$, by dropping all the redundant points. We define the *winding number* of a legal loop $\mathcal{P} = (v_1, \dots, v_d)$ to be the rotation number of the vectors $v_1, \dots, v_d$ around the origin.

Joining successive points in a legal loop $\mathcal{P} = (v_1, \dots, v_d)$ by straight lines forms a lattice polygon which may have a self-intersection. A unimodular sequence $v_1, \dots, v_d$ determines a reduced legal loop. Conversely, a reduced legal loop $\mathcal{P} = (v_1, \dots, v_d)$ determines a unimodular sequence by adding all the lattice points on the line segment $v_i v_{i+1}$ (called a *side* of $\mathcal{P}$) connecting v_i and v_{i+1} for every i. To each side $v_i v_{i+1}$ with $v_i \neq v_{i+1}$, we assign the sign of $\det(v_i, v_{i+1})$, denoted $\mathrm{sgn}(v_i, v_{i+1})$.

For a reduced legal loop $\mathcal{P} = (v_1, \dots, v_d)$, we set

$$w_i = \frac{v_i - v_{i-1}}{\det(v_{i-1}, v_i)} \quad \text{for } i = 1, \dots, d, \tag{2.1}$$

where $v_0 = v_d$. Note that w_i is integral and primitive, and define $\mathcal{P}^\vee = (w_1, \dots, w_d)$ following [11] (see also [2]). It is not difficult to see that $\mathcal{P}^\vee = (w_1, \dots, w_d)$ is again a legal loop, although it may not be reduced (see the proof of Theorem 2.3 below). If a legal loop $\mathcal{P}$ is not reduced, then we define $\mathcal{P}^\vee$ to be $(\mathcal{P}_{\mathrm{red}})^\vee$. When the vectors $v_1, \dots, v_d$ are the vertices of a convex lattice polygon P with only the origin as an interior lattice point and are arranged in counterclockwise order, the sequence $w_1, \dots, w_d$ is also in counterclockwise order and the convex hull of $w_1, \dots, w_d$ is the 180 degree rotation of the polygon $P^\vee$ dual to P.

EXAMPLE 2.2

Let us consider $\mathcal{P}$ and $\mathcal{Q}$ as described in Example 1.1. Then those are reduced legal loops.

(a) We have

$$\mathcal{P}^\vee = \big((2,1), (-1,1), (-1,-1), (1,-1), (1,0)\big).$$

(b) Similarly,

$$\mathcal{Q}^\vee = \big((0,1), (-2,1), (1,-2), (1,2), (-2,-1), (2,-1)\big).$$

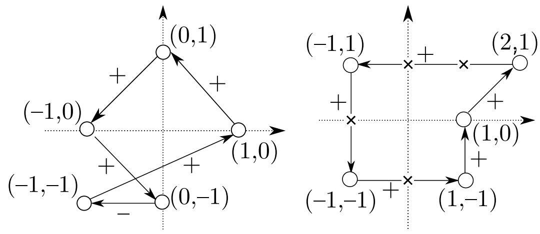

Figure 1. Legal loops $\mathcal{P}$ and $\mathcal{P}^\vee$ and sides with signs.

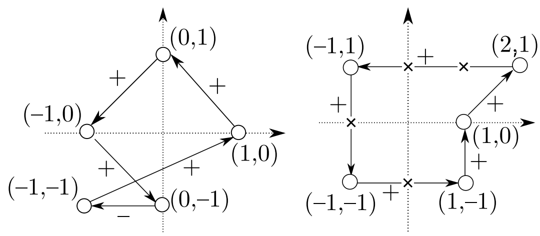

Figure 2. Legal loops $\mathcal{Q}$ and $\mathcal{Q}^\vee$.

DEFINITION

Let $|v_iv_{i+1}|$ be the number of lattice points on the side v_iv_{i+1} minus 1, so $|v_iv_{i+1}| = 0$ when $v_i = v_{i+1}$. Then we define

$$B(\mathcal{P}) = \sum_{i=1}^{d} \operatorname{sgn}(v_i, v_{i+1})|v_iv_{i+1}|.$$

Clearly, $B(\mathcal{P}) = B(\mathcal{P}_{\mathrm{red}})$.

THEOREM 2.3 (GENERALIZED TWELVE-POINT THEOREM [11, SECTION 9.1])

Let $\mathcal{P}$ be a legal loop, and let r be the winding number of $\mathcal{P}$. Then $B(\mathcal{P}) + B(\mathcal{P}^\vee) = 12r$.

Proof

We may assume that $\mathcal{P}$ is reduced. As remarked before, the reduced legal loop $\mathcal{P} = (v_1, \ldots, v_d)$ determines a unimodular sequence by adding all the lattice points on the side v_iv_{i+1} for every i, and the unimodular sequence determines a reduced legal loop, say, $\mathcal{Q}$. Clearly, $B(\mathcal{P}) = B(\mathcal{Q})$ and $(\mathcal{P}^\vee)_{\mathrm{red}} = (\mathcal{Q}^\vee)_{\mathrm{red}}$. In the rest of the section, we may assume that the vectors $v_1, \ldots, v_d$ in our legal loop $\mathcal{P}$ form a unimodular sequence.

