Holomorphic endomorphisms of $\mathbb{P}^3(\mathbb{C})$ related to a Lie algebra of type A_3 and catastrophe theory

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Abstract The typical chaotic maps $f(x)=4x(1-x)$ and $g(z)=z^2-2$ are well known. Veselov generalized these maps. We consider a class of maps $P^d_{A_3}$ of those generalized maps, view them as holomorphic endomorphisms of $\mathbb{P}^3(\mathbb{C})$, and make use of methods of complex dynamics in higher dimension developed by Bedford, Fornaess, Jonsson, and Sibony. We determine Julia sets J_1, J_2, J_3, J_Π and the global forms of external rays. Then we have a foliation of the Julia set J_2 formed by stable disks that are composed of external rays.

We also show some relations between those maps and catastrophe theory. The set of the critical values of each map restricted to a real three-dimensional subspace decomposes into a tangent developable of an astroid in space and two real curves. They coincide with a cross section of the set obtained by Poston and Stewart where binary quartic forms are degenerate. The tangent developable encloses the Julia set J_3 and joins to a Möbius strip, which is the Julia set J_Π in the plane at infinity in $\mathbb{P}^3(\mathbb{C})$. Rulings of the Möbius strip correspond to rulings of the surface of J_3 by external rays.

1. Introduction

The typical chaotic map $f(x)=4x(1-x)$ is well known (see, e.g., [21]). Its complex version is a Chebyshev map $g(z)=z^2-2$. It is also a chaotic map. Generalized Chebyshev functions and maps in several variables were studied by several researchers (see Koornwinder [14], Lidl [15], Beerends [2], Veselov [22], Hoffman and Withers [11], and Uchimura [19]).

A polynomial endomorphism $P^d_{A_3}(z_1,z_2,z_3)$ of degree d on $\mathbb{C}^3$ is defined by the following. We consider the jth elementary symmetric function in t_1,t_2,t_3,t_4 with $t_4=1/(t_1t_2t_3)$ for $j=1,2,3$. Let

$$z_1 = t_1+t_2+t_3+\frac{1}{t_1t_2t_3},$$

$$z_2 = t_1t_2+t_1t_3+t_2t_3+\frac{1}{t_1t_2}+\frac{1}{t_1t_3}+\frac{1}{t_2t_3}, \tag{1.1}$$

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$$z_3 = \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 t_2 t_3 \quad \big(t_j \in \mathbb{C} \setminus \{0\}\big).$$

Set

$$\Phi_1(t_1, t_2, t_3) = (z_1, z_2, z_3).$$

Then $P^d_{A_3}$ satisfies the following commutative diagram:

$$\begin{array}{ccc} (t_1, t_2, t_3) & \longrightarrow & (t_1^d, t_2^d, t_3^d) \\ {\scriptstyle \Phi_1}\downarrow & & {\scriptstyle \Phi_1}\downarrow \\ (z_1, z_2, z_3) & \longrightarrow & P^d_{A_3}(z_1, z_2, z_3) \end{array} \tag{1.2}$$

Clearly, Φ_1 is a branched covering map. We show two examples:

$$\begin{aligned} P^2_{A_3}(z_1, z_2, z_3) &= (z_1^2 - 2z_2, z_2^2 - 2z_1 z_3 + 2, z_3^2 - 2z_2), \\ P^3_{A_3}(z_1, z_2, z_3) &= (z_1^3 - 3z_1 z_2 + 3z_3, z_2^3 - 3z_1 z_2 z_3 + 3z_3^2 + 3z_1^2 - 3z_2, \\ &\qquad z_3^3 - 3z_3 z_2 + 3z_1). \end{aligned}$$

These are based on the definition given by Veselov [22]. Veselov [22] defined generalized Chebyshev maps as follows. Let G be a simple complex Lie algebra of rank n, H be its Cartan subalgebra, H^* be its dual space, $\mathcal{L}$ be a lattice of weights in H^* generated by the fundamental weights $\varpi_1, \dots, \varpi_n$, and L be the dual lattice in H. One defines

$$\phi_G : H/L \to \mathbb{C}^n, \quad \phi_G = (\varphi_1, \dots, \varphi_n), \quad \varphi_k = \sum_{w \in W} \exp\big[2\pi i w(\varpi_k)\big],$$

where W is the Weyl group acting on the space H^*.

To each G of rank n is associated an infinite series of integrable polynomial mappings P^d_G from $\mathbb{C}^n$ to $\mathbb{C}^n, d = 2, 3, \dots$, determined by the condition

$$\phi_G(dx) = P^d_G\big(\phi_G(x)\big).$$

For $n = 1$ there is a unique simple algebra A_1. Here $\phi_{A_1} = 2\cos(2\pi x)$ and the $P^d_{A_1}$ are, within a linear substitution, Chebyshev polynomials of a single variable. Here A_n is the Lie algebra of $\mathrm{SL}(n+1, \mathbb{C})$.

The dynamics of $P^d_{A_2}$ was studied in [20]. In this article, we consider maps $P^d_{A_3}$, view them as holomorphic endomorphisms of $\mathbb{P}^3(\mathbb{C})$, and make use of methods of complex dynamics in higher dimension developed by Fornaess and Sibony [9] and Bedford and Jonsson [1].

In this article we will provide a typical example of complex dynamics in higher dimension. In this higher-dimensional dynamics, classical geometrical figures, for example, a Möbius strip and a special ruled surface (tangent developable), which is called the Holy Grail in catastrophe theory, appear with their chaotic dynamical structures.

The main tools used in this article are Julia sets and external rays. We present some background on Julia sets. The main references are [1], [9], and [18]. Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a regular polynomial endomorphism of degree d (see the paragraph before Proposition 2.1). Set

$$K(f) := \{z \in \mathbb{C}^k : \{f^n(z)\} \text{ is bounded}\}.$$

We define the Green function of f as

$$G(z) := \lim_{n\to\infty} d^{-n} \log^+ \|f^n(z)\|, \quad z \in \mathbb{C}^k.$$

The Green current $T_{\mathbb{C}^k} := \frac{1}{2\pi} dd^c G$ is a positive closed $(1,1)$-current. A regular polynomial endomorphism f extends to a holomorphic endomorphism of $\mathbb{P}^k$, still denoted by f.

The Green current $T_{\mathbb{C}^k}$ has an extension as a positive closed current to $\mathbb{P}^k$ in the following manner. Every holomorphic endomorphism f of $\mathbb{P}^k$ has a lift $F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$. The projection $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ semiconjugates F to f: $\pi \circ F = f \circ \pi$. The Green function G_F of F is defined by

$$G_F := \lim_{n\to\infty} d^{-n} \log\|F^n(z)\|.$$

The Green current $T = T_{\mathbb{P}^k}$ of f is defined by

$$\pi^* T = \frac{1}{2\pi} dd^c G_F.$$

We can define the currents $T^l := T \wedge \cdots \wedge T$ (l terms). The lth Julia set $J_l(f)$ is the support of T^l. The Green measure μ_f of f is defined by

$$\mu_f := (T)^k.$$

The measure μ_f is a probability measure that is invariant under f and maximizes entropy.

In our case we consider four kinds of Julia sets, $J_1(f)$, $J_2(f)$, $J_3(f)$, and $J_2(f_\Pi)$, where f_Π denotes the restriction of f to the hyperplane Π at infinity. We will determine these four kinds of Julia sets in Theorems 2.7, 3.2, and 4.2.

We will determine the Julia set $J_3(f)$ and the maximal entropy measure μ_f in Theorem 2.7. The Julia set $J_3(f)$ coincides with the set $K(f)$. To obtain Theorem 2.7 we use Briend and Duval's theorem in complex dynamics and some results of the theory of Lie groups.

We will determine the Julia set $J_2(f_\Pi)$ and the maximal entropy measure μ_{f_Π} in Theorem 3.2. The Julia set $J_2(f_\Pi)$ is a Möbius strip $\mathcal{M}$. On the Möbius strip $\mathcal{M}$ we give a dynamical measure. The map f_Π restricted to $\mathbb{C}^2$ is a polynomial skew product map of $\mathbb{C}^2$. The maximal entropy measure for f_Π restricted to the base curve which is a unit circle is $d\theta/2\pi$, and that restricted to each ruling is the invariant measure of Chebyshev maps in one variable.

Next we provide some background on external rays. External rays play an important role in the theory of dynamics in one complex variable. Let $f : \mathbb{P} \to \mathbb{P}$ be a monic polynomial map of degree $d \geq 2$. Suppose that the set $K = K(f)$ is connected. Then the complement $\mathbb{C} \setminus K$ is conformally isomorphic to the complement $\mathbb{C} \setminus \bar{\mathbb{D}}$ under the Böttcher map ϕ. The external rays for K are defined by

$$\{z : \arg(\phi(z)) = \text{const}\}.$$

The image of an external ray under f is also another external ray.

Bedford and Jonsson [1] defined external rays for holomorphic endomorphisms of $\mathbb{P}^k$. We will determine the global forms of external rays of our maps $f = P^d_{A_3}$. The image of each external ray under the extended map f on $\mathbb{P}^3$ is also an external ray. We will show in Theorem 4.2 that the Julia set $J_2(f)$ is a foliated space and leaves of the space are stable disks composed of external rays. The image of a stable disk under the map f is another stable disk.

Next we consider the dynamics of $P^d_{A_3}$ restricted to a real three-dimensional subspace. The map $P^d_{A_3} : \mathbb{C}^3 \to \mathbb{C}^3$ admits an invariant space

$$R_3 := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 = \bar{z}_3 \text{ and } z_2 \text{ is real}\}.$$

We consider the dynamics of $P^d_{A_3}$ restricted to R_3. The set $J_3(f) = K(f)$ lies in the space R_3. Sometimes we may regard R_3 as $\mathbb{R}^3$. Then $J_3(f)$ is isomorphic to a closed domain in $\mathbb{R}^3$ bounded by the ruled surface $\mathcal{A}$ whose base curve is an astroid in space (see Proposition 2.4). In particular, $\mathcal{A}$ is a part of the tangent developable of an astroid in space, and so, we call it an *astroidalhedron*. A ruled surface is called a *tangent developable* if its rulings are tangent lines to its base curve. The ruled surface $\mathcal{A}$ has a relationship to the root system of a Lie algebra of type A_3 and a $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron.

The external rays included in R_3 are half-lines that connect the ruled surface $\mathcal{A}$ and the Möbius strip $\mathcal{M} = J_2(f_\Pi)$. By this fact, we will show that rulings of $\mathcal{M}$ correspond to rulings of $\mathcal{A}$ by external rays in Proposition 4.9.

Next we will show some relations between those maps and catastrophe theory. The dynamics of the maps $P^d_{A_2}$ on $\mathbb{C}^2$ was studied in [20]. The set of critical values of $P^d_{A_2}$ restricted to $\{z_1 = \bar{z}_2\}$ is proved to be a deltoid. The deltoid coincides with a cross section of the bifurcation set (caustics) of the elliptic umbilic catastrophe map (D_4^-). In [20], it was shown that the external rays and their extensions constitute a family of lines whose envelope is the deltoid. Hence, these lines are real "rays" of caustics.

In addition to the caustics, the deltoid has relations with binary cubic forms

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3, \quad a, b, c, d \in \mathbb{R}.$$

Let V be the set where the discriminant of $f(x, y)$ vanishes. To understand the geometry of the set V, Zeeman [23] pursued a different tack. Zeeman [23] showed that $V \cap S^3$ is mapped diffeomorphically to the "umbilic bracelet." It has a deltoid section that rotates 1/3 twist going once round the bracelet.

We return to the study of the maps $P^d_{A_3}$. In this case we will show that the set of critical values of $P^d_{A_3}$ restricted to R_3 has relations with binary quartic forms.

Poston and Stewart [16], [17] studied quartic forms in two variables,

$$f(x, y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4, \quad a, b, c, d, e \in \mathbb{R}.$$

Let $\triangle$ be the discriminant of $f(x,y)$, and let $\mathscr{D} \subset \mathbb{R}^5$ be the algebraic set given by $\triangle = 0$. The set $\mathscr{W} = \mathscr{D} \cap S^5$ is decomposed into $\mathscr{W}_1$ and $\mathscr{W}_\infty$. Then $\mathscr{W}_1$ is diffeomorphic to $\mathscr{U}$. They considered a cross section $\mathscr{Q}$ of $\mathscr{U}$. The shape for $\mathscr{Q}$ is called the Holy Grail in catastrophe theory. We will show in Proposition 5.8 that the set $\mathscr{Q}$ coincides with the set of critical values of $P^d_{A_3}$ restricted to R_3 by a coordinate transformation. We will show that the set decomposes into a tangent developable $\mathcal{T}$ of an astroid in space and two real curves in Proposition 5.5. The astroidalhedron $\mathcal{A}$ is a part of $\mathcal{T}$.

In Proposition 5.6, we will show that the rims of $\mathcal{T}$ join simply to the boundary of $\mathcal{M}$ in the hyperplane Π at infinity in $\mathbb{P}^3(\mathbb{C})$. Poston and Stewart [16], [17] dealt with the same situation by analyzing $\mathscr{W}_\infty$ in $\mathbb{R}^5$. It is complicated. But we consider the situation in $\mathbb{P}^3(\mathbb{C})$, and so, our description is simpler. We will show that any ruling of $\mathcal{T}$, that is, any tangent line to the astroid, consists of two external rays and their extension and that any external ray which is not a ruling connects the astroidalhedron $\mathcal{A}$ and Möbius strip $\mathcal{M}$.

In this article, we will show not only static aspects of catastrophe theory but also dynamical aspects of catastrophe theory. We know that the sets of critical values of $P^d_{A_2}$ and $P^d_{A_3}$ restricted to the real subspaces have relations with binary cubic forms and quartic forms, respectively. These relations will be generalized for general maps $P^d_{A_n}$.

