Constructing Lefschetz fibrations via daisy substitutions

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Abstract We construct new families of nonhyperelliptic Lefschetz fibrations by applying the daisy substitutions to the families of words $(c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1}{}^2 c_{2g} \cdot c_{2g-1} \cdots c_2 c_1)^2 = 1$, $(c_1 c_2 \cdots c_{2g} c_{2g+1})^{2g+2} = 1$, and $(c_1 c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)} = 1$ in the mapping class group Γ_g of the closed orientable surface of genus g, and we study the sections of these Lefschetz fibrations. Furthermore, we show that the total spaces of some of these Lefschetz fibrations are irreducible exotic 4-manifolds, and we compute their Seiberg–Witten invariants. By applying the knot surgery to the family of Lefschetz fibrations obtained from the word $(c_1 c_2 \cdots c_{2g} c_{2g+1})^{2g+2} = 1$ via daisy substitutions, we also construct an infinite family of pairwise nondiffeomorphic irreducible symplectic and nonsymplectic 4-manifolds homeomorphic to $(g^2 - g + 1)\mathbb{CP}^2 \# (3g^2 - g(k - 3) + 2k + 3)\overline{\mathbb{CP}}^2$ for any $g \geq 3$ and $k = 2, \ldots, g + 1$.

1. Introduction

The Lefschetz fibrations are fundamental objects to study in 4-dimensional topology. In his remarkable works, Simon Donaldson [9] showed that every closed symplectic 4-manifold admits a structure of Lefschetz pencil, which can be blown up at its base points to yield a Lefschetz fibration, and conversely, Robert Gompf [19] showed that the total space of a genus g Lefschetz fibration admits a symplectic structure, provided that the homology class of the fiber is nontrivial. Given a Lefschetz fibration over $\mathbb{S}^2$, one can associate to it a word in the mapping class group of the fiber composed solely of right-handed Dehn twists, and conversely, given such a factorization in the mapping class group, one can construct a Lefschetz fibration over $\mathbb{S}^2$ (see, e.g., [19]).

Recently, there has been much interest in trying to understand the topological interpretation of various relations in the mapping class group. A particularly well-understood case is the daisy relation, which corresponds to the symplectic operation of rational blowdown (see [11], [12]). Another interesting problem, which is still open, is whether any Lefschetz fibration over $\mathbb{S}^2$ admits a section

(see, e.g., [33]). Furthermore, one would like to determine how many disjoint sections the given Lefschetz fibration admits. The later problem has been studied for the standard family of hyperelliptic Lefschetz fibrations (with total spaces $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}^2}$) in [25], [28], and [35], using the computations in the mapping class group, and such results are useful in constructing (exotic) Stein fillings (see [1], [2]).

Motivated by these results and problems, our goal in this article is to construct new families of Lefschetz fibrations over $\mathbb{S}^2$ by applying the sequence of daisy substitutions and conjugations to the hyperelliptic words $(c_1 c_2 \cdots c_{2g-1} \cdot c_{2g} c_{2g+1}{}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1$, $(c_1 c_2 \cdots c_{2g} c_{2g+1})^{2g+2} = 1$, and $(c_1 c_2 \cdots c_{2g-1} \cdot c_{2g})^{2(2g+1)} = 1$ in the mapping class group of the closed orientable surface of genus g for any $g \geq 3$ and study the sections of these Lefschetz fibrations (cf. Theorems 24–28). Furthermore, we show that the total spaces of our Lefschetz fibrations given by the last two words are irreducible exotic symplectic 4-manifolds, and we compute their Seiberg–Witten invariants (cf. Theorem 31). Analogous (but weaker) results for the special case of $g = 2$, using the lantern substitutions only, were obtained in [11] and [3]. We would like to remark that the mapping class group computations in our article are more involved and subtle than those in [11] and [3]. One family of examples, obtained from the fiber sums of the Lefschetz fibrations by using daisy relations, were studied in [12]. However, the examples obtained in [12] have larger topology, and the computations of Seiberg–Witten invariants and study of sections were not addressed in [12]. Moreover, we prove the nonhyperellipticity of our Lefschetz fibrations and provide some criteria for nonhyperellipticity under the daisy substitutions (cf. Theorem 29). Some of our examples can be used to produce the families of nonisomorphic Lefschetz fibrations over $\mathbb{S}^2$ with the same total spaces and exotic Stein fillings. We hope to return to these examples in future work.