Since the sequence $v_1, \ldots, v_d$ is unimodular, $\operatorname{sgn}(v_i, v_{i+1}) = \epsilon_i$ and $|v_iv_{i+1}| = 1$ for any i. Therefore,

$$B(\mathcal{P}) = \sum_{i=1}^{d} \operatorname{sgn}(v_i, v_{i+1})|v_iv_{i+1}| = \sum_{i=1}^{d} \epsilon_i. \tag{2.2}$$

On the other hand, it follows from (2.1) and (1.2) that

$$\begin{aligned} w_{i+1} - w_i &= \epsilon_i(v_{i+1} - v_i) - \epsilon_{i-1}(v_i - v_{i-1}) \\ &= \epsilon_i v_{i+1} + \epsilon_{i-1} v_{i-1} - (\epsilon_i + \epsilon_{i-1}) v_i \\ &= -(a_i + \epsilon_i + \epsilon_{i-1}) v_i \end{aligned} \tag{2.3}$$

and that

$$\begin{aligned} \det(w_i, w_{i+1}) &= \epsilon_{i-1}\epsilon_i \det(v_i - v_{i-1}, v_{i+1} - v_i) \\ &= \epsilon_{i-1}\epsilon_i \det(v_i - v_{i-1}, -\epsilon_{i-1}\epsilon_i v_{i-1} - \epsilon_i a_i v_i - v_i) \\ &= \epsilon_{i-1}\epsilon_i \big(\det(v_i, -\epsilon_{i-1}\epsilon_i v_{i-1}) + \det(-v_{i-1}, -\epsilon_i a_i v_i - v_i)\big) \\ &= \epsilon_{i-1} + a_i + \epsilon_i. \end{aligned} \tag{2.4}$$

Since v_i is primitive, (2.3) shows that $|w_i w_{i+1}| = |\epsilon_{i-1} + \epsilon_i + a_i|$, and this together with (2.4) shows that

$$\operatorname{sgn}(w_i, w_{i+1})|w_i w_{i+1}| = \epsilon_{i-1} + \epsilon_i + a_i.$$

Therefore,

$$B(\mathcal{P}^\vee) = \sum_{i=1}^{d} \operatorname{sgn}(w_i, w_{i+1})|w_i w_{i+1}| = \sum_{i=1}^{d} (\epsilon_{i-1} + \epsilon_i + a_i). \tag{2.5}$$

It follows from (2.2) and (2.5) that

$$\begin{aligned} B(\mathcal{P}) + B(\mathcal{P}^\vee) &= \sum_{i=1}^{d} \epsilon_i + \sum_{i=1}^{d} (\epsilon_{i-1} + \epsilon_i + a_i) \\ &= 3\sum_{i=1}^{d} \epsilon_i + \sum_{i=1}^{d} a_i, \end{aligned}$$

which is equal to $12r$ by Theorem 1.2, proving the theorem. □

EXAMPLE 2.4

Let us consider again the legal loops $\mathcal{P}$ and $\mathcal{Q}$ in the previous example.

(a) On the one hand, $B(\mathcal{P}) = 1 + 1 + 1 - 1 + 1 = 3$. On the other hand, $B(\mathcal{P}^\vee) = 3 + 2 + 2 + 1 + 1 = 9$. Thus, we have $B(\mathcal{P}) + B(\mathcal{P}^\vee) = 12$. The left-hand side (resp., right-hand side) of Figure 1 depicted in Example 2.2 shows $\mathcal{P}$ (resp., $\mathcal{P}^\vee$) together with signs, where the symbols $\circ$ and $\times$ stand for lattice points in $\mathbb{Z}^2$.

(b) On the one hand, $B(\mathcal{Q}) = 6$. On the other hand, $B(\mathcal{Q}^\vee) = 18$. Hence, $B(\mathcal{Q}) + B(\mathcal{Q}^\vee) = 24$. The left-hand side (resp., right-hand side) of Figure 2 shows $\mathcal{Q}$ (resp., $\mathcal{Q}^\vee$). Note that the signs on the sides of $\mathcal{Q}$ and $\mathcal{Q}^\vee$ are all $+$.

REMARK

Kasprzyk and Nill [7, Corollary 2.7] pointed out that the generalized twelve-point theorem can further be generalized to what are called *ℓ-reflexive loops*, where ℓ is a positive integer and a 1-reflexive loop is a legal loop.

3. Generalized Pick's formula for lattice multipolygons

In this section, we introduce the notion of a lattice multipolygon and state the generalized Pick's formula for lattice multipolygons, which is essentially proved in [8, Theorem 8.1]. Moreover, from this formula, we can define the Ehrhart polynomials for lattice multipolygons.

We begin with the well-known Pick's [10] formula for lattice polygons. Let P be a (not necessarily convex) lattice polygon, let ∂P be the boundary of P, and let $P^\circ = P\backslash\partial P$. We define

$$A(P) = \text{the area of } P, \qquad B(P) = |\partial P \cap \mathbb{Z}^2|, \qquad \sharp P^\circ = |P^\circ \cap \mathbb{Z}^2|,$$

where $|X|$ denotes the cardinality of a finite set X. Then Pick's formula says that

$$A(P) = \sharp P^\circ + \frac{1}{2}B(P) - 1. \tag{3.1}$$

We may rewrite (3.1) as

$$\sharp P^\circ = A(P) - \frac{1}{2}B(P) + 1 \qquad \text{or} \qquad \sharp P = A(P) + \frac{1}{2}B(P) + 1,$$

where $\sharp P = |P \cap \mathbb{Z}^2|$.