2. The sets $K(P^d_{A_3})$ and $J_3(P^d_{A_3})$

In this section we determine the set $K(P^d_{A_3})$ of bounded orbits and the third Julia set $J_3(P^d_{A_3})$. We will show that the surface of $K(P^d_{A_3})$ is a part of the tangent developable of an astroid in space.

We consider the map $P^d_{A_3}$ defined by (1.1) and (1.2). Let

$$P^d_{A_3} = \big(g_1^{(d)}(z_1,z_2,z_3), g_2^{(d)}(z_1,z_2,z_3), g_3^{(d)}(z_1,z_2,z_3)\big).$$

Then, from [15, pp. 183–184] we know that the set of polynomials $\{g_j^{(d)}(z_1,z_2,z_3)\}$ satisfies the following recurrence formulas:

$$\begin{aligned} &g_1^{(k)} = z_1 g_1^{(k-1)} - z_2 g_1^{(k-2)} + z_3 g_1^{(k-3)} - g_1^{(k-4)}, \\ &g_1^{(j)} = \sum_{r=1}^{j} (-1)^{r-1} z_r g_1^{(j-r)} + (-1)^j (4-j) z_j \quad (j=0,1,2,3), z_0 = 1, \end{aligned} \tag{2.1}$$

$$g_3^{(k)}(z_1,z_2,z_3) = g_1^{(k)}(z_3,z_2,z_1), \tag{2.2}$$

$$\begin{aligned} &g_2^{(k+6)} - z_2 g_2^{(k+5)} + (z_1 z_3 - 1) g_2^{(k+4)} - (z_1^2 - 2z_2 + z_3^2) g_2^{(k+3)} \\ &\quad + (z_1 z_3 - 1) g_2^{(k+2)} - z_2 g_2^{(k+1)} + g_2^{(k)} = 0. \end{aligned} \tag{2.3}$$

Note that the formula in [15, p. 184] corresponding to (2.3) is incorrect. The correct coefficient of $g_2^{(k+3)}$ is equal to $-(z_1^2 - 2z_2 + z_3^2)$. And the correct initial values are given by

$$g_2^{(-2)} = z_2^2 - 2z_1z_2 + 2, \qquad g_2^{(-1)} = z_2, \qquad g_2^{(0)} = 6, \qquad g_2^{(1)} = z_2,$$
$$g_2^{(2)} = g_2^{(-2)}, \qquad g_2^{(3)} = z_2^3 - 3z_1z_2z_3 + 3z_3^2 + 3z_1^2 - 3z_2.$$

A polynomial endomorphism f of degree d is called *regular* if the homogeneous part f_h of degree d satisfies $f_h^{-1}(0) = \{0\}$.

PROPOSITION 2.1
We have that $P_{A_3}^d(z_1, z_2, z_3)$ is a regular polynomial endomorphism.

Proof
Let $f := P_{A_3}^d(z_1, z_2, z_3)$. From (2.1), (2.2), and (2.3), we have $f_h = (z_1^d, h_2^{(d)}, z_3^d)$, where $h_2^{(d)}(z_1, z_2, z_3)$ is a polynomial satisfying the recurrence formulas

$$h_2^{(d+2)} = z_2 h_2^{(d+1)} - z_1 z_3 h_2^{(d)}, \qquad h_2^{(1)} = z_2, \qquad h_2^{(2)} = z_2^2 - 2z_1z_3. \tag{2.4}$$

Then we deduce $f_h^{-1}(0) = \{0\}$. □

Next we study the set

$$K(P_{A_3}^d) = \big\{ z \in \mathbb{C}^3 : \text{the orbit } \big\{ (P_{A_3}^d)^n(z) \big\} \text{ is bounded} \big\}.$$

Then $K(P_{A_3}^d)$ is described in the following form.

PROPOSITION 2.2 ([22])
We have that $K(P_{A_3}^d) = \{\Phi_1(t_1, t_2, t_3) : |t_1| = |t_2| = |t_3| = 1\}$.

The set $K(P_{A_3}^d(z_1, z_2, z_3))$ is given by (see [8])

$$\begin{cases} z_1 = e^{i\alpha} + e^{i\beta} + e^{i\gamma} + e^{i(-\alpha-\beta-\gamma)}, \\ z_2 = e^{i(\alpha+\beta)} + e^{i(\alpha+\gamma)} + e^{i(\gamma+\beta)} + e^{-i(\beta+\gamma)} + e^{-i(\gamma+\alpha)} + e^{-i(\alpha+\beta)}, \\ z_3 = e^{-i\alpha} + e^{-i\beta} + e^{-i\gamma} + e^{i(\alpha+\beta+\gamma)}, \end{cases} \tag{2.5}$$
$$-\alpha - \beta - \gamma \le \alpha \le \beta \le \gamma \le 2\pi - \alpha - \beta - \gamma.$$

We call $R' := \{(\alpha, \beta, \gamma) : -\alpha - \beta - \gamma \le \alpha \le \beta \le \gamma \le 2\pi - \alpha - \beta - \gamma\}$ the *natural domain* (see Figure 1).

We denote the real three-dimensional subspace $\{(z_1, z_2, \bar{z}_1) : z_1 \in \mathbb{C}, z_2 \in \mathbb{R}\}$ by R_3. Then $K(P_{A_3}^d) \subset R_3$, and R_3 is invariant under the maps $P_{A_3}^d$. Sometimes we regard R_3 as $\mathbb{R}^3$.

To facilitate computations we transform the Euclidean coordinates (α, β, γ) into new coordinates (s_1, s_2, s_3) concerning the root system of type A_3. A base $\{\alpha_j\}$ for the root system and fundamental weights ϖ_j of type A_3 are given by

$$\alpha_1 = \Big(-\frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}\Big), \qquad \alpha_2 = (\sqrt{2}, 0, 0), \qquad \alpha_3 = \Big(-\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}\Big),$$

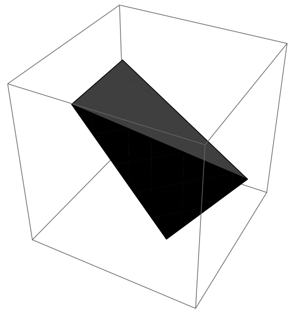

Figure 1. The natural domain R'.

$$\varpi_1 = \Big(0, -\frac{1}{2}, \frac{1}{\sqrt{2}}\Big), \qquad \varpi_2 = \Big(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\Big), \qquad \varpi_3 = \Big(0, \frac{1}{2}, \frac{1}{\sqrt{2}}\Big).$$

One of the alcoves of A_3 is the closed region R bounded by the polyhedron $\sqrt{2}\pi\ (O, \varpi_1, \varpi_2, \varpi_3)$. We call the region R the *fundamental region*. The region R' is transformed to R by a transformation T. The matrix associated with the transformation T from the (α, β, γ) space to the (s_1, s_2, s_3) space is given by

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}. \tag{2.6}$$

The region R is a closed region bounded by a $(\sqrt{3}, \sqrt{3}, 2)$-*tetrahedron*. That is, it has four faces which are congruent with each other and the ratios of whose edge lengths are equal to $\sqrt{3} : \sqrt{3} : 2$. Coxeter [6] proved that there exist only seven types of reflective space-fillers. It is one of them. A convex polyhedron P is called a *reflective space-filler* if its congruent copies tile the 3-space in such a way that

(1) the tilling is face to face,
(2) if the intersection $P_1 \cap P_2$ of two of those copies has a face in common, then P_1 is the mirror image of P_2 in the common face, and
(3) each of the dihedral angles of P is π/k for integer $k \geq 2$.

We consider the tiling of the (s_1, s_2, s_3) space by $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedrons. The region R (see Figure 2) is a closed region bounded by one of these tetrahedrons with vertices

$$O = (0,0,0), \qquad A_1 = (0, -\pi/\sqrt{2}, \pi), \qquad A_2 = (\pi, 0, \pi), \qquad A_3 = (0, \pi/\sqrt{2}, \pi).$$

Let $\mathcal{G}$ be the group of isometrics which is generated by the reflections in the faces of these tetrahedrons.

The reflection in the hyperplane through the origin orthogonal to α_i is given by

$$w_{\alpha_i}(x) = x - \frac{2(x, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \quad (i = 1, 2, 3), x \in \mathbb{R}^3.$$

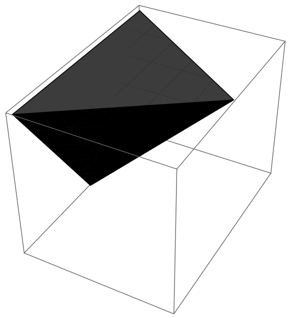

Figure 2. The fundamental region R.

Set $J_i := w_{\alpha_i}$. Then J_i is the reflection in the face $\triangle OA_jA_k$ of the tetrahedron ∂R with $\{i,j,k\} = \{1,2,3\}$. Set $J_0(s_1, s_2, s_3) = (s_1, s_2, 2\pi - s_3)$. Then J_0 is the reflection in the face $\triangle A_1A_2A_3$. It is known, for example, from [3] that the reflections J_0, J_1, J_2, and J_3 generate the group $\mathcal{G}$. Set $X = \{e^{i\alpha}, e^{i\beta}, e^{i\gamma}, e^{-i(\alpha+\beta+\gamma)}\}$. Then by the direct computations using (2.6) we can prove that each J_k acts on the set X as a permutation, for $k = 0,1,2,3$. For any element (s_1, s_2, s_3) in the space, there exists an element J in the group $\mathcal{G}$ such that $J(s_1, s_2, s_3) \in R$.

PROPOSITION 2.3

For $k = 0,1,2,3$, let the images of (s_1, s_2, s_3) and $J_k(s_1, s_2, s_3)$ under the inverse of the transformation T be (α, β, γ) and $(\alpha', \beta', \gamma')$. Then we have

$$\Phi_1(e^{i\alpha}, e^{i\beta}, e^{i\gamma}) = \Phi_1(e^{i\alpha'}, e^{i\beta'}, e^{i\gamma'}).$$

Proof

The terms in z_i $(i = 1,2,3)$ in (2.5) are invariant under any J_k. □

We study the surface of $K(P^d_{A_3})$. We define a coordinate system (p_1, p_2, q) of R_3 by

$$p_1(1,0,0,0,1,0) + p_2(0,1,0,0,0,-1) + q(0,0,1,0,0,0).$$

We consider the map Φ_1 restricted to R' onto $K(f) \subset R_3$. We denote it by φ_1. The mapping $\varphi_1 : R' \to K(f)$ is given by

$$\begin{aligned} p_1 &= \mathrm{Re}(e^{i\alpha} + e^{i\beta} + e^{i\gamma} + e^{i(-\alpha-\beta-\gamma)}), \\ p_2 &= \mathrm{Im}(e^{i\alpha} + e^{i\beta} + e^{i\gamma} + e^{i(-\alpha-\beta-\gamma)}), \\ q &= e^{i(\alpha+\beta)} + e^{i(\alpha+\gamma)} + e^{i(\gamma+\beta)} + e^{-i(\beta+\gamma)} + e^{-i(\gamma+\alpha)} + e^{-i(\alpha+\beta)}. \end{aligned} \tag{2.7}$$

So φ_1 is a diffeomorphism from $\mathrm{int}(R')$ to $\mathrm{int}(K(f))$, and $\partial R'$ is mapped onto $\partial K(f)$ injectively.

PROPOSITION 2.4

The surface of $K(P^d_{A_3})$ is a part of the tangent developable of an astroid in space.

The surface is given by

$$\chi(u,v) = (4\cos^3 u, 4\sin^3 u, 6\cos 2u) + v(\cos u, -\sin u, 2),$$
$$-2 - 2\cos 2u \le v \le 2 - 2\cos 2u.$$

Proof

To get the surface, we substitute an inequality sign for an equality sign in the definition of R'. That is, we set $-\alpha - \beta - \gamma = \alpha$. By (2.7) and the above equality, we have

$$(p_1, p_2, q) = 2(\cos\alpha, \sin\alpha, \cos 2\alpha) + 2\cos(\alpha+\beta)(\cos\alpha, -\sin\alpha, 2)$$
$$(0 \le \alpha < 2\pi, 0 \le \alpha + \beta < \pi). \tag{2.8}$$

From the properties of reflections of R, we see that (2.8) represents the surface of $K(P^d_{A_3})$. It is a ruled surface. Using a striction curve (see [10, Lemma 17.7]), we reparameterize the ruled surface. Set

$$\tilde{\chi}(u,v) = 2(\cos u, \sin u, \cos 2u) + 2v(\cos u, -\sin u, 2).$$

Then from [10, Lemma 17.7], we have a reparameterization

$$\chi(u,v) = (4\cos^3 u, 4\sin^3 u, 6\cos 2u) + v(\cos u, -\sin u, 2)$$
$$(-2 - 2\cos 2u \le v \le 2 - 2\cos 2u).$$

The base curve $\{(4\cos^3 u, 4\sin^3 u, 6\cos 2u) : 0 \le u < 2\pi\}$ is an astroid in space, and $\chi(u,v)$ is a part of the tangent developable of the astroid. $\square$

The astroid consists of edges of the surface. We call the ruled surface an *astroidal-hedron* and denote it by $\mathcal{A}$ (see Figure 3). By [13], we see that those edges except for four vertices of $\mathcal{A}$ are cuspidal edges (see Figure 4).