The organization of our article is as follows. In Sections 2 and 3 we recall the main definitions and results that will be used throughout the article. In Section 4, we prove some technical lemmas, important in the proofs of our main theorems. In Sections 5 and 6, we construct new families of Lefschetz fibrations by applying the daisy substitutions to the words given above, study the sections, and prove the nonhyperellipticity of our Lefschetz fibrations (Theorems 24–29). Finally, in Section 7, we prove that the total spaces of some of these Lefschetz fibrations are exotic symplectic 4-manifolds, which we verify by computing their Seiberg–Witten invariants and obtaining an infinite family of exotic 4-manifolds via knot surgery (Theorems 31, 32), and make some remarks and raise questions. We would like to remark that the main technical content of our article is more algebraic since our proofs rely heavily on mapping class group techniques. It is possible to pursue a more geometric approach (see Example 12), but such an approach alone does not yield the optimal results as presented here.

2. Mapping class groups

Let Σ_g^n be a 2-dimensional, compact, oriented, and connected surface of genus g with n boundary components. Let $\mathrm{Diff}^+(\Sigma_g^n)$ be the group of all orientation-preserving self-diffeomorphisms of Σ_g^n which are the identity on the boundary, and let $\mathrm{Diff}_0^+(\Sigma_g)$ be the subgroup of $\mathrm{Diff}^+(\Sigma_g)$ consisting of all orientation-preserving self-diffeomorphisms that are isotopic to the identity. The isotopies are also assumed to fix the points on the boundary. The *mapping class group* Γ_g^n of Σ_g^n is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g^n; that is,

$$\Gamma_g^n = \mathrm{Diff}^+(\Sigma_g^n)/\mathrm{Diff}_0^+(\Sigma_g^n).$$

For simplicity, we write $\Sigma_g = \Sigma_g^0$ and $\Gamma_g = \Gamma_g^0$. The hyperelliptic mapping class group H_g of Σ_g is defined as the subgroup of Γ_g consisting of all isotopy classes commuting with the isotopy class of the hyperelliptic involution $\iota : \Sigma_g \to \Sigma_g$.

DEFINITION 1

Let α be a simple closed curve on Σ_g^n. A *right-handed* (or positive) *Dehn twist* about α is a diffeomorphism of $t_\alpha : \Sigma_g^n \to \Sigma_g^n$ obtained by cutting the surface Σ_g^n along α and gluing the ends back after rotating one of the ends 2π to the right.

It is well known that the mapping class group Γ_g^n is generated by Dehn twists. It is an elementary fact that the conjugate of a Dehn twist is again a Dehn twist: if $\phi : \Sigma_g^n \to \Sigma_g^n$ is an orientation-preserving diffeomorphism, then $\phi \circ t_\alpha \circ \phi^{-1} = t_{\phi(\alpha)}$. The following lemma is easy to verify (see [22] for a proof).

LEMMA 2

Let α and β be two simple closed curves on Σ_g^n. If α and β are disjoint, then their corresponding Dehn twists satisfy the commutativity relation $t_\alpha t_\beta = t_\beta t_\alpha$. If α and β transversely intersect at a single point, then their corresponding Dehn twists satisfy the braid relation $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$.

2.1. Daisy relation and daisy substitution

We recall the definition of the daisy relation (see [30], [12], [7]).

DEFINITION 3

Let Σ_0^{p+2} denote a sphere with $p+2$ boundary components ($p \geq 2$). Let $\delta_0, \delta_1, \delta_2, \ldots, \delta_{p+1}$ be the p boundary curves of Σ_0^{p+2}, and let $x_1, x_2, \ldots, x_{p+1}$ be the interior curves as shown in Figure 1. Then, we have the *daisy relation of type p*

$$t_{\delta_0}^{p-1} t_{\delta_1} t_{\delta_2} \cdots t_{\delta_{p+1}} = t_{x_1} t_{x_2} \cdots t_{x_{p+1}}.$$

We call the following relator the *daisy relator of type p*:

$$t_{\delta_{p+1}}^{-1} \cdots t_{\delta_2}^{-1} t_{\delta_1}^{-1} t_{\delta_0}^{-p+1} t_{x_1} t_{x_2} \cdots t_{x_{p+1}} (= 1).$$
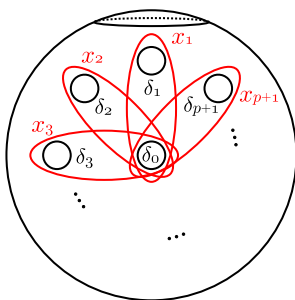
Figure 1. Daisy relation.