In [5], the notion of *shaven lattice polygon* is introduced, and Pick's formula (3.1) is generalized to shaven lattice polygons. The generalization of Pick's formula discussed in [8] is similar to that in [5] but a bit more general, which we shall explain.

Let $\mathcal{P} = (v_1, \dots, v_d)$ be a sequence of points $v_1, \dots, v_d$ in $\mathbb{Z}^2$. One may regard $\mathcal{P}$ as an *oriented* piecewise linear loop by connecting all successive points from v_i to v_{i+1} in $\mathcal{P}$ by straight lines as before, where $v_{d+1} = v_1$. To each side v_iv_{i+1}, we assign a sign $+$ or $-$, denoted $\epsilon(v_iv_{i+1})$. In Section 2, we assigned $\operatorname{sgn}(v_i, v_{i+1})$, which is the sign of $\det(v_i, v_{i+1})$, to v_iv_{i+1}, but $\epsilon(v_iv_{i+1})$ may be different from $\operatorname{sgn}(v_i, v_{i+1})$. However, we require that the assignment ϵ of signs satisfies the following condition $(\star)$:

$(\star)$ when there are consecutive three points v_{i-1}, v_i, v_{i+1} in $\mathcal{P}$ lying on a line, we have

(1) $\epsilon(v_{i-1}v_i) = \epsilon(v_iv_{i+1})$ if v_i is in between v_{i-1} and v_{i+1};
(2) $\epsilon(v_{i-1}v_i) \neq \epsilon(v_iv_{i+1})$ if v_{i-1} lies on v_iv_{i+1} or v_{i+1} lies on $v_{i-1}v_i$.

A *lattice multipolygon* is $\mathcal{P}$ equipped with the assignment ϵ satisfying $(\star)$. We need to express a lattice multipolygon as a pair $(\mathcal{P}, \epsilon)$ to be precise, but we omit ϵ and express a lattice multipolygon simply as $\mathcal{P}$ in the following. Reduced legal loops introduced in Section 2 are lattice multipolygons.

REMARK

Lattice multipolygons such that three consecutive points are not on the same line are introduced in [8, Section 8]. But if we require the condition ($\star$), then the argument developed there works for any lattice multipolygon. A shaven polygon introduced in [5] is a lattice multipolygon with $\epsilon = +$ in our terminology, so that v_i is allowed to lie on the line segment $v_{i-1}v_{i+1}$ but v_{i-1} (resp., v_{i+1}) is not allowed to lie on v_iv_{i+1} (resp., $v_{i-1}v_i$) by (2) of ($\star$); that is, there is no *whisker*.

Let $\mathcal{P}$ be a multipolygon with a sign assignment ϵ. We think of $\mathcal{P}$ as an oriented piecewise linear loop with signs attached to sides. For $i = 1, \dots, d$, let n_i denote a normal vector to each side v_iv_{i+1} such that the 90 degree rotation of $\epsilon(v_iv_{i+1})n_i$ has the same direction as v_iv_{i+1}. The winding number of $\mathcal{P}$ around a point $v \in \mathbb{R}^2 \setminus \mathcal{P}$, denoted $d_{\mathcal{P}}(v)$, is a locally constant function on $\mathbb{R}^2 \setminus \mathcal{P}$, where $\mathbb{R}^2 \setminus \mathcal{P}$ means the set of elements in $\mathbb{R}^2$ which does not belong to any side of $\mathcal{P}$.

Following [8, Section 8], we define

$$\begin{aligned}
A(\mathcal{P}) &:= \int_{v\in\mathbb{R}^2\setminus\mathcal{P}} d_{\mathcal{P}}(v)\,dv,\\
B(\mathcal{P}) &:= \sum_{i=1}^{d} \epsilon(v_iv_{i+1})|v_iv_{i+1}|,\\
C(\mathcal{P}) &:= \text{the rotation number of the sequence of } n_1,\dots,n_d.
\end{aligned}$$

Notice that $A(\mathcal{P})$ and $B(\mathcal{P})$ can be 0 or negative. If $\mathcal{P}$ arises from a lattice polygon P, namely, $\mathcal{P}$ is a sequence of the vertices of P arranged in counterclockwise order and $\epsilon = +$, then $A(\mathcal{P}) = A(P)$, $B(\mathcal{P}) = B(P)$, and $C(\mathcal{P}) = 1$.

Now, we define $\sharp\mathcal{P}$ in such a way that if $\mathcal{P}$ arises from a lattice polygon P, then $\sharp\mathcal{P} = \sharp P$. Let $\mathcal{P}_+$ be an oriented loop obtained from $\mathcal{P}$ by pushing each side v_iv_{i+1} slightly in the direction of n_i. Since $\mathcal{P}$ satisfies the condition ($\star$), $\mathcal{P}_+$ misses all lattice points, so the winding numbers $d_{\mathcal{P}_+}(u)$ can be defined for any lattice point u by using $\mathcal{P}_+$. Then we define

$$\sharp\mathcal{P} := \sum_{u\in\mathbb{Z}^2} d_{\mathcal{P}_+}(u).$$

As remarked before, the lattice multipolygons treated in [8] require that three consecutive points v_{i-1}, v_i, v_{i+1} do not lie on the same line. But if the sign assignment ϵ satisfies the condition ($\star$) above, then the argument developed in [8, Section 8] works, and we obtain the following generalized Pick's formula for lattice multipolygons.