Now we begin with the study of Julia sets. In Section 1 we define the lth Julia set J_l. In our situation we have three kinds of Julia sets J_1, J_2, and J_3. Clearly, $J_1 \supset J_2 \supset J_3$. We begin with the study of J_3. We will show that $J_3 = K(P^d_{A_3})$. To show this we use Briend and Duval [4, Theorem 2]. It reads as follows. Let P_n denote the set of repelling periodic points of period n. The number of the elements in P_n is d^{3n}. Let $f = P^d_{A_3}$. Set $\mu = (T_f)^3$. Then the sequence of measures $\mu_n := d^{-3n}\sum_{a \in P_n} \delta_a$ converges weakly to μ.

From the above diagram (1.2), we have the following lemma.

LEMMA 2.5

Any periodic point of f in $\mathrm{int}(K(f))$ is repelling.

Next we consider the distribution of repelling periodic points. Using a conjugacy from $K(f)$ to R, we study the distribution of repelling periodic points. We will show that the repelling periodic points are dense and equidistributed in R.

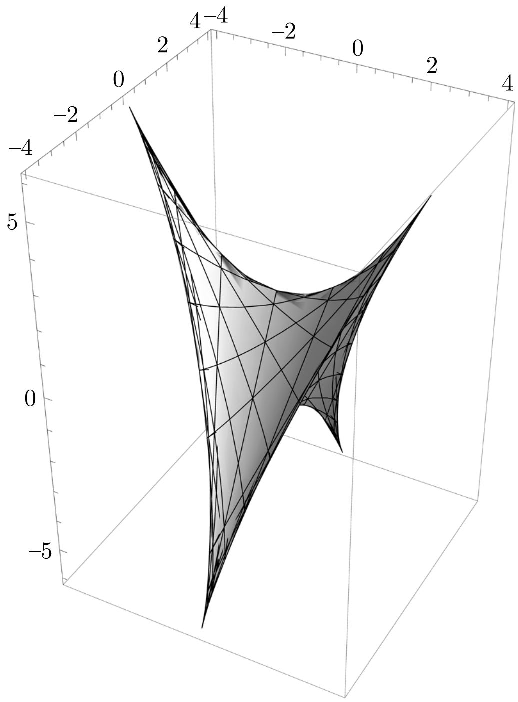

Figure 3. An astroidalhedron.

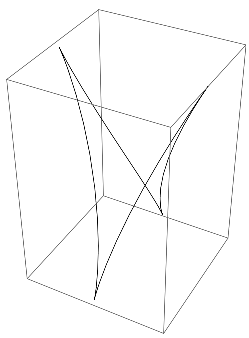

Figure 4. An astroid in space.

Combining the inverse of φ_1 with the coordinate transformation T, we get a continuous map φ from $K(f)$ to R such that φ restricted to $\mathrm{int}(K(f))$ is a diffeomorphism. We set $\rho := \varphi \circ f \circ \varphi^{-1}$. Then $\rho(s_1, s_2, s_3) = d(s_1, s_2, s_3)$.

To study the distribution of periodic points of ρ, we use an argument similar to that used in [20, Proposition 2.2]. We first consider the case $d = 2$. The image of the fundamental region R under ρ and its division into eight $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedrons are depicted in Figure 5.

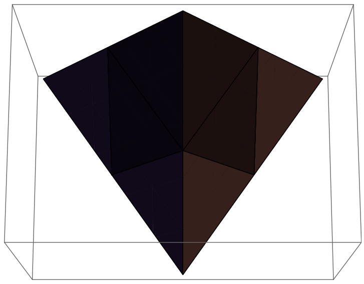

Figure 5. Eight tetrahedrons.

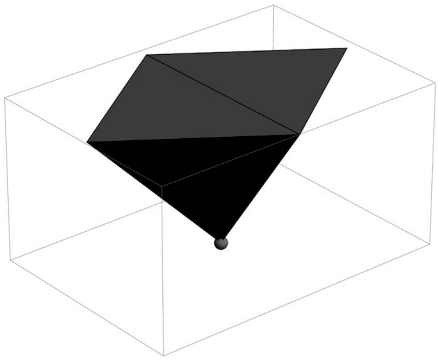

Figure 6. A triangular prism.

For any $d \geq 3$, we combine the three adjacent $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedrons which yield a triangular prism. A small ball denotes the origin (see Figure 6).

The triangular prism plays the same role as the equilateral triangle plays in [20, Proposition 2.2]. Then the image of the fundamental region R under ρ^n consists of d^{3n} regions, each of which is congruent to R. Each region is mapped to R by some sequence of reflections in $\mathcal{G}$.

Conversely, we consider the subdivision of R. We can divide the fundamental region R into d^{3n} regions, each D_n of which is congruent to a region bounded by a smaller $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron. Combining ρ^n and the sequence of reflections, we have a continuous map from D_n onto R. Then by the fixed point theorem, we can prove the following lemma.

LEMMA 2.6

Each region D_n has a periodic point of period n of ρ.

All the repelling periodic points are dense and equidistributed in R. Hence, we can prove the following theorem.

THEOREM 2.7

(1) *We have* $J_3(P^d_{A_3}) = K(P^d_{A_3})$.

(2) *The maximal entropy measure* μ *of* $P^d_{A_3}(z_1, z_2, z_3)$ *is given by*

$$\mu = \frac{36}{\pi^3}\frac{1}{\sqrt{d_3}}dp_1dp_2dq,$$

where

$$d_3 = 256 - 27(z_1^4 + \bar{z}_1^4) + (z_1^2 + \bar{z}_1^2)(144z_2 - 4z_2^3 + 18z_1\bar{z}_1z_2) \\ - 80z_1\bar{z}_1z_2^2 + z_1^2\bar{z}_1^2z_2^2 - 192z_1\bar{z}_1 - 4z_1^3\bar{z}_1^3 - 6z_1^2\bar{z}_1^2 - 128z_2^2 + 16z_2^4,$$

with $z_1 = p_1 + ip_2$ *and* $z_2 = q$.

(3) *The Lyapunov exponents of* $P^d_{A_3}$ *with respect to the measure* μ *are given by* $\lambda_1 = \lambda_2 = \lambda_3 = \log d$.

Proof

(1) From Briend and Duval's theorem and Lemmas 2.5 and 2.6, we have $J_3(P^d_{A_3}) = K(P^d_{A_3})$.

(2) By pulling back the Lebesgue measure on R we will obtain the invariant measure μ. Set $\tilde{\mu}_n := -\varphi_*\mu_n$. From Lemma 2.6 we deduce that the sequence $\{\tilde{\mu}_n\}$ converges weakly to $\tilde{\mu} = \frac{3\sqrt{2}}{\pi^3}ds_1 \wedge ds_2 \wedge ds_3$. Hence,

$$\mu = -\frac{3\sqrt{2}}{\pi^3}\varphi^* ds_1 \wedge ds_2 \wedge ds_3.$$

From (2.6), we have

$$T^* ds_1 \wedge ds_2 \wedge ds_3 = \frac{1}{\sqrt{2}} d\alpha \wedge d\beta \wedge d\gamma.$$

Using [8, Lemma 3], we can compute the Jacobian determinant

$$\det \frac{\partial(p_1, p_2, q)}{\partial(\alpha, \beta, \gamma)}.$$

Then

$$\left(\det \frac{\partial(p_1, p_2, q)}{\partial(\alpha, \beta, \gamma)}\right)^2 = \frac{d_3}{4},$$

where

$$d_3 = 256 - 27(z_1^4 + \bar{z}_1^4) + (z_1^2 + \bar{z}_1^2)(144z_2 - 4z_2^3 + 18z_1\bar{z}_1z_2) \\ - 80z_1\bar{z}_1z_2^2 + z_1^2\bar{z}_1^2z_2^2 - 192z_1\bar{z}_1 - 4z_1^3\bar{z}_1^3 - 6z_1^2\bar{z}_1^2 - 128z_2^2 + 16z_2^4,$$

with $z_1 = p_1 + ip_2$ and $z_2 = q$. (Note that the formula from [8, p. 98] corresponding to the above formula for d_3 is incorrect.) Hence,

$$(\varphi_1^{-1})^* d\alpha \wedge d\beta \wedge d\gamma = \frac{1-2}{\sqrt{d_3}} dp_1 \wedge dp_2 \wedge dq.$$

Since $\varphi^* = (\varphi_1^{-1})^* T^*$, the assertion (2) follows.

(3) This assertion follows from the fact that $\rho(s_1, s_2, s_3) = d(s_1, s_2, s_3)$. $\square$

3. Julia set J_Π and stable sets

In this section we continue to study Julia sets. Set $f := P^d_{A_3}(z_1, z_2, z_3)$. From Proposition 2.1 we know that f is a regular polynomial endomorphism. So f extends continuously and holomorphically to $\mathbb{P}^3$, still denoted by f. We will study the Julia sets $J_2(f)$, $J_1(f)$, and $J_2(f_\Pi)$, where f_Π denotes the restriction of f to the hyperplane Π at infinity. Note that Π is completely invariant under f.

The Böttcher coordinate is useful in holomorphic dynamics in one complex variable. We try to construct analogous maps to the Böttcher coordinate.

Let f_h denote the homogeneous part of degree d of $f(z_1, z_2, z_3)$. Set

$$\Phi_2(x, y, z) = \Big(x^2, x\Big(y + \frac{1}{y}\Big)/z, 1/z^2\Big).$$

PROPOSITION 3.1

We have that f and f_h satisfy the following commutative diagram:

$$\begin{array}{ccc} (z_1,z_2,z_3) & \xrightarrow{f} & (z_1^{(d)}, z_2^{(d)}, z_3^{(d)}) \\ \uparrow \Phi_1 & & \uparrow \Phi_1 \\ (t_1,t_2,t_3) & \rightarrow & (t_1^d, t_2^d, t_3^d) \\ \uparrow & & \uparrow \\ (\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3}) & \rightarrow & (\sqrt{t_1}^d, \sqrt{t_2}^d, \sqrt{t_3}^d) \\ \downarrow \Phi_2 & & \downarrow \Phi_2 \\ (t_1, \frac{\sqrt{t_1}}{\sqrt{t_3}}(\sqrt{t_2} + \frac{1}{\sqrt{t_2}}), \frac{1}{t_3}) & \xrightarrow{f_h} & (t_1^d, (\frac{\sqrt{t_1}}{\sqrt{t_3}})^d(\sqrt{t_2}^d + \frac{1}{\sqrt{t_2}^d}), \frac{1}{t_3^d}) \end{array} \tag{3.1}$$

where $t_j \in \mathbb{C} \setminus \{0\}$, $\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3}$ are arbitrary branches, and

$$\begin{aligned} z_1^{(d)} &= t_1^d + t_2^d + t_3^d + \frac{1}{t_1^d t_2^d t_3^d}, \\ z_2^{(d)} &= t_1^d t_2^d + t_1^d t_3^d + t_2^d t_3^d + \frac{1}{t_1^d t_2^d} + \frac{1}{t_1^d t_3^d} + \frac{1}{t_2^d t_3^d}, \\ z_3^{(d)} &= \frac{1}{t_1^d} + \frac{1}{t_2^d} + \frac{1}{t_3^d} + t_1^d t_2^d t_3^d. \end{aligned} \tag{3.2}$$

Proof

The upper half of the commutative diagram is shown in (1.2). We prove the lower half of the diagram by induction on d. If $d = 2$ or 3, we can directly prove that the diagram is commutative. The function f_h is considered in the proof of Proposition 2.1:

$$f_h(x, y, z) = \big(x^d, h_2^{(d)}(x, y, z), z^d\big).$$

Set

$$\Phi_2(\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3}) = (x, y, z).$$

Then

$$h_2^{(d+2)} \circ \Phi_2 = y h_2^{(d+1)} \circ \Phi_2 - xz h_2^{(d)} \circ \Phi_2.$$

Hence, the diagram is commutative for any d. □

We use the definitions and notation in [1]. Let $\Pi := \mathbb{P}^3 - \mathbb{C}^3$, the plane at infinity. It is isomorphic to $\mathbb{P}^2$. Clearly, Π is completely invariant. Let f_Π denote the restriction of f to Π. We may define the current $T_\Pi := T|_\Pi$ as the slice current. Set

$$\mu_\Pi := T_\Pi^2 \qquad \text{and} \qquad J_2(f_\Pi) := \operatorname{supp}(\mu_\Pi).$$

Bedford and Jonsson [1] used the symbol J_Π for $J_2(f_\Pi)$. We have the following statements for J_Π and μ_Π.

THEOREM 3.2

(1) *The Julia set $J_2(f_\Pi)$ is a Möbius strip $\mathcal{M}$,*

$$\mathcal{M} = \{(e^{\theta i}, xe^{\frac{\theta}{2}i}) : 0 \le \theta < 2\pi, -2 \le x \le 2\}.$$

(2) *The maximal entropy measure $\mu = \mu_\Pi$ is given by*

$$\sigma_*(\mu) = \frac{d\theta}{2\pi} \quad \text{on } \{e^{i\theta} : 0 \le \theta < 2\pi\} \text{ in the } \xi\text{-plane},$$

$$\mu(\cdot \mid \sigma^{-1}(\xi)) = \frac{1}{\pi}\frac{dx}{\sqrt{4-x^2}} \quad \text{on } \{xe^{\frac{\theta}{2}i} : -2 \le x \le 2\}.$$

Here $f_\Pi(z_1 : z_2 : z_3) = f_\Pi(\xi : \eta : 1)$, and $\sigma(\xi,\eta) = \xi$.