THEOREM 3.1 (CF. [8, THEOREM 8.1])
We have that $\sharp\mathcal{P} = A(\mathcal{P}) + \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P})$.

Proof
Let $\mathcal{P} = (v_1, \dots, v_d)$ be a lattice multipolygon. Similarly to the proof of [8, Theorem 8.1], we construct the multifan from $\mathcal{P}$ and apply the results in [8, Section 7].

Assume that $\mathcal{P}$ contains three consecutive points lying on a line, say, v_1, v_2, and v_3. Let n_i denote the primitive normal vector to each side v_iv_{i+1} such that 90 degree rotation of $\epsilon(v_iv_{i+1})n_i$ has the same direction as v_iv_{i+1}. Then the condition $(\star)$ implies that $n_1 = n_2$. Let n_{12} denote the primitive vector such that n_{12} is orthogonal to n_1. We add the new lattice vector n_{12} between n_1 and n_2, and the remaining method for the construction of the multifan associated with $\mathcal{P}$ is the same as in the proof of [8, Theorem 8.1]. Now, by applying the results in [8, Section 7], we can see that the required formula also holds for $\mathcal{P}$. □

If we define $\mathcal{P}^\circ$ to be $\mathcal{P}$ with $-\epsilon$ as a sign assignment, then

$$\sharp\mathcal{P}^\circ = A(\mathcal{P}) - \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P}), \tag{3.2}$$

and if $\mathcal{P}$ arises from a lattice polygon P, then $\sharp\mathcal{P}^\circ = \sharp P^\circ$.

Given a positive integer m, we dilate $\mathcal{P}$ by m times, denoted $m\mathcal{P}$; in other words, if $\mathcal{P}$ is $(v_1,\dots,v_d)$ with a sign assignment ϵ, then $m\mathcal{P}$ is $(mv_1,\dots,mv_d)$ with $\epsilon(v_iv_{i+1})$ as the sign of the side mv_imv_{i+1} of $m\mathcal{P}$ for each i. Then we have

$$\sharp(m\mathcal{P}) = A(\mathcal{P})m^2 + \frac{1}{2}B(\mathcal{P})m + C(\mathcal{P}); \tag{3.3}$$

that is, $\sharp(m\mathcal{P})$ is a polynomial in m of degree at most 2 whose coefficients are as above. Moreover, the equality

$$\sharp(m\mathcal{P}^\circ) = A(\mathcal{P})m^2 - \frac{1}{2}B(\mathcal{P})m + C(\mathcal{P}) = (-1)^2\sharp(-m\mathcal{P})$$

holds, so that the reciprocity holds for lattice multipolygons. We call the polynomial (3.3) the *Ehrhart polynomial* of a lattice multipolygon $\mathcal{P}$. We refer the reader to [1] for an introduction to the theory of Ehrhart polynomials of general convex lattice polytopes.

REMARK

In [6], lattice multipolytopes $\mathcal{P}$ of dimension n are defined, and it is proved that $\sharp(m\mathcal{P})$ is a polynomial in m of degree at most n which satisfies $\sharp(m\mathcal{P}^\circ) = (-1)^n\sharp(-m\mathcal{P})$, whose leading coefficient and constant term have similar geometric meanings to those in the 2-dimensional case above.

4. Ehrhart polynomials of lattice multipolygons

In this section, we will discuss which polynomials appear as the Ehrhart polynomials of lattice multipolygons. By virtue of (3.3), studying whether a polynomial $am^2 + bm + c$ is the Ehrhart polynomial of some lattice multipolygon is equivalent to classifying the triple $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ for lattice multipolygons $\mathcal{P}$. In the remainder of the section, we will discuss this triple for lattice multipolygons and their natural subfamilies.

If the triple (a,b,c) is equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of some lattice multipolygon $\mathcal{P}$, then (a,b,c) must be in the set

$$\mathcal{A} = \left\{(a,b,c) \in \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \mathbb{Z} : a+b \in \mathbb{Z}\right\},$$

because

$$B(\mathcal{P}) \in \mathbb{Z}, \qquad C(\mathcal{P}) \in \mathbb{Z}, \qquad A(\mathcal{P}) + \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P}) = \sharp\mathcal{P} \in \mathbb{Z}.$$

The following theorem shows that this condition is sufficient.

THEOREM 4.1

The triple (a,b,c) is equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of some lattice multipolygon $\mathcal{P}$ if and only if $(a,b,c) \in \mathcal{A}$.

Proof

It suffices to prove the "if" part. We pick up $(a,b,c) \in \mathcal{A}$. Then one has an expression

$$(a,b,c) = a'(1,0,0) + b'\Big(\frac{1}{2}, \frac{1}{2}, 0\Big) + c'(0,0,-1) \tag{4.1}$$

with integers a', b', c' because $a' = a - b$, $b' = 2b$, and $c' = -c$. One can easily check that $(1,0,0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, and $(0,0,-1)$ are, respectively, equal to $(A(\mathcal{P}_j), \frac{1}{2}B(\mathcal{P}_j), C(\mathcal{P}_j))$ of the lattice multipolygons $\mathcal{P}_j$ $(j = 1,2,3)$ shown in Figure 3, where the sign of $v_i v_{i+1}$ is given by the sign of $\det(v_i, v_{i+1})$ for $\mathcal{P}_j$.

Moreover, reversing both the order of the points and the signs on the sides for $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, we obtain lattice multipolygons $\mathcal{P}_1'$, $\mathcal{P}_2'$, and $\mathcal{P}_3'$ whose triples are, respectively, $(-1,0,0)$, $(-\frac{1}{2}, -\frac{1}{2}, 0)$, and $(0,0,1)$. Since all six of these lattice multipolygons have a common lattice point $(1,1)$, one can produce a lattice multipolygon by joining as many of them as we want at the common point, and since the triples behave additively with respect to the join operation, this together with (4.1) shows the existence of a lattice multipolygon with the desired (a,b,c). $\square$

In the rest of the article, we shall consider several natural subfamilies of lattice multipolygons and discuss the characterization of their triples. We note that if $(a,b,c) = (A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ for some lattice multipolygon $\mathcal{P}$, then (a,b,c) must be in the set $\mathcal{A}$.

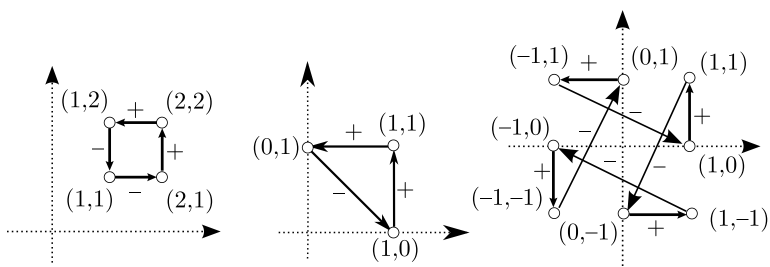

Figure 3. Lattice multipolygons $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ from the left.

4.1. Lattice polygons

One of the most natural subfamilies of lattice multipolygons is the family of convex lattice polygons. Their triples were essentially characterized by Scott [13] as follows.

THEOREM 4.2 ([13])

A triple $(a,b,c) \in \mathcal{A}$ is equal to $(A(P), \frac{1}{2}B(P), C(P))$ of a convex lattice polygon P if and only if $c = 1$ and (a,b) satisfies one of the following:

(1) $a+1 = b \geq \frac{3}{2}$;
(2) $\frac{a}{2}+2 \geq b \geq \frac{3}{2}$;
(3) $(a,b) = (\frac{9}{2}, \frac{9}{2})$.

If we do not require convexity, then the characterization becomes simpler than Theorem 4.2.

PROPOSITION 4.3

A triple $(a,b,c) \in \mathcal{A}$ is equal to $(A(P), \frac{1}{2}B(P), C(P))$ of a (not necessarily convex) lattice polygon P if and only if $c = 1$ and $a+1 \geq b \geq \frac{3}{2}$.

Proof

If P is a lattice polygon, then we have

$$C(P) = 1, \qquad B(P) \geq 3, \qquad A(P) - \frac{1}{2}B(P) + 1 = \sharp P^{\circ} \geq 0,$$

and this implies the "only if" part. On the other hand, let $(a,b,1) \in \mathcal{A}$ with $a+1 \geq b \geq \frac{3}{2}$. Thanks to Theorem 4.2, we may assume that $b > \frac{a}{2}+2$; that is, $4b-2a-6 > 2$. Let P be the lattice polygon shown in Figure 4. Then, one has

$$A(P) = 2(a-b+2) + \frac{1}{2}(4b-2a-8) = a$$

and

$$B(P) = (a-b+2) + 2 + (a-b+1) + 1 + 4b - 2a - 6 = 2b.$$

This shows that $(A(P), \frac{1}{2}B(P), C(P)) = (a,b,c)$, as desired. ☐

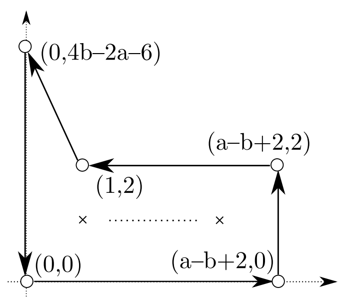

Figure 4. Lattice polygon P with $(A(P), \frac{1}{2}B(P), C(P)) = (a,b,c)$.

4.2. Unimodular lattice multipolygons

We say that a lattice multipolygon $\mathcal{P}=(v_1,\dots,v_d)$ is *unimodular* if the sequence $(v_1,\dots,v_d)$ is unimodular and the sign assignment ϵ is defined by $\epsilon(v_iv_{i+1})=\det(v_i,v_{i+1})$ for $i=1,\dots,d$, where $v_{d+1}=v_1$. When a unimodular lattice multipolygon $\mathcal{P}$ arises from a convex lattice polygon, $\mathcal{P}$ is essentially the same as a so-called *reflexive polytope* of dimension 2, which is completely classified (16 polygons up to equivalence; see, e.g., [11, Figure 2]) and the triples $(A(P),\frac{1}{2}B(P), C(P))$ of reflexive polytopes P are characterized by the condition that $c=1$ and $a=b\in\{\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},4,\frac{9}{2}\}$.