(3) *The Lyapunov exponents of f_Π with respect to μ are given by $\lambda_1 = \lambda_2 = \log d$.*

To prove this theorem we use Jonsson's results (see [12]). Jonsson [12] studied polynomial skew product maps on $\mathbb{C}^2$. A polynomial skew product of $\mathbb{C}^2$ of degree $d \ge 2$ is a map of the form $f(z,w) = (p(z), q(z,w))$, where p and q are polynomials of degree d. Let $G_p(z)$ be the Green function of p, and let $G(z,w)$ be the Green function of f on $\mathbb{C}^2$. Set

$$K_p := \{G_p = 0\} \qquad \text{and} \qquad J_p := \partial K_p.$$

Define $G_z(w) := G(z,w) - G_p(z)$. Let

$$K_z := \{G_z = 0\} \qquad \text{and} \qquad J_z := \partial K_z.$$

Proof of Theorem 3.2

(1) Let π be the projection from $\mathbb{C}^3 - \{0\}$ to Π. Then $\pi \circ f_h = f_\Pi \circ \pi$. Since $f_h(z,w,v) = (z^d, h_2^{(d)}(z,w,v), v^d)$, it follows that $f_\Pi(z : w : v) = (z^d : h_2^{(d)}(z,w,v) : v^d)$.

Case 1: $v = 0$. The line $\{v = 0\}$ at infinity in Π is an attracting set of $f_\Pi(z : w : v)$. Hence, there is a neighborhood of $\{v = 0\}$ which does not have any repelling periodic points of f_Π. Therefore,

$$\{v = 0\} \cap J_2(f_\Pi) = \emptyset.$$

Case 2: $v \neq 0$. Then $f_\Pi(z:w:1) = (z^d : h_2^{(d)}(z,w,1):1)$ and so we consider a polynomial skew product on $\mathbb{C}^2$, still denoted by f_Π,

$$f_\Pi(z,w) = \big(z^d, h_2^{(d)}(z,w,1)\big).$$

Set $z = t_1$ and $w = \sqrt{t_1}(\sqrt{t_2} + \frac{1}{\sqrt{t_2}})$. Then from (3.1) we see that

$$f_\Pi\Big(t_1, \sqrt{t_1}\Big(\sqrt{t_2} + \frac{1}{\sqrt{t_2}}\Big)\Big) = \Big(t_1^d, \sqrt{t_1}^d\Big(\sqrt{t_2}^d + \frac{1}{\sqrt{t_2}^d}\Big)\Big). \tag{3.3}$$

We use Jonsson's results. In our case $p(z) = z^d$ and so $J_p = \{|z| = 1\}$. Hence, we may assume that $z = t_1 \neq 0$. To use [12, Corollary 4.4], we consider K_a for any $a = e^{i\theta} \in J_p$. Let $t_1 = e^{i\theta}$. Since $G_p(a) = 0$, we have $G_a(w) = G(a,w)$, where

$$G(a,w) = \lim_{n\to\infty} d^{-n} \log^+ \big|f_\Pi^n(a,w)\big|.$$

From (3.3) and the definition of K_a, we see that $w \in K_a$ if and only if $w = e^{i\theta/2}(e^{i\phi} + e^{-i\phi})$ with $0 \le \phi \le 2\pi$. Hence,

$$K_a = \{2\cos\phi e^{\frac{i\theta}{2}} : 0 \le \phi \le 2\pi\}.$$

Therefore,

$$J_a = \partial K_a = K_a.$$

By [12, Corollary 4.4], we conclude that

$$J_2(f_\Pi) = \overline{\bigcup_{a \in J_p} \{a\} \times J_a} = \big\{(e^{i\theta}, 2\cos\phi e^{i\theta/2}) : 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\big\}.$$

(2) To prove this assertion, we use [12, Theorem 4.2]. The action of μ on a test function φ is given by

$$\int \varphi\mu = \int \Big(\int \varphi(z,w)\mu_z(w)\Big)\mu_p(z),$$

where

$$\mu_p := \frac{1}{2\pi} dd^c G_p \qquad \text{and} \qquad \mu_z := \frac{1}{2\pi} dd^c G_z.$$

Since $p(z) = z^d$, it follows that $\mu_p = \frac{1}{2\pi} d\theta$ and $\mathrm{supp}(\mu_p)$ is the unit circle S^1. We will compute

$$G_z(w) := G(z,w) - G_p(z) \qquad \text{and} \qquad \mu_z \quad \text{for } z \in S^1.$$

Let $a = e^{i\theta}$.

As before, we set $z = t_1 = a$ and $w = \sqrt{t_1}(\sqrt{t_2} + \frac{1}{\sqrt{t_2}})$. From (3.3), we have

$$\begin{aligned}
\big|f_\Pi^n(a,w)\big|^2 &= |a^{d^n}|^2 + \Big|(\sqrt{a})^{d^n}\Big(\sqrt{t_2}^{d^n} + \frac{1}{\sqrt{t_2}^{d^n}}\Big)\Big|^2 \\
&= 1 + \Big|\sqrt{t_2}^{d^n} + \frac{1}{\sqrt{t_2}^{d^n}}\Big|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}G(a,w) &= \lim_{n\to\infty}\frac{1}{2d^n}\log\Big(1+\Big|\sqrt{t_2}^{d^n}+\frac{1}{\sqrt{t_2}^{d^n}}\Big|^2\Big)\\ &= \lim_{n\to\infty}\frac{1}{2d^n}\log^+\Big|\sqrt{t_2}^{d^n}+\frac{1}{\sqrt{t_2}^{d^n}}\Big|^2\\ &= \lim_{n\to\infty}\frac{1}{d^n}\log^+\Big|\sqrt{t_2}^{d^n}+\frac{1}{\sqrt{t_2}^{d^n}}\Big|\\ &= \lim_{n\to\infty}\frac{1}{d^n}\log^+\big|T_d^n(u)\big|\\ &= G_T(u).\end{aligned}$$

Here $T_d(u)$ is the Chebyshev polynomial of degree d of a single variable $u = (\sqrt{t_2}+\frac{1}{\sqrt{t_2}})$ and $G_T(u)$ is the Green function of $T_d(u)$.

Since $w = e^{\frac{i\theta}{2}}u$ and $G_T(u) = G(a,w) = G_a(w)$, we have

$$\frac{\partial^2}{\partial u\,\partial\bar u}G_T(u) = e^{-\frac{i\theta}{2}}\cdot e^{\frac{i\theta}{2}}\frac{\partial^2}{\partial w\,\partial\bar w}G(e^{i\theta},w) = \frac{\partial^2}{\partial w\,\partial\bar w}G_a(w).$$

It is known from [21] that the maximal entropy measure $(1/2\pi)dd^cG_T(u)$ of $T_d(u)$ is equal to $\frac{1}{\pi}\frac{du_1}{\sqrt{4-u_1}}$ supported on the segment $\{u_1 : -2\le u_1\le 2\}$, where $u_1 = \mathrm{Re}(u)$. Hence, the current μ_a is given by

$$\frac{1}{\pi}\frac{dx}{\sqrt{4-x^2}}\quad \text{on } \{xe^{\frac{\theta}{2}i} : -2\le x\le 2\}.$$

(3) We have proved that J_p is connected and each J_a is connected for all $a\in J_p$. Hence, from [12, Theorem 6.5] we have $\lambda_1=\lambda_2=\log d$. □

We continue to study Julia sets. We consider orbits of f and classify all the points of $\mathbb{C}^3$ into four categories. We begin by finding invariant sets of f in $\mathbb{P}^3$. We already have two invariant sets $K(f)$ and $J_2(f_\Pi)$. Besides these sets, there are two circles:

$$S_1 := \big\{(1:e^{i\theta}:0:0) : 0\le\theta<2\pi\big\},\qquad S_2 := \big\{(0:e^{i\theta}:1:0) : 0\le\theta<2\pi\big\},$$

and three attracting fixed points:

$$P_1 = (1:0:0:0),\qquad P_2 = (0:1:0:0),\qquad P_3 = (0:0:1:0).$$

We define the stable set of an invariant set X by

$$W^s(X,f) = \big\{x\in\mathbb{P}^3 : d(f^nx,X)\to 0 \text{ as } n\to\infty\big\}.$$

Then we have the following proposition.

PROPOSITION 3.3

Let a,b,c,d be a permutation of the set $\{|t_1|,|t_2|,|t_3|,|t_4|\}$, where $t_4=\frac{1}{t_1t_2t_3}$.

(1) *If $a=b=c=d=1$, then $\Phi_1(t_1,t_2,t_3)\in K(f)$.*
(2) *If $a>b=c=1>d=\frac{1}{a}$, then $\Phi_1(t_1,t_2,t_3)\in W^s(J_2(f_\Pi),f)$.*

(3) *If* $a > b = 1 > c \geq d$ *or* $a \geq b > c = 1 > d$, *then* $\Phi_1(t_1, t_2, t_3) \in W^s(S_1 \cup S_2, f)$.

(4) *If* $(a-1)(b-1)(c-1)(d-1) \neq 0$, *then* $\Phi_1(t_1, t_2, t_3) \in W^s(P_1 \cup P_2 \cup P_3, f)$.

Proof

(1) The assertion (1) is already shown in Proposition 2.2.

(2) Let $r_j = |t_j|$, $j = 1, 2, 3, 4$. We assume that

$$r_1 = r, \qquad r_3 = \frac{1}{r}, \qquad r_2 = r_4 = 1, \quad r > 1.$$

Then

$$z_1 = re^{i\alpha} + e^{i\beta} + \frac{e^{i\gamma}}{r} + e^{i(-\alpha-\beta-\gamma)},$$

$$z_2 = re^{i(\alpha+\beta)} + e^{i(\alpha+\gamma)} + re^{i(-\gamma-\beta)} + \frac{1}{r}e^{i(\beta+\gamma)} + e^{i(-\alpha-\gamma)} + \frac{1}{r}e^{-i(\alpha+\beta)},$$

$$z_3 = \frac{1}{r}e^{-i\alpha} + e^{-i\beta} + re^{-i\gamma} + e^{i(\alpha+\beta+\gamma)}.$$

The dominant terms of z_1, z_2, z_3 are $re^{i\alpha}, re^{i(\alpha+\beta)} + re^{i(-\beta-\gamma)}, re^{-i\gamma}$, respectively. Then for large n,

$$\begin{aligned} f^n(z_1 : z_2 : z_3 : 1) &\simeq \Big(\exp(i\alpha d^n) : \exp\big(i(\alpha+\beta)d^n\big) \\ &\qquad + \exp\big(-i(\beta+\gamma)d^n\big) : \exp(-i\gamma d^n) : \frac{1}{r^{d^n}}\Big) \\ &= \Big(\exp\big(i(\alpha+\gamma)d^n\big) : \exp\Big(i(\alpha+\gamma)\frac{d^n}{2}\Big) \\ &\qquad \cdot 2\cos\Big(\Big(\frac{\alpha+\gamma}{2} + \beta\Big)d^n\Big) : 1 : \exp(i\gamma d^n)/r^{d^n}\Big). \end{aligned}$$

Hence,

$$\begin{aligned} &(z_1 : z_2 : z_3 : 1) \in W^s\big(\{(e^{i\sigma} : 2\cos\tau e^{\frac{i\sigma}{2}} : 1 : 0) : 0 \leq \sigma < 2\pi, 0 \leq \tau < \pi\}, f\big) \\ &\quad = W^s\big(J_2(f_\Pi), f\big). \end{aligned}$$

Then assertion (2) follows.

(3) We assume that $r_1 \geq r_2 \geq r_3$. If $a > b = 1 > c \geq d$, then there are four cases:

$$\text{(i)} \quad r_4 > r_1 = 1 > r_2 \geq r_3, \qquad \text{(ii)} \quad r_1 > r_4 = 1 > r_2 \geq r_3,$$

$$\text{(iii)} \quad r_1 > r_2 = 1 > r_4 \geq r_3, \qquad \text{(iv)} \quad r_1 > r_2 = 1 > r_3 \geq r_4.$$

Let

$$M(z_1) := \max\{r_1, r_2, r_3, r_4\},$$

$$M(z_2) := \max\{r_1r_2, r_1r_3, r_1r_4, r_2r_3, r_2r_4, r_3r_4\},$$

$$M(z_3) := \max\Big\{\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r_4}\Big\}.$$

Let $\mathrm{dom}(z_j)$ be the set of the maximum elements that are equal to $M(z_j)$.

Case (i). Then $\mathrm{dom}(z_1) = \{r_4\}, \mathrm{dom}(z_2) = \{r_1 r_4\}, M(z_3) = \frac{1}{r_3}$. Hence, $M(z_1) = M(z_2) > M(z_3)$.

For the other cases, we can show that $\mathrm{dom}(z_1)$ and $\mathrm{dom}(z_2)$ are singletons and that $M(z_1) = M(z_2) > M(z_3)$. Hence, if we set $r := M(z_1) = M(z_2)$, then

$$f^n(z_1 : z_2 : z_3 : 1) \simeq \Big(\exp(i\sigma d^n) : \exp(i\tau d^n) : \varepsilon_n : \frac{1}{r^{d^n}}\Big), \quad \text{with } \varepsilon_n \to 0 \ (n \to \infty).$$

Hence,

$$(z_1 : z_2 : z_3 : 1) \in W^s\big(\{(1 : e^{i\theta} : 0 : 0) : 0 \le \theta < 2\pi\}, f\big).$$

Similarly, we can prove that if $a \ge b > c = 1 > d$, then

$$(z_1 : z_2 : z_3 : 1) \in W^s\big(\{(0 : e^{i\theta} : 1 : 0) : 0 \le \theta < 2\pi\}, f\big).$$

Then assertion (3) follows.

(4) If $(a-1)(b-1)(c-1)(d-1) \neq 0$, then (see (3) on p. 2.13) there are three cases:

(i) $a > 1 > b \ge c \ge d$, (ii) $a \ge b > 1 > c \ge d$, (iii) $a \ge b \ge c > 1 > d$.

Case (i). Then we see that $M(z_1) > M(z_2), M(z_3)$ and $\mathrm{dom}(z_1)$ is a singleton. Hence,

$$(z_1 : z_2 : z_3 : 1) \in W^s\big((1 : 0 : 0 : 0), f\big).$$

Case (ii). Then we see that $M(z_2) > M(z_1), M(z_3)$ and $\mathrm{dom}(z_2)$ is a singleton. Hence,

$$(z_1 : z_2 : z_3 : 1) \in W^s\big((0 : 1 : 0 : 0), f\big).$$

Case (iii). Then we see that $M(z_3) > M(z_1), M(z_2)$ and $\mathrm{dom}(z_3)$ is a singleton. Hence,

$$(z_1 : z_2 : z_3 : 1) \in W^s\big((0 : 0 : 1 : 0), f\big). \qquad \square$$

4. Julia sets J_1, J_2 and external rays

External rays for holomorphic endomorphisms of $\mathbb{P}^k$ were introduced by Bedford and Jonsson [1]. We review some results from [1]. Global stable manifolds at each point of a in J_Π are defined by

$$W^s(a) = \big\{x \in \mathbb{P}^k : d(f^j x, f^j a) \to 0 \text{ as } j \to \infty\big\}.$$

Note that $W^s(a)$ contains all the local stable manifolds $W^s_{\mathrm{loc}}(b)$ for $b \in J_\Pi$ with $f^n_\Pi b = f^n_\Pi a$, $n \ge 0$. Divide $W^s(a)$ into stable disks W_a. Let $\mathcal{E}_a$ denote the set of all gradient lines in W_a, and let the set $\mathcal{E}$ of external rays be the union of all $\mathcal{E}_a$'s. Note that f maps gradient lines to gradient lines.

In this article, using the Böttcher coordinate we construct global external rays. We consider $\Phi_1(re^{i\alpha}, e^{i\beta}, \frac{1}{r}e^{i\gamma})$,

$$z_1 = re^{i\alpha} + e^{i\beta} + \frac{e^{i\gamma}}{r} + e^{i(-\alpha-\beta-\gamma)},$$

$$\begin{aligned} z_2 &= re^{i(\alpha+\beta)} + e^{i(\alpha+\gamma)} + re^{i(-\gamma-\beta)} \\ &\quad + \frac{1}{r}e^{i(\beta+\gamma)} + e^{i(-\alpha-\gamma)} + \frac{1}{r}e^{-i(\alpha+\beta)}, \\ z_3 &= \frac{1}{r}e^{-i\alpha} + e^{-i\beta} + re^{-i\gamma} + e^{i(\alpha+\beta+\gamma)}. \end{aligned} \tag{4.1}$$

Let $R(\alpha,\beta,\gamma;r)$ denote this point $\Phi_1(re^{i\alpha}, e^{i\beta}, \frac{1}{r}e^{i\gamma})$ in $\mathbb{P}^3$. Then using an argument similar to that from the proof of Proposition 3.3(2), we can prove that

$$R(\alpha,\beta,\gamma;\infty) = \Big(e^{i(\alpha+\gamma)} : \Big(2\cos\Big(\frac{\alpha+\gamma}{2}+\beta\Big)\Big)e^{i\frac{\alpha+\gamma}{2}} : 1 : 0\Big) \in J_\Pi,$$

where

$$R(\alpha,\beta,\gamma;\infty) := \lim_{r\to\infty} R(\alpha,\beta,\gamma;r).$$

Clearly, $R(\alpha,\beta,\gamma;1) \in K(f)$ and $R(\alpha,\beta,\gamma;r) = R(\alpha,-\alpha-\beta-\gamma,\gamma;r)$.

Define an *external ray* by $R(\alpha,\beta,\gamma) := \{R(\alpha,\beta,\gamma;r) : r > 1\}$. (External rays of f_h are given by $\{\Phi_2(re^{i\alpha}, e^{i\beta}, \frac{1}{r}e^{i\gamma}) : r > 1\}$.) Clearly,

$$f\big(R(\alpha,\beta,\gamma;r)\big) = R(d\alpha, d\beta, d\gamma; r^d).$$

Then

$$f\big(R(\alpha,\beta,\gamma)\big) = R(d\alpha, d\beta, d\gamma),$$

and

$$\text{if } r > 1, \quad \lim_{n\to\infty} f^n\big(R(\alpha,\beta,\gamma;r)\big) \in J_\Pi.$$

We set

$$D(\alpha+\gamma,\beta) := \bigcup_{0\le\theta<2\pi} R(\alpha-\theta,\beta,\gamma+\theta).$$

By the above equality, we have $f(D(\alpha+\gamma,\beta)) = D(d(\alpha+\gamma), d\beta)$. The next lemma shows that $D(\alpha+\gamma,\beta)$ is a stable disk passing through $R(\alpha,\beta,\gamma;\infty)$.

LEMMA 4.1
We have that $D(\alpha+\gamma,\beta) \subset W^s(R(\alpha,\beta,\gamma;\infty))$.

Proof
Let (z_1,z_2,z_3) be any point of $R(\alpha-\theta,\beta,\gamma+\theta)$. The dominant terms of z_1, z_2, and z_3 are $re^{i(\alpha-\theta)}$, $re^{i(\alpha+\beta-\theta)} + re^{i(-\beta-\gamma-\theta)}$, and $re^{-i(\gamma+\theta)}$, respectively. As in the proof of Proposition 3.3(2), we can prove that

$$\begin{aligned} f^n(z_1 : z_2 : z_3 : 1) \simeq \Big(&\exp\big(i(\alpha+\gamma)d^n\big) : \exp\Big(i(\alpha+\gamma)\frac{d^n}{2}\Big) \\ &\cdot 2\cos\Big(\Big(\frac{\alpha+\gamma}{2}+\beta\Big)d^n\Big) : 1 : \exp\big(i(\gamma+\theta)d^n\big)/r^{d^n}\Big). \end{aligned}$$

On the other hand, by Proposition 3.1, we have

$$
\begin{aligned}
&f_{\Pi}^n\big(R(\alpha,\beta,\gamma;\infty)\big)\\
&\quad = f_{\Pi}^n\big(e^{i(\alpha+\gamma)} : e^{\frac{\alpha+\gamma}{2}i}(e^{(\frac{\alpha+\gamma}{2}+\beta)i} + e^{-(\frac{\alpha+\gamma}{2}+\beta)i}) : 1 : 0\big)\\
&\quad = \Big(\exp\big(i(\alpha+\gamma)d^n\big) : \exp\Big(i(\alpha+\gamma)\frac{d^n}{2}\Big)\cdot 2\cos\Big(\Big(\frac{\alpha+\gamma}{2}+\beta\Big)d^n\Big) : 1 : 0\Big).
\end{aligned}
$$

Then the lemma follows. □

From Proposition 3.3, we deduce that the set $\{D(\alpha+\gamma,\beta)\}$ forms a foliation of $W^s(J_\Pi, f)$.

Now we will determine the Julia sets $J_2(f)$ and $J_1(f)$. Using a result from [1] we will determine $J_2(f)$. Corollary 8.5 of [1] reads as follows. For almost every $a \in J_\Pi$, we have $\overline{W^s(a)} = \operatorname{supp}(T^{k-1} \llcorner \{G > 0\})$. Here G is the Green function of f.

Using this and Proposition 3.3, we have the following. Let $F(f)$ denote the Fatou set of f.

THEOREM 4.2

We have that $\mathbb{P}^3$ decomposes into the following sets:

(1) $J_3(f) = K(f)$,
(2) $J_2(f) \setminus J_3(f) = W^s(J_2(f_\Pi), f) = \bigcup D(\alpha+\beta,\beta)$,
(3) $J_1(f) \setminus J_2(f) = W^s(S_1 \cup S_2, f)$,
(4) $F(f) = W^s(P_1 \cup P_2 \cup P_3, f)$.

Proof

(1) This assertion is shown in Theorem 2.7(1).

(2) To prove this, we need [1, Corollary 8.5]. We know from Theorem 3.2 that

$$
J_2(f_\Pi) = \mathcal{M} = \big\{(e^{i\theta}, xe^{\frac{i\theta}{2}}) : 0 \le \theta < 2\pi, -2 \le x \le 2\big\}.
$$

And the maximal entropy measure μ_Π is given there. By [1, Corollary 8.5], we see that there is an element a in $\mathcal{M}$ such that

$$
\overline{W^s(a)} = \operatorname{supp}\big(T^2 \llcorner \{G > 0\}\big). \tag{4.2}
$$

Set $a = (e^{i\theta}, xe^{\frac{\theta}{2}i})$.

We claim that

$$
J_2(f_\Pi) = \overline{\bigcup_n f_\Pi^{-n}\big(f_\Pi^n(a)\big)}. \tag{4.3}
$$

To see this, we note that, in the proof of Theorem 3.2,

$$
f_\Pi(z,w) = \big(z^d, h_2^{(d)}(z,w,1)\big).
$$

Since $e^{i\theta} \in J_p$ with $p(z) = z^d$, $\bigcup_n p^{-n}(e^{i\theta})$ is dense in $J_p = S^1$. Also the set $\bigcup_n p^{-n}(p^n(e^{i\theta}))$ is dense in J_p. From Theorem 3.2(2) we know that, on the fibers

$\{\sigma^{-1}(z) : z \in \bigcup_n p^{-n}(p^n(e^{i\theta}))\}, h_2^{(d)}$ acts as the Chebyshev map T_d. Then (4.3) follows.

For any $c \in \overline{\bigcup_n f_\Pi^{-n}(f_\Pi^n(a))}$, there is a sequence $\{b_m\}$ with

$$b_m \in \bigcup_n f_\Pi^{-n}\left(f_\Pi^n(a)\right)$$

such that $b_m \to c$ as $m \to \infty$. Since $b_m \in W^s(a)$, it follows that $c \in \overline{W^s(a)}$. Set $c = R(\alpha, \beta, \gamma; \infty)$ and $b_m = R(\alpha_m, \beta_m, \gamma_m; \infty)$. Then we have $(\alpha_m + \gamma_m, \beta_m) \to (\alpha + \gamma, \beta)$.

We claim that

$$D(\alpha + \gamma, \beta) \subset \overline{W^s(a)}. \tag{4.4}$$

Indeed, we have shown that the center $R(\alpha, \beta, \gamma; \infty)$ of the disk $D(\alpha + \gamma, \beta)$ is in $\overline{W^s(a)}$. For any point $R(\alpha - \theta, \beta, \gamma + \theta; r)$ in $D(\alpha + \gamma, \beta)$, we can select a sequence $\{R(\alpha - \theta, \beta_m, \alpha_m + \gamma_m - \alpha + \theta; r)\}$ such that

$$R(\alpha - \theta, \beta_m, \alpha_m + \gamma_m - \alpha + \theta; r) \to R(\alpha - \theta, \beta, \gamma + \theta; r) \quad \text{as } m \to \infty.$$

Hence, from Lemma 4.1, we have

$$R(\alpha - \theta, \beta_m, \alpha_m + \gamma_m - \alpha + \theta; r) \in D(\alpha_m + \gamma_m, \beta_m) \subset W^s\big(R(\alpha_m, \beta_m, \gamma_m; \infty)\big).$$

Since

$$W^s\big(R(\alpha_m, \beta_m, \gamma_m; \infty)\big) = W^s(b_m) = W^s(a),$$

it follows that $R(\alpha - \theta, \beta_m, \alpha_m + \gamma_m - \alpha + \theta; r) \in W^s(a)$. Then $R(\alpha - \theta, \beta, \gamma + \theta; r) \in \overline{W^s(a)}$. Therefore, (4.4) follows. Hence, from (4.3) we deduce that

$$\bigcup_{\alpha+\gamma,\beta} D(\alpha + \gamma, \beta) \subset \overline{W^s(a)}. \tag{4.5}$$

Conversely, we claim that

$$\bigcup_{\alpha+\gamma,\beta} D(\alpha + \gamma, \beta) \supset W^s(a). \tag{4.6}$$

In the first place we consider any element b of $W^s(a) \cap \Pi$. From the proof of Theorem 3.2(1), we may assume that $b = (z : w : v)$ with $v \neq 0$. By case 2 of the proof of Theorem 3.2(1), we see that $b \in J_\Pi$. Then $b \in \bigcup_{\alpha+\gamma,\beta} D(\alpha + \gamma, \beta)$.

Next we assume that (z_1, z_2, z_3) is an element of $W^s(a)$ in $\mathbb{C}^3$. Then from Proposition 3.3, we see that (z_1, z_2, z_3) is written as $\Phi_1(t_1, t_2, t_3)$ in Proposition 3.3(2). Then we may set $(z_1, z_2, z_3) = \Phi_1(re^{i\alpha}, e^{i\beta}, \frac{1}{r}e^{i\gamma})$. Hence, $(z_1, z_2, z_3) \in R(\alpha, \beta, \gamma) \subset D(\alpha + \gamma, \beta)$. Then (4.6) follows.