We can characterize $(A(P),\frac{1}{2}B(P),C(P))$ of unimodular lattice multipolygons P as follows.

THEOREM 4.4

A triple $(a,b,c)\in\mathcal{A}$ is equal to $(A(\mathcal{P}),\frac{1}{2}B(\mathcal{P}),C(\mathcal{P}))$ of a unimodular lattice multipolygon $\mathcal{P}$ if and only if $a=b$.

Proof

If $\mathcal{P}$ is a unimodular lattice multipolygon arising from a unimodular sequence $v_1,\dots,v_d$, then one sees that

$$A(\mathcal{P})=\frac{1}{2}\sum_{i=1}^{d}\det(v_i,v_{i+1}),$$

$$B(\mathcal{P})=\sum_{i=1}^{d}\det(v_i,v_{i+1})|v_iv_{i+1}|=\sum_{i=1}^{d}\det(v_i,v_{i+1}),$$

and this implies the "only if" part. Conversely, if $(a,b,c)\in\mathcal{A}$ satisfies $a=b$, then one has an expression

$$(a,b,c)=a'\Big(\frac{1}{2},\frac{1}{2},0\Big)+c'(0,0,-1)$$

with integers a',c' because $a'=2a$ and $c'=-c$. We note that the lattice multipolygons $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_2'$, and $\mathcal{P}_3'$ in the proof of Theorem 4.1 are unimodular lattice multipolygons. Therefore, joining as many of them as we want at the common point $(1,1)$, we can find a unimodular lattice multipolygon $(A(P),\frac{1}{2}B(P), C(P))=(a,b,c)$, as required. □

EXAMPLE 4.5

The $\mathcal{P}$ and $\mathcal{Q}$ in Example 1.1 are unimodular lattice multipolygons, and we have

$$\Big(A(\mathcal{P}),\frac{1}{2}B(\mathcal{P}),C(\mathcal{P})\Big)=\Big(\frac{3}{2},\frac{3}{2},1\Big)\text{ and }\Big(A(\mathcal{Q}),\frac{1}{2}B(\mathcal{Q}),C(\mathcal{Q})\Big)=(3,3,2).$$

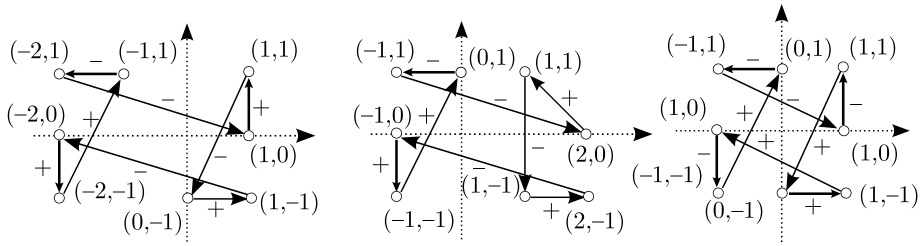

Figure 5. Lattice multipolygons $\mathcal{P}_4$, $\mathcal{P}_5$, and $\mathcal{P}_6$ from the left.

4.3. Some other subfamilies of lattice multipolygons

EXAMPLE 4.6 (LEFT-TURNING (RIGHT-TURNING) LATTICE MULTIPOLYGONS)
We say that a lattice multipolygon $\mathcal{P}$ is *left-turning* (resp., *right-turning*) if $\det(v-u, w-u)$ is always positive (resp., negative) for three consecutive points u, v, w in $\mathcal{P}$ arranged in this order and not lying on the same line. In other words, w lies on the left-hand side (resp., right-hand side) with respect to the direction from u to v. For example, $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ in Figure 3 and $\mathcal{Q}$ in Example 1.1(b) are all left-turning.

Somewhat surprisingly, the left-turning (or right-turning) condition does not give any restriction on the triple $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$; that is, every $(a,b,c) \in \mathcal{A}$ can be equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of a left-turning (or right-turning) lattice multipolygon $\mathcal{P}$. A proof is given by using the lattice multipolygons $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ shown in Figure 3 together with $\mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6$ shown in Figure 5. We remark that the signs of $\mathcal{P}_4$, $\mathcal{P}_5$, and $\mathcal{P}_6$ do not always coincide with the sign of $\det(v_i, v_{i+1})$.

EXAMPLE 4.7 (LEFT-TURNING LATTICE MULTIPOLYGONS WITH ALL $+$ SIGNS)
We consider left-turning lattice multipolygons $\mathcal{P}$ and impose one more restriction that the signs on the sides of $\mathcal{P}$ are all $+$. In this case, some interesting phenomena happen. For example, a simple observation shows that

$$B(\mathcal{P}) \geq 2C(\mathcal{P}) + 1 \qquad \text{and} \qquad C(\mathcal{P}) \geq 1. \tag{4.2}$$

We note that $C(\mathcal{P}) = 1$ if and only if $\mathcal{P}$ arises from a convex lattice polygon, and those $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ are characterized by Theorem 4.2. Therefore, it suffices to treat the case where $C(\mathcal{P}) \geq 2$, and we can see that a triple $(a,b,c) \in \mathcal{A}$ is equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of a left-turning lattice multipolygon $\mathcal{P}$ with all $+$ signs if

$$b \geq c + 1 \qquad \text{and} \qquad c \geq 2.$$

This condition is equivalent to $B(\mathcal{P}) \geq 2C(\mathcal{P}) + 2$ for a lattice multipolygon. On the other hand, we have $B(\mathcal{P}) \geq 2C(\mathcal{P}) + 1$ for a left-turning lattice multipolygon $\mathcal{P}$ with all $+$ signs by (4.2). Therefore, the case where $B(\mathcal{P}) = 2C(\mathcal{P}) + 1$ is not covered above, and this extreme case is exceptional. In fact, one can observe that if $\mathcal{P}$ is a left-turning lattice multipolygon with all $+$ signs and $B(\mathcal{P}) = 2C(\mathcal{P}) + 1$, then $\sharp\mathcal{P}^\circ \geq 0$; that is, $A(\mathcal{P}) \geq \frac{1}{2}$.

EXAMPLE 4.8 (LATTICE MULTIPOLYGONS WITH ALL $+$ SIGNS)

Finally, we consider lattice multipolygons $\mathcal{P}$ with all $+$ signs; namely, we do not assume that $\mathcal{P}$ is either left-turning or right-turning. However, this case is similar to the previous one (left-turning lattice multipolygons with all $+$ signs). For example, when $C(\mathcal{P}) \neq 0$, we still have $B(\mathcal{P}) \geq 2|C(\mathcal{P})| + 1$. Thus, we also have that a triple $(a, b, c) \in \mathcal{A}$ is equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of a lattice multipolygon $\mathcal{P}$ with all $+$ signs if

$$b \geq |c| + 1 \qquad \text{and} \qquad |c| \geq 2.$$

Moreover, when $B(\mathcal{P}) = 2|C(\mathcal{P})| + 1$, $\mathcal{P}$ must be left-turning or right-turning depending on whether $C(\mathcal{P}) > 0$ or $C(\mathcal{P}) < 0$. Hence, we can say that, when we discuss $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of lattice multipolygons $\mathcal{P}$ with all $+$ signs, it suffices to consider those of left-turning or right-turning ones when $C(\mathcal{P}) \notin \{-1, 0, 1\}$.

On the other hand, on the remaining exceptional cases where $C(\mathcal{P}) = 0$ or $C(\mathcal{P}) = \pm 1$, we can characterize the triples completely as follows. Let $(a, b, c) \in \mathcal{A}$.

(a) When $c = 0$, (a, b, c) is equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of a lattice multipolygon $\mathcal{P}$ with all $+$ signs if and only if $b \geq 2$ (see Figure 6).

(b) When $c = 1$, (a, b, c) is equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of a lattice multipolygon $\mathcal{P}$ with all $+$ signs if and only if either $b \geq \frac{5}{2}$ or $\frac{3}{2} \leq b \leq 2$ and $a - b + 1 \geq 0$ (see Figure 7 and Proposition 4.3).

(c) When $c = -1$, (a, b, c) is equal to $(A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ of a lattice multipolygon $\mathcal{P}$ with all $+$ signs if and only if either $b \geq \frac{5}{2}$ or $\frac{3}{2} \leq b \leq 2$ and $a + b - 1 \leq 0$. One can simply reverse the order of the vertices and flip the sign of a of the example in Figure 7.

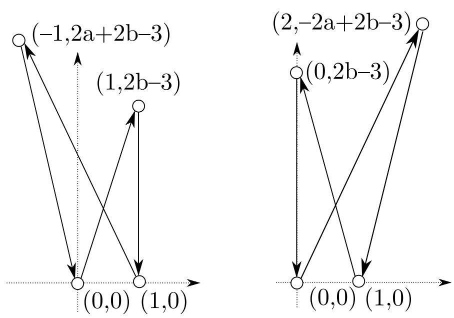

Figure 6. Lattice multipolygons with all $+$ signs whose triples equal $(a, b, 0)$ when $a + b \geq 2$ and $a + b \leq 2$, respectively.

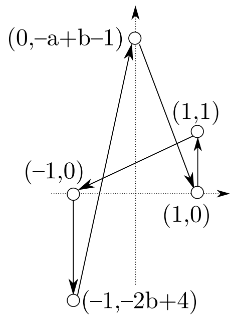

Figure 7. Lattice multipolygon with all + signs whose triple equals $(a,b,1)$ when $b \geq \frac{5}{2}$.

Appendix: Proof of Theorem 1.2 using toric topology

Theorem 1.2 was originally proved using toric topology. In fact, it is proved in [8, Section 5] when $\epsilon_i = 1$ for every i, and the argument there works in our general setting with a little modification, which we shall explain.