From (4.5) and (4.6), it follows that $\overline{W^s(a)} = \overline{\bigcup D(\alpha + \gamma, \beta)}$. The set $\bigcup \overline{D(\alpha + \gamma, \beta)}$ is a union of closed disks, each of which is centered at a point of the Möbius strip. Hence, $\bigcup \overline{D(\alpha + \gamma, \beta)}$ is a closed set. Then $\overline{\bigcup D(\alpha + \gamma, \beta)} = \bigcup \overline{D(\alpha + \gamma, \beta)}$. Thus, from (4.2) we have

$$\operatorname{supp}\big(T^2 \llcorner \{G > 0\}\big) = \overline{\bigcup_{\alpha+\gamma,\beta} D(\alpha + \gamma, \beta)} = \bigcup_{\alpha+\gamma,\beta} \overline{D(\alpha + \gamma, \beta)}.$$

Set $A := \{G > 0\}$. Let U_1 and U_2 be the maximal open sets in which $T^2 = 0$ and $T^2 \llcorner A = 0$, respectively. Then $\operatorname{supp} T^2 = \mathbb{P}^3 \setminus U_1$ and $\operatorname{supp}(T^2 \llcorner A) = \mathbb{P}^3 \setminus U_2$. Since $K(f) = J_3 \subset \operatorname{supp} T^2$ and $\bigcup R(\alpha, \beta, \gamma; 1) = K(f) \subset \operatorname{supp}(T^2 \llcorner A)$, we have

$$U_i \cap K(f) = \emptyset, \quad i = 1, 2. \tag{4.7}$$

Let ψ be any 2-form of class C^∞ with compact support in U_1. Then by the definition of U_1 and (4.7), we have

$$0 = \langle T^2, \psi \rangle = \langle T^2, \psi \wedge \chi_A \rangle = \langle T^2 \llcorner A, \psi \rangle,$$

where χ_A is a characteristic function of A. Then we have $U_1 \subset U_2$. Similarly, we can prove that $U_2 \subset U_1$. Then it follows that $\operatorname{supp} T^2 = \operatorname{supp}(T^2 \llcorner A)$. Since $K(f) = J_3(f)$, we have $J_2(f) \backslash J_3(f) = \bigcup D(\alpha + \gamma, \beta)$. Assertion (2) follows.

(3) and (4) To prove these two statements we note that if f is a holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$, then the Julia set $J_1(f)$ is the complement of the Fatou set of f (see [18, Théorème 3.3.2]).

Note that $\mathbb{P}^k = \mathbb{C}^3 \cup \Pi$. In the first place we consider the set $\mathbb{C}^3$. We have shown in Proposition 3.3 that $\mathbb{C}^3$ decomposes into four categories. Only Proposition 3.3(4) corresponds to the Fatou set $F(f)$.

Next we consider a decomposition of Π. We have shown in the proof of Theorem 3.2 that

$$f_\Pi(z : w : v) = \big(z^d : h_2^{(d)}(z, w, v) : v^d\big).$$

Case 1: $v \neq 0$. If $z = 0$, then

$$f_\Pi(0 : w : v) = \big(0 : h_2^{(d)}(0, w, v) : v^d\big).$$

From (2.4), we see that $h_2^{(d)}(0, w, v) = w^d$. If $|w| = |v|$, then $(0 : w : v) \in S_2$. If $|w| \neq |v|$, then $(0 : w : v) \in W^s(P_2 \cup P_3, f_\Pi)$. Next we assume that $z \neq 0$. Then

$$f_\Pi(z, w) = \big(z^d, h_2^{(d)}(z, w, 1)\big).$$

We use the argument in the proof of Theorem 3.2. Set $z = t_1$ and $w = \sqrt{t_1}(\sqrt{t_2} + \frac{1}{\sqrt{t_2}})$. Set $t_1 = r_1 e^{i\sigma}$ and $t_2 = r_2 e^{i\tau}$. Then from (3.3) we have

$$\begin{aligned} f_\Pi^n(z, w) = \big(&r_1^{d^n} \exp(i\sigma d^n), r_1^{d^n/2} \exp(i\sigma d^n/2)\big(r_2^{d^n/2} \exp(i\tau d^n/2) \\ &+ r_2^{-d^n/2} \exp(-i\tau d^n/2)\big)\big). \end{aligned}$$

Hence, if $r_1 = r_2 = 1$, then (z, w) is an element of the Möbius strip $\mathcal{M}$. If $r_1 \neq 1$ and ($r_1 = r_2$ or $r_1 r_2 = 1$), then $(z : w : 1) \in W^s(S_1 \cup S_2, f_\Pi)$. If $r_1 \neq r_2$ and $r_1 r_2 \neq 1$, then $(z : w : 1) \in W^s(P_1 \cup P_2 \cup P_3, f_\Pi)$.

Case 2: $v = 0$. Using an argument similar to the proof of the case $z = 0$, we have the following results.

$$\begin{aligned} &\text{If } |z| = |w|, \quad \text{then } (z : w : 0) \in S_1, \\ &\text{If } |z| \neq |w|, \quad \text{then } (z : w : 0) \in W^s(P_1 \cup P_2, f_\Pi). \end{aligned}$$

Now we combine the results on $\mathbb{C}^3$ and Π. Since the Fatou set of f is $W^s(P_1 \cup P_2 \cup P_3, f)$, assertions (3) and (4) follow. $\square$

By direct computations, we can prove that $J_1(f)$ is a foliated space and that leaves of the space are topological polydisks in $\mathbb{C}^2$.

Next we consider external rays in R_3 $(=\{(z_1,z_2,\bar{z}_1): z_1\in\mathbb{C}, z_2\in\mathbb{R}\})$. Recall that any point $R(\alpha,\beta,\gamma;\infty)\in\mathcal{M}$ has a disk $D(\alpha+\gamma,\beta)$ centered at itself.

PROPOSITION 4.3

If $R(\alpha,\beta,\gamma)\subset R_3$, then $\alpha=\gamma$. Here $R(\alpha,\beta,\alpha)$ is a half-line and lands at a point of the astroidalhedron $\mathcal{A}$. Hence, an external ray in $D(\alpha+\gamma,\beta)$ included in R_3 is only the external ray $R(\frac{\alpha+\gamma}{2},\beta,\frac{\alpha+\gamma}{2})$.

Proof

By (4.1), we have $z_1-\bar{z}_3=(e^{i\alpha}-e^{i\gamma})(r-\frac{1}{r})$. If $z_1=\bar{z}_3$, then $\alpha=\gamma$. In this case, $R(\alpha,\beta,\alpha;r)$ is expressed as

$$z_1=\left(r+\frac{1}{r}\right)e^{i\alpha}+e^{i\beta}+e^{i(-2\alpha-\beta)},\qquad z_2=2\left(r+\frac{1}{r}\right)\cos(\alpha+\beta)+2\cos 2\alpha. \tag{4.8}$$

Therefore, $R(\alpha,\beta,\alpha)$ is a half-line and lands at a point of the astroidalhedron $\mathcal{A}$. □

We extend the half-line $R(\alpha,\beta,\alpha)$ to the interior of $K(f)$. In (4.8), we substitute $e^{i\theta}$ for r. That is,

$$\begin{aligned} z_1&=e^{i(\alpha+\theta)}+e^{i(\alpha-\theta)}+e^{i\beta}+e^{i(-2\alpha-\beta)},\\ z_2&=4\cos\theta\cos(\alpha+\beta)+2\cos 2\alpha,\quad 0\le\theta<2\pi. \end{aligned} \tag{4.9}$$

We call this the *internal ray* of $R(\alpha,\beta,\alpha)$ and denote it by $R_0(\alpha,\beta,\alpha)$.

PROPOSITION 4.4

Internal rays $R_0(\alpha,\beta,\alpha)$ are classified into two categories.

(1) *If $\alpha+\beta=0$ or $\alpha+\beta=\pi$, then the internal ray is a ruling of $\mathcal{A}$.*

(2) *If $\alpha+\beta\neq 0,\pi$, then the internal ray $R_0(\alpha,\beta,\alpha)$ links two external rays $R(\alpha,\beta,\alpha)$ and $R(\alpha+\pi,\beta,\alpha+\pi)$. And the internal ray touches the surface $\mathcal{A}$.*

Proof

(1) If $\alpha+\beta=0$, then

$$z_1=2\cos\theta e^{i\alpha}+2e^{-i\alpha},\qquad z_2=4\cos\theta+2\cos 2\alpha,\quad 0\le\theta<2\pi.$$

Hence, from (2.8) we know that this is a ruling of $\mathcal{A}$. The same holds for $\alpha+\beta=\pi$.

(2) If $\alpha+\beta\neq 0,\pi$, then the four terms of z_1 in (4.9) are distinct except for the cases

$$\theta=0,\qquad \theta=\pi,\qquad \theta=\pm(\alpha-\beta),\qquad \text{and}\qquad \theta=\pm(3\alpha+\beta).$$

Then the internal ray is not included in $\mathcal{A}$ and touches the surface at two points $\theta=\pm(\alpha-\beta)$ and $\theta=\pm(3\alpha+\beta)$. □

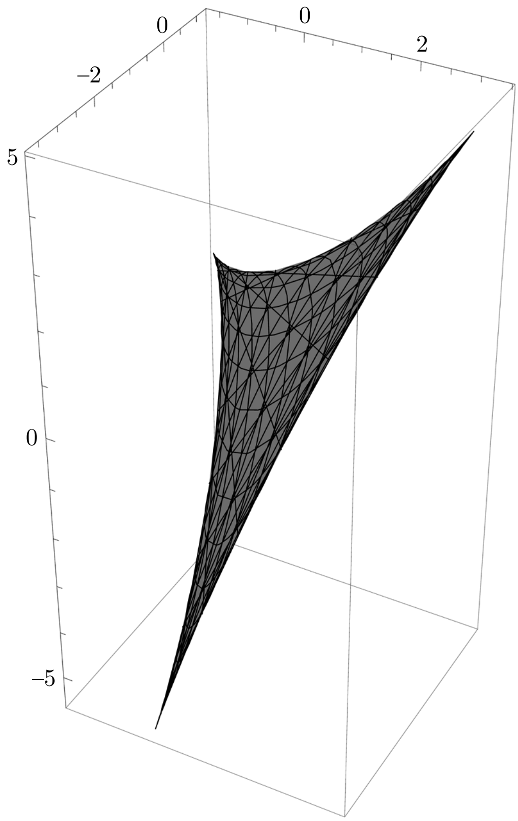

Figure 7. A face $\varphi_1(H)$.

COROLLARY 4.5

The rulings of the astroidalhedron are internal rays.

Next we study "inscribed faces" of $\mathcal{A}$. Using the notation from Section 2, we consider a face H in the natural domain R' in the space (α, β, γ) defined by $H := \{\alpha = c\} \cap R'$, where c is a constant. Recall that φ_1 is the map from R' onto $K(f)$.

PROPOSITION 4.6

We have that $\varphi_1(H)$ is a face on the plane in the (p_1, p_2, q) space given by

$$p_1 \cos c - p_2 \sin c - q/2 = \cos 2c.$$

Proof

By direct computations, we have this proposition. □

The face $\varphi_1(H)$ is depicted in Figure 7.

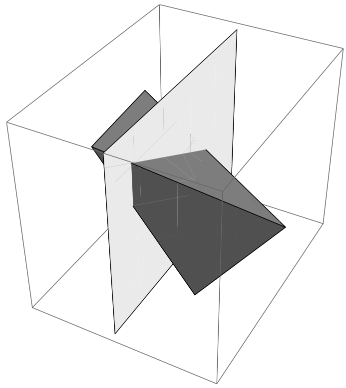

Figure 8. A line segment L and a face H.

We denote four vertices of the polyhedron $\partial R'$ by $O(0,0,0)$, $B_1(\pi/2,\pi/2,\pi/2)$, $B_2(-\pi,\pi,\pi)$, and $B_3(-\pi/2,-\pi/2,3\pi/2)$. We consider the triangle $\triangle OB_2B_3$. It lies on the plane $2\alpha+\beta+\gamma=0$. Set $L:=H\cap\triangle OB_2B_3$ (see Figure 8). The line segment L is given by $\{(c,\beta,-2c-\beta)\}$. The image of L under the transformation T is a line segment which is parallel to the root α_3. The image of $\triangle OB_2B_3$ under φ_1 is a part of the surface $\mathcal{A}$.

PROPOSITION 4.7

We have that $\varphi_1(L)$ is a ruling of $\mathcal{A}$. At any point of $\varphi_1(L)$, the face $\varphi_1(H)$ is tangent to $\varphi_1(\triangle OB_2B_3)$.

Proof

Let $(p_1,p_2,q):=\varphi_1(c,\beta,-2c-\beta)$. Then as in the proof of (2.8), we have

$$(p_1,p_2,q)=2(\cos c,\sin c,\cos 2c)+2\cos(\beta+c)(\cos c,-\sin c,2).$$

Hence from (2.8), we see that $\varphi_1(L)$ is a ruling of $\mathcal{A}$.

Since $\triangle OB_2B_3=\{(\alpha,\beta,\gamma)\in R' : 2\alpha+\beta+\gamma=0\}$, we have that $\varphi_1(\triangle OB_2B_3)$ is given by

$$p_1(\alpha,\beta)=2\cos\alpha+2\cos(\alpha+\beta)\cos\alpha,\qquad p_2(\alpha,\beta)=2\sin\alpha-2\sin\alpha\cos(\alpha+\beta),$$

$q(\alpha,\beta)=2(\cos 2\alpha+2\cos(\alpha+\beta))$. Set $\chi(\alpha,\beta)=(p_1(\alpha,\beta),p_2(\alpha,\beta),q(\alpha,\beta))$. Let $N:=(\cos c,-\sin c,-1/2)$ be the normal to $\varphi_1(H)$ at $\varphi_1(c,\beta,-2c-\beta)$. We see that the normal vector N is also orthogonal to the tangent vectors

$$\frac{\partial\chi}{\partial\alpha}\quad\text{and}\quad\frac{\partial\chi}{\partial\beta}\quad\text{at } \varphi_1(c,\beta,-2c-\beta).$$ □

We describe the "inscribed face" $\varphi_1(H)$ in Proposition 4.6 in terms of internal rays. Set $D_0(\beta)=\bigcup_\alpha R_0(\alpha,\beta,\alpha)$. Then we have the following proposition.

PROPOSITION 4.8

We have that $D_0(\beta)$ is equal to $\varphi_1(\{\beta=\text{const}\})$.