We identify $\mathbb{Z}^2$ with $H_2(BT)$, where $T = (S^1)^2$ and BT is the classifying space of T. We may think of BT as $(\mathbb{C}P^\infty)^2$. For each i $(i = 1, \dots, d)$, we form a cone $\angle v_i v_{i+1}$ in $\mathbb{R}^2$ spanned by v_i and v_{i+1} and attach the sign ϵ_i to the cone. The collection of cones $\angle v_i v_{i+1}$ with signs ϵ_i attached form a multifan a_j, and the same construction as in [8, Section 5] produces a real 4-dimensional closed connected smooth manifold M with an action of T satisfying the following conditions:

(1) $H^{\mathrm{odd}}(M) = 0$.

(2) M admits a unitary (or weakly complex) structure preserved under the T-action, and the multifan associated to M with this unitary structure is the given a_j.

(3) Let M_i $(i = 1, \dots, d)$ be the characteristic submanifold of M corresponding to the edge vector v_i; that is, M_i is a real codimension 2 submanifold of M fixed pointwise under the circle subgroup determined by the v_i. Then M_i does not intersect with M_j unless $j = i-1, i, i+1$ and the intersection numbers of M_i with M_{i-1} and M_{i+1} are ϵ_{i-1} and ϵ_i, respectively.

Choose an arbitrary element $v \in \mathbb{R}^2$ not contained in any 1-dimensional cone in the multifan a_j. Then [8, Theorem 4.2] says that the Todd genus $T[M]$ of M is given by

$$T[M] = \sum_i \epsilon_i, \tag{A.1}$$

where the sum above runs over all i's such that the cone $\angle v_i v_{i+1}$ contains the vector v. Clearly, the right-hand side of (A.1) agrees with the rotation number of the sequence $v_1, \dots, v_d$ around the origin. In the rest of the section, we compute the Todd genus $T[M]$.

Let $ET \to BT$ be the universal principal T-bundle, and let M_T be the quotient of $ET \times M$ by the diagonal T-action. The space M_T is called the *Borel construction of* M, and the equivariant cohomology $H^q_T(M)$ of the T-space M is defined to be $H^q(M_T)$. The first projection from $ET \times M$ onto ET induces a fibration

$$\pi\colon M_T \to ET/T = BT$$

with fiber M. The inclusion map ι of the fiber M to M_T induces a surjective homomorphism $\iota^*\colon H^q_T(M) \to H^q(M)$.

Let $\xi_i \in H^2_T(M)$ be the Poincaré dual to the cycle M_i in the equivariant cohomology. Then ξ_i restricts to the ordinary Poincaré dual $x_i \in H^2(M)$ to the cycle M_i through ι^*. By [8, Lemma 1.5], we have

$$\pi^*(u) = \sum_{j=1}^{d} \langle u, v_j \rangle \xi_j \quad \text{for any } u \in H^2(BT), \tag{A.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between cohomology and homology. Multiplying both sides of (A.2) by ξ_i and restricting the resulting identity to the ordinary cohomology by ι^*, we obtain

$$0 = \langle u, v_{i-1} \rangle x_{i-1} x_i + \langle u, v_i \rangle x_i^2 + \langle u, v_{i+1} \rangle x_{i+1} x_i \quad \text{for all } u \in H^2(BT), \tag{A.3}$$

because M_i does not intersect with M_j unless $j = i-1, i, i+1$, where $x_{d+1} = x_1$. We evaluate both sides of (A.3) on the fundamental class $[M]$ of M. Since the intersection numbers of M_i with M_{i-1} and M_{i+1} are, respectively, ϵ_{i-1} and ϵ_i as mentioned above, the identity (A.3) reduces to

$$0 = \langle u, v_{i-1} \rangle \epsilon_{i-1} + \langle u, v_i \rangle \bigl\langle x_i^2, [M] \bigr\rangle + \langle u, v_{i+1} \rangle \epsilon_i \quad \text{for all } u \in H^2(BT) \tag{A.4}$$

and further reduces to

$$0 = \epsilon_{i-1} v_{i-1} + \bigl\langle x_i^2, [M] \bigr\rangle v_i + \epsilon_i v_{i+1}, \tag{A.5}$$

because (A.4) holds for any $u \in H^2(BT)$. Comparing (A.5) with (1.2), we conclude that $\langle x_i^2, [M] \rangle = a_i$. Summing up the above argument, we have

$$\bigl\langle x_i x_j, [M] \bigr\rangle = \begin{cases} \epsilon_{i-1} & \text{if } j = i-1, \\ a_i & \text{if } j = i, \\ \epsilon_i & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{A.6}$$

By [8, Theorem 3.1] the total Chern class $c(M)$ of M with the unitary structure is given by $\prod_{i=1}^{d}(1 + x_i)$. Therefore,

$$c_1(M) = \sum_{i=1}^{d} x_i, \qquad c_2(M) = \sum_{i<j} x_i x_j,$$

and hence,

$$T[M] = \frac{1}{12}\langle c_1(M)^2 + c_2(M), [M]\rangle$$
$$= \frac{1}{12}\Big\langle \Big(\sum_{i=1}^{d} x_i\Big)^2 + \sum_{i<j} x_i x_j, [M]\Big\rangle$$
$$= \frac{1}{12}\Big(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d} \epsilon_i\Big),$$

where the first identity is known as Noether's formula when M is an algebraic surface and is known to hold even for unitary manifolds, and we used (A.6) at the last identity. This proves the theorem because $T[M]$ agrees with the desired rotation number as remarked in (A.1).

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