Proof

If we regard $\alpha+\theta$ as α' and $\alpha-\theta$ as γ' in (4.9), then we have $z_1 = e^{i\alpha'} + e^{i\gamma'} + e^{i\beta} + e^{-i(\alpha'+\beta+\gamma')}$. We fix $\beta = \text{const}$ and move α and θ. Then we have $\varphi_1(\{\beta = \text{const}\}) = D_0(\beta)$. □

Using external rays in R_3 whose internal rays are like those from Proposition 4.4(2), we construct a map E from $\mathcal{M}_0$ to $\mathcal{A}_0$, where

$$\mathcal{M}_0 = \big\{(e^{\theta i}, xe^{\frac{\theta}{2}i}) : 0 \le \theta < 2\pi, -2 < x < 2\big\},$$
$$\mathcal{A}_0 = \big\{(4\cos^3 u, 4\sin^3 u, 6\cos 2u) + v(\cos u, -\sin u, 2) :$$
$$0 \le u < 2\pi, -2-2\cos 2u < v < 2-2\cos 2u\big\}.$$

The external ray $R(\alpha,\beta,\alpha)$ with $\alpha+\beta \ne 0, \pi$ has two endpoints. One is in $\mathcal{M}_0$ and the other is in $\mathcal{A}_0$. Using these two endpoints, we define a map E from $\mathcal{M}_0$ to $\mathcal{A}_0$ by

$$\begin{aligned} &E\big((e^{2i\alpha} : 2\cos(\alpha+\beta)e^{i\alpha} : 1 : 0)\big) \\ &\quad = \big(2e^{i\alpha} + e^{i\beta} + e^{i(-2\alpha-\beta)}, 4\cos(\alpha+\beta) + 2\cos 2\alpha\big). \end{aligned} \tag{4.10}$$

PROPOSITION 4.9

The image of any ruling of $\mathcal{M}_0$ under the map E is also a ruling of $\mathcal{A}_0$.

Proof

In (4.10), we fix α and move β. Then by the same argument used in the proof of Proposition 2.4, we can prove that the image $(2e^{i\alpha} + e^{i\beta} + e^{i(-2\alpha-\beta)}, 4\cos(\alpha+\beta) + 2\cos 2\alpha)$ is written as (2.8). □

5. The set of critical values and catastrophe theory

In this section we show some relations between $P^d_{A_3}$ and catastrophe theory. Before we start studying the relations, we review some results on maps $P^d_{A_2}$ on $\mathbb{C}^2$ related to Lie algebras of type A_2. We show in [20] the following results. The set of critical values of $P^d_{A_2}$ restricted to $\{z_1 = \bar{z}_2\}$ is a deltoid. The deltoid coincides with a cross section of the bifurcation set (caustics) of the elliptic umbilic catastrophe map (D_4^-). The external rays and their extensions constitute a family of lines whose envelope is the deltoid. These lines are real "rays" of caustics (see Figure 9). In addition to the caustics, the deltoid has relations with binary cubic forms

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3, \quad a,b,c,d \in \mathbb{R}.$$

The discriminant D is given by

$$D = 4(ac^3 + b^3d) + 27a^2d^2 - b^2c^2 - 18abcd,$$
$$\text{Set } V = \big\{(a,b,c,d) \in \mathbb{R}^4 : D(a,b,c,d) = 0\big\}.$$

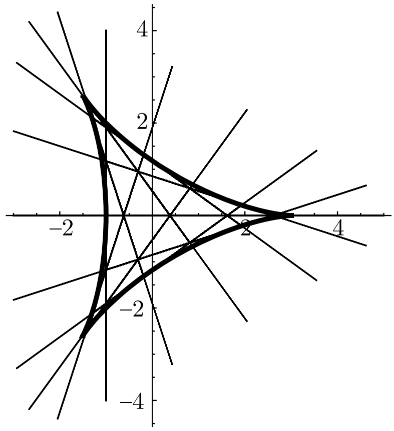

Figure 9. A deltoid and external rays.

Zeeman [23] showed that $V \cap S^3$ is mapped diffeomorphically to the "umbilic bracelet." It has a deltoid section that rotates 1/3 twist going once round the bracelet.

Now we return to the study of the maps $P^d_{A_3}$. We will show that the set of critical values of $P^d_{A_3}$ restricted to R_3 decomposes into the tangent developable of an astroid and two real curves. The set coincides with a cross section of the set obtained by Poston and Stewart [16], [17] where binary quartic forms are degenerate. The shape for the cross section is called the Holy Grail.

We begin with the study of the critical set of $P^d_{A_3}$. Let $t_4 = 1/(t_1t_2t_3)$. We use the notation from (1.1).

PROPOSITION 5.1

The critical set C_d of $P^d_{A_3}(z_1,z_2,z_3)$ is equal to

$$\big\{(z_1,z_2,z_3) \in \mathbb{C}^3 : t_1 = \varepsilon t_2 \ \textit{or}\ t_1 = \varepsilon t_3 \ \textit{or}\ t_1 = \varepsilon t_4 \ \textit{or}$$
$$t_2 = \varepsilon t_3 \ \textit{or}\ t_2 = \varepsilon t_4 \ \textit{or}\ t_3 = \varepsilon t_4, \varepsilon = e^{2j\pi\sqrt{-1}/d}\ (1 \le j \le d-1)\big\}.$$

Proof

Recall the map $\Phi_1(t_1,t_2,t_3) = (z_1,z_2,z_3)$. Then

$$\det D\Phi_1 = t_4 \prod_{1\le i<j\le 4} (t_i - t_j)$$

and

$$\det D(P^d_{A_3} \circ \Phi_1) = d^3 t_4 \prod_{1\le i<j\le 4} (t_i^d - t_j^d).$$

The proposition follows because

$$\det DP^d_{A_3} = \det D(P^d_{A_3} \circ \Phi_1)/\det D\Phi_1. \qquad \square$$

Clearly, the sets $P^d_{A_3}(C_d)$ $(d=2,3,4,\dots)$ are the same. The set $P^d_{A_3}(C_d)$ is an algebraic surface in $\mathbb{P}^3$ invariant under $P^d_{A_3}$, that is,

$$P^d_{A_3}\big(P^d_{A_3}(C_d)\big) = P^d_{A_3}(C_d).$$

Then $P^d_{A_3}$ is a critically finite map (see [7]).

We will determine the set $P^d_{A_3}(C_d)\cap R_3$. We may set $f := P^2_{A_3}(z_1,z_2,z_3)$ and $C := C_2$. If $(z_1,z_2,z_3)\in C$, then without loss of generality we may assume that $t_1 = -t_4$, where $t_4 = 1/(t_1t_2t_3)$. Then

$$z_1 = t_2+t_3, \qquad z_2 = t_2t_3 + \frac{1}{t_2t_3}, \qquad z_3 = \frac{1}{t_2}+\frac{1}{t_3},$$

and the image of (z_1,z_2,z_3) under f is written as

$$\begin{aligned} z_1^{(2)} &= t_2^2+t_3^2-2\frac{1}{t_2t_3},\\ z_2^{(2)} &= t_2^2t_3^2 - 2\Big(\frac{t_2}{t_3}+\frac{t_3}{t_2}\Big)+\frac{1}{t_2^2t_3^2},\\ z_3^{(d)} &= \frac{1}{t_2^2}+\frac{1}{t_3^2}-2t_2t_3. \end{aligned}$$

Set $t_2 = re^{i\alpha}$ and $t_3 = \mathrm{Re}^{i\beta}$. Then to determine the set $f(C)\cap R_3$ we need the following.

PROPOSITION 5.2
The point $(z_1^{(2)}, z_2^{(1)}, z_3^{(2)})$ belongs to the set R_3 if and only if the following three conditions are satisfied:

(1) $(r^2R^4-r^2)\cos 2b + 2(r^3R^3-rR)\cos(a+b) = R^2 - r^4R^2$,
(2) $(r^2R^4-r^2)\sin 2b + 2(r^3R^3-rR)\sin(a+b) = 0$,
(3) $(r^4R^4-1)\sin a - 2(r^3R-rR^3)\sin b = 0$,

where $a = 2\alpha+2\beta, b = \alpha-\beta$.

Proof
We may check the conditions

$$z_1^{(2)} = \overline{z_3^{(2)}} \qquad \text{and} \qquad z_2^{(2)}\in\mathbb{R}.$$

The former condition is equivalent to

$$\Big(r^2-\frac{1}{r^2}\Big)+\Big(R^2-\frac{1}{R^2}\Big)e^{2(\alpha-\beta)i}+2\Big(rR-\frac{1}{rR}\Big)e^{(3\alpha+\beta)i} = 0.$$

The latter condition is equivalent to

$$r^2R^2e^{2(\alpha+\beta)i}+\frac{1}{r^2R^2}e^{-2(\alpha+\beta)i}-2\Big(\frac{r}{R}e^{i(\alpha-\beta)}+\frac{R}{r}e^{i(\beta-\alpha)}\Big)\in\mathbb{R}.$$

Then the proposition follows. □

Next we will show a refinement of Proposition 5.2. We consider four cases:

(i) $r = R = 1$,
(ii) $rR = 1$ and $r \neq R$,
(iii) $rR \neq 1$ and $r = R$,
(iv) $rR \neq 1$ and $r \neq R$.

If $r = R = 1$, then the conditions (1), (2), and (3) are trivially satisfied.

LEMMA 5.3

We assume that the conditions (1), (2), and (3) in Proposition 5.2 are satisfied.

(i) *If $rR = 1$ and $r \neq R$, then $b = 0, \pi$.*
(ii) *If $rR \neq 1$ and $r = R$, then $(a, b) = (0, \pi), (\pi, 0)$.*

The proof is straightforward.

LEMMA 5.4

We assume that $rR \neq 1$ and $r \neq R$. Then there are not any numbers $0 < r, R$ and $0 \leq a, b < 2\pi$ satisfying (1), (2), and (3) in Proposition 5.2.

Proof

Suppose that there exist numbers $0 < r, R$ and $0 \leq a, b < 2\pi$ satisfying (1), (2), and (3). From (3) we have

$$\sin a = c_1 \sin b, \quad \text{where } c_1 := \frac{2(r^3R - rR^3)}{r^4R^4 - 1}. \tag{5.1}$$

We square both sides of (1) and (2). Then we add the left-hand sides and add the right-hand sides. Hence, if $R \neq 1$, then

$$\begin{aligned} \cos(a-b) = \frac{1}{2pq}\left(R^4(1-r^4)^2 - p^2 - q^2\right) =: c_2, \\ \text{where } p = r^2R^4 - r^2 \text{ and } q = 2(r^3R^3 - rR). \end{aligned} \tag{5.2}$$

(We denote the right-hand side of (5.2) by c_2.) Applying the addition theorem to $\cos(a-b)$ and using (5.1), we obtain

$$\sin^2 b = \frac{1 - c_2^2}{1 + c_1^2 - 2c_1c_2}. \tag{5.3}$$

From (2) and (5.1), it follows that

$$\cos a \sin b = c_3 \cos b \sin b, \quad \text{where } c_3 = \frac{-r(1+R^4)}{R(1+r^2R^2)}.$$

Case 1: $\sin b \neq 0$. Then

$$\cos a = c_3 \cos b. \tag{5.4}$$

Substituting $\sin a$ in (5.1) and $\cos a$ in (5.4) for those in (1) and then substituting $\sin^2 b$ in (5.3) for the result, we have

$$\frac{(r-R)(r+R)(-1+r^2R^2)^2}{1+r^2R^2} = 0,$$

which is a contradiction.

Case 2: $\sin b = 0$. Then $\sin a = 0$.

$$\text{If } (a,b) = (0,0) \quad \text{or} \quad (\pi,\pi), \quad \text{then } (r+R)^2(r^2R^2-1) = 0,$$

$$\text{If } (a,b) = (0,\pi) \quad \text{or} \quad (\pi,0), \quad \text{then } (r-R)^2(r^2R^2-1) = 0.$$

In any case, we have a contradiction.

If $R = 1$, we also have a contradiction. ☐

From Lemma 5.4, we know that $f(C) \cap R_3$ decomposes into three cases:

(i) $r = R = 1$,
(ii) $rR = 1$ and $r \neq R$,
(iii) $rR \neq 1$ and $r = R$.

Case i: $r = R = 1$. The set $\{(z_1^{(2)}, z_2^{(2)}, z_3^{(2)}) : r = R = 1\}$ is equal to the astroidalhedron $\mathcal{A}$. This is a central part of the tangent developable in Figure 10.

Case ii: $rR = 1$ *and* $r \neq R$. From Lemma 5.3, it follows that $b = 0$ or π. If $b = \pi$, then $\alpha - \beta = \pi$ and so $t_2 = re^{i\alpha}$, $t_3 = -\frac{1}{r}e^{i\alpha}$. Set $\theta = -2\alpha$. Then we have a top bowl. This is the upper part of the tangent developable in Figure 10. The top bowl is given as

$$(5.5)\qquad \begin{aligned} z_1^{(2)} &= \left(r^2 + \frac{1}{r^2}\right)e^{-i\theta} + 2e^{i\theta}, \qquad z_2^{(2)} = 2\left(r^2 + \frac{1}{r^2}\right) + 2\cos 2\theta, \\ z_3^{(2)} &= \left(r^2 + \frac{1}{r^2}\right)e^{i\theta} + 2e^{-i\theta}. \end{aligned}$$

If $b = 0$, then $\alpha - \beta = 0$ and so $t_2 = re^{i\alpha}$, $t_3 = \frac{1}{r}e^{i\alpha}$. Set $\theta = -2\alpha$. Then we have a lower bowl. This is the lower part of the tangent developable in Figure 10. The lower bowl is given as

$$(5.6)\qquad \begin{aligned} z_1^{(2)} &= \left(r^2 + \frac{1}{r^2}\right)e^{-i\theta} - 2e^{i\theta}, \qquad z_2^{(2)} = -2\left(r^2 + \frac{1}{r^2}\right) + 2\cos 2\theta, \\ z_3^{(2)} &= \left(r^2 + \frac{1}{r^2}\right)e^{i\theta} - 2e^{-i\theta}. \end{aligned}$$

Case iii: $rR \neq 1$ *and* $r = R$. Then $(a,b) = (0,\pi)$ or $(\pi,0)$. If $a = 0$ and $b = \pi$, then $t_2 = ir$, $t_3 = -ir$. Then we have top whiskers (see Figure 10). Top whiskers are given as

$$(5.7)\qquad z_1^{(2)} = -2\left(r^2 + \frac{1}{r^2}\right), \qquad z_2^{(2)} = r^4 + \frac{1}{r^4} + 4, \qquad z_3^{(2)} = -2\left(r^2 + \frac{1}{r^2}\right).$$

If $a = \pi$ and $b = 0$, then $t_2 = t_3 = re^{i\pi/4}$. Then we have lower whiskers (see Figure 10). Lower whiskers are given as

$$(5.8)\qquad z_1^{(2)} = 2i\left(r^2 + \frac{1}{r^2}\right), \qquad z_2^{(2)} = -r^4 - \frac{1}{r^4} - 4, \qquad z_3^{(2)} = -2i\left(r^2 + \frac{1}{r^2}\right).$$

Hence, $f(C) \cap R_3$ decomposes into the astroidalhedron $\mathcal{A}$, a top bowl, a lower bowl, top whiskers, and lower whiskers.

Next we consider relations between $f(C) \cap R_3$ and external rays. The half-lines (5.5) and (5.6) with $1 \leq r \leq \infty$ are external rays $R(-\theta, \theta, -\theta)$ and $R(-\theta, \theta +$

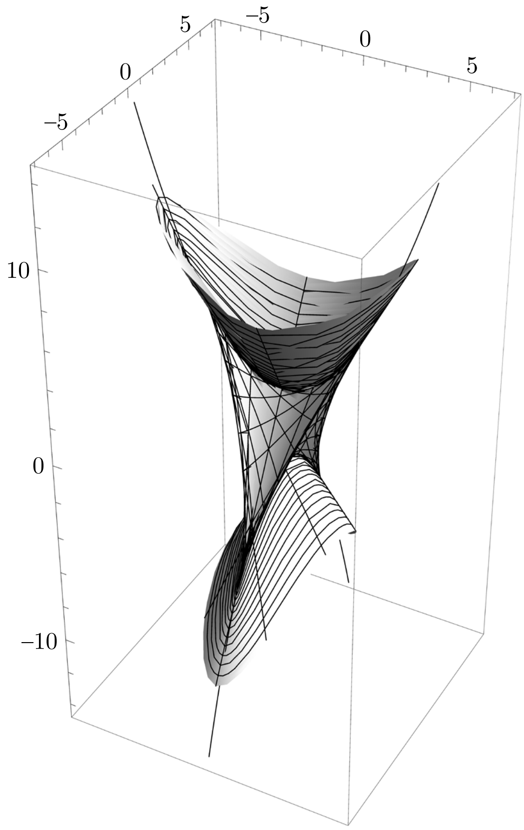

Figure 10. The tangent developable of an astroid in space and whiskers.

$\pi, -\theta)$ and land at points on the upper and lower self-intersection lines, respectively. By Propositions 2.4 and 4.4, we know that, by adding an internal ray to the half-lines, we have a tangent line to the astroid.

Then we have the following proposition.

PROPOSITION 5.5

We have that $f(C) \cap R_3 \setminus \{\textit{top and lower whiskers}\}$ *is the tangent developable* $\mathcal{T}$ *of an astroid in space given by*

$$\chi(u, v) = (4\cos^3 u, 4\sin^3 u, 6\cos 2u) + v(\cos u, -\sin u, 2) \quad (-\infty < v < \infty).$$

The tangent developable $\mathcal{T}$ consists of $\mathcal{A}$, the top bowl, and the lower bowl. Any ruling of $\mathcal{T}$, that is, any tangent line to the astroid, consists of two external rays and an intermediate internal ray.

PROPOSITION 5.6

(1) *The rims of the bowls join to the boundary of the Möbius strip* $\mathcal{M}$ *in* Π.

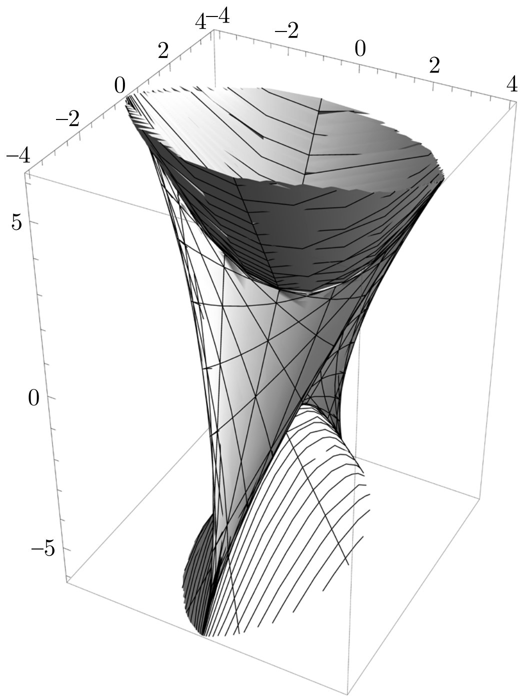

Figure 11. The tangent developable of an astroid in space.

(2) *The images of the two self-intersection lines under the map* φ *from* $K(f)$ *to* R *defined in Section 2 are the two edges of longest length of the* $(\sqrt{3}, \sqrt{3}, 2)$*-tetrahedron* ∂R.

Proof
(1) The external rays in the top bowl and the lower bowl are given in (5.5) and (5.6). Making $r \to \infty$, we see that

$$\text{top bowl}: (z_1^{(2)} : z_2^{(2)} : z_3^{(2)} : 1) \to (e^{-i\theta} : 2 : e^{i\theta} : 0) \in \mathcal{M},$$

$$\text{lower bowl}: (z_1^{(2)} : z_2^{(2)} : z_3^{(2)} : 1) \to (e^{-i\theta} : -2 : e^{i\theta} : 0) \in \mathcal{M}.$$

(2) We denote four vertices of the $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron ∂R by $O = (0,0,0)$, $A_1 = (0, -\pi/\sqrt{2}, \pi)$, $A_2 = (\pi, 0, \pi)$, and $A_3 = (0, \pi/\sqrt{2}, \pi)$ (see Figure 2). The lengths of OA_2 and A_1A_3 are equal to $\sqrt{2}\pi$ and the lengths of the other edges are equal to $\sqrt{3}\pi/\sqrt{2}$. The images of OA_2 and A_1A_3 under the map φ^{-1} are the upper self-intersection line and the lower self-intersection line, respectively (see Figure 4). □

Recall that $J_3(f)$ is the closed domain bounded by $\mathcal{A}$. We have shown in Proposition 4.9 that the image of any ruling of $\mathcal{M}_0$ under the map E is also a ruling of $\mathcal{A}_0$ (see Figures 11 and 12).

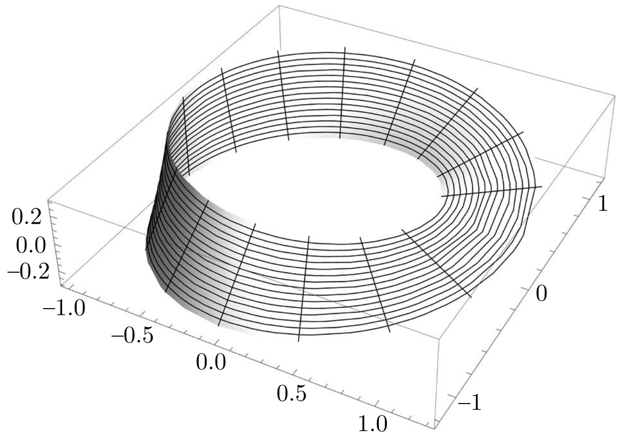

Figure 12. A Möbius strip.

Last, we consider relations between $f(C) \cap R_3$ and binary quartic forms. Poston and Stewart [16], [17] studied quartic forms in two variables,

$$f(x,y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4, \quad a,b,c,d,e \in \mathbb{R}.$$

Here, $f(x,y)$ can be expressed uniquely as

$$f(x,y) = \mathrm{Re}(\alpha z^4 + \beta z^3\bar{z} + \gamma z^2\bar{z}^2), \quad \alpha,\beta \in \mathbb{C}, \gamma \in \mathbb{R}. \tag{5.9}$$

We use the results and notation in [17, pp. 268–269]. Let $\triangle$ be the discriminant of $f(x,y)$, and let $\mathscr{Q} \subset \mathbb{R}^5$ be the algebraic set given by $\triangle = 0$. To understand the geometry of $\mathscr{Q}$ they pursued a different tack. The set $\mathscr{W} = \mathscr{Q} \cap S^4$ is decomposed into $\mathscr{W}_1$ and $\mathscr{W}_\infty$, and $\mathscr{W}_1$ is diffeomorphic to $\mathscr{U}$. Then $\mathscr{U}$ is the orbit of $\mathscr{Q}$ under a maximal tours $\mathbb{T}$ of $\mathrm{GL}_2(\mathbb{R})$, and $\mathscr{Q}_0$ is the main part of $\mathscr{Q}$. We consider the set $\mathscr{Q}_0$. Lemma 3.3 in [16] states that $\mathscr{Q}_0$ is given parametrically by

$$\beta = \frac{1}{2}(-3e^{i\phi} + e^{-3i\phi} - 2\gamma e^{-i\phi}), \quad 0 \le \phi < 2\pi. \tag{5.10}$$

The shape for $\mathscr{Q}$ (or $\mathscr{Q}_0$) is called the Holy Grail in [5] and depicted in [17, Figure 5]. We compare the shape with Figure 11. We show relations between $\mathscr{Q}_0$ and the tangent developable $\mathcal{T}$ in Proposition 5.5 of this article.

LEMMA 5.7

The set $\mathscr{Q}_0$ coincides with $\mathcal{T}$ by a coordinate transformation.

Proof

As in the proof of [16, Lemma 3.3], we put $\alpha = 1$ and $z = e^{i\theta}$ in the right-hand side of (5.9). That is, we consider the equation

$$e^{4i\theta} + e^{-4i\theta} + \beta e^{2i\theta} + \bar{\beta} e^{-2i\theta} + 2\gamma = 0. \tag{5.11}$$

The equation (5.10) follows from the condition that (5.11) has a double root in θ. We will find the same condition in our situation. From (5.11), we have

$$(e^{2i\theta})^4 + \beta(e^{2i\theta})^3 + 2\gamma(e^{2i\theta})^2 + \bar{\beta}e^{2i\theta} + 1 = 0. \tag{5.12}$$

Hence, we consider the equation

$$T^4 - z_1T^3 + z_2T^2 - z_3T + 1 = 0. \tag{5.13}$$

Let the solutions of (5.13) be t_1, t_2, t_3, and t_4. Then the condition that (5.11) has a double root in θ is described as follows. From (5.12), we assume that $z_1 = \bar{z}_3$ and z_2 is real. That is, $(z_1, z_2, z_3) \in R_3$. Under this assumption, (5.13) has a solution $\{t_1, t_2, t_3, t_4\}$ such that $t_1 = t_2 = e^{i\theta}$. Set $t_3 = re^{i\phi}$. Then $t_4 = (1/r)e^{-i(2\theta+\phi)}$. Relations between t_j's and z_j's are given in (1.1) with $t_4 = 1/(t_1t_2t_3)$. Then we can express the condition that such an element (z_1, z_2, z_3) lies in R_3 in terms of the variables r, ϕ, and θ. If $r = 1$, then $(z_1, z_2, z_3) \in \mathcal{A}$. Next we assume that $r \neq 1$. Then by an argument similar to that used in the proof of Lemma 5.3(i), we see that if such an element (z_1, z_2, z_3) lies in R_3, then $\phi + \theta = 0$ or $\phi + \theta = \pi$. If $\phi + \theta = 0$, then (z_1, z_2, z_3) belongs to the top bowl in (5.5). If $\phi + \theta = \pi$, then (z_1, z_2, z_3) belongs to the lower bowl in (5.6). The coordinate transformation is given by $\beta = -z_1$ and $2\gamma = z_2$. □

We can also prove this lemma by reparameterizing the ruled surface given by (5.10) using a striction curve.

The set $\mathscr{Q} \setminus \mathscr{Q}_0$ constitutes two whiskers in [17]. We can show that the whiskers in [17] coincide with the whiskers in (5.7) and (5.8) by the above coordinate transformation. Each whisker in this article joins to an attracting fixed point $P_2 = (0:1:0:0)$ of f.

PROPOSITION 5.8
The set $\mathscr{Q}$ coincides with $f(C) \cap R_3$ by a coordinate transformation.

In Proposition 5.6, we show that the rims of the bowls join to the boundary of $\mathcal{M}$. Poston and Stewart [16], [17] deal with the same situation by considering the attaching map to $\mathscr{W}_\infty \subset S^2 = \{\alpha = 0\} \subset S^4$. But it is complicated in $\mathbb{R}^5$. However, we consider the situation in $\mathbb{P}^3(\mathbb{C})$. Hence, the tangent developable $\mathcal{T}$ joins simply to the boundary of $\mathcal{M}$. We have studied the external rays that connect $\mathcal{T}$ and $\mathcal{M}$, and any ruling of $\mathcal{T}$ consists of two external rays and their intermediate internal ray.

We have shown the static aspect of catastrophe theory and also the dynamical aspect of catastrophe theory.

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