REMARK 4

When $p = 2$, the daisy relation is commonly known as the *lantern relation* (see [8], [23]).

We next introduce a daisy substitution, a substitution technique introduced by T. Fuller.

DEFINITION 5

Let $d_1, \ldots, d_m$ and $e_1, \ldots, e_n$ be simple closed curves on Γ_g^n, and let R be a product $R = t_{d_1} t_{d_2} \cdots t_{d_l} t_{e_m}^{-1} \cdots t_{e_2}^{-1} t_{e_1}^{-1}$. Suppose that $R = 1$ in Γ_g^n. Let ϱ be a word in Γ_g^n including $t_{d_1} t_{d_2} \cdots t_{d_l}$ as a subword:

$$\varrho = U \cdot t_{d_1} t_{d_2} \cdots t_{d_l} \cdot V,$$

where U and V are words. Thus, we obtain a new word in Γ_g^n, denoted by ϱ', as follows:

$$\varrho' := U \cdot t_{e_1} t_{e_2} \cdots t_{e_m} \cdot V.$$

Then, we say that ϱ' is obtained by applying an *R-substitution* to ϱ. In particular, if R is a daisy relator of type p, then we say that ϱ' is obtained by applying a *daisy substitution of type p* to ϱ.

3. Lefschetz fibrations

DEFINITION 6

Let X be a closed, oriented smooth 4-manifold. A smooth map $f : X \to \mathbb{S}^2$ is a genus g *Lefschetz fibration* if it satisfies the following conditions:

(i) f has finitely many critical values $b_1, \ldots, b_m \in S^2$, and f is a smooth Σ_g-bundle over $\mathbb{S}^2 - \{b_1, \ldots, b_m\}$;

(ii)   for each i $(i = 1, \ldots, m)$, there exists a unique critical point p_i in the *singular fiber* $f^{-1}(b_i)$ such that about each p_i and b_i there are local complex coordinate charts agreeing with the orientations of X and $\mathbb{S}^2$ on which f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$;

(iii) f is relatively minimal (i.e., no fiber contains a (-1)-sphere).

Each singular fiber is obtained by collapsing a simple closed curve (the *vanishing cycle*) in the regular fiber. The monodromy of the fibration around a singular fiber is given by a right-handed Dehn twist along the corresponding vanishing cycle. For a genus g Lefschetz fibration over $\mathbb{S}^2$, the product of right-handed Dehn twists t_{v_i} about the vanishing cycles v_i, for $i = 1, \ldots, m$, gives us the global monodromy of the Lefschetz fibration, the relation $t_{v_1} t_{v_2} \cdots t_{v_m} = 1$ in Γ_g. This relation is called the *positive relator*. Conversely, such a positive relator defines a genus g Lefschetz fibration over $\mathbb{S}^2$ with the vanishing cycles $v_1, \ldots, v_m$.

According to Kas [24] and Matsumoto [27], if $g \geq 2$, then the isomorphism class of a Lefschetz fibration is determined by a positive relator modulo simultaneous conjugations

$$t_{v_1} t_{v_2} \cdots t_{v_m} \sim t_{\phi(v_1)} t_{\phi(v_2)} \cdots t_{\phi(v_m)} \quad \text{for all } \phi \in M_g$$

and elementary transformations

$$t_{v_1} \cdots t_{v_{i-1}} t_{v_i} t_{v_{i+1}} t_{v_{i+2}} \cdots t_{v_m} \sim t_{v_1} \cdots t_{v_{i-1}} t_{t_{v_i}(v_{i+1})} t_{v_i} t_{v_{i+2}} \cdots t_{v_m},$$

$$t_{v_1} \cdots t_{v_{i-2}} t_{v_{i-1}} t_{v_i} t_{v_{i+1}} \cdots t_{v_m} \sim t_{v_1} \cdots t_{v_{i-2}} t_{v_i} t_{t_{v_i}^{-1}(v_{i-1})} t_{v_{i+1}} \cdots t_{v_m}.$$

Note that $\phi t_{v_i} \phi^{-1} = t_{\phi(v_i)}$. We denote a Lefschetz fibration associated to a positive relator $\varrho \in \Gamma_g$ by f_ϱ.

For a Lefschetz fibration $f : X \to \mathbb{S}^2$, a map $\sigma : \mathbb{S}^2 \to X$ is called a *section* of f if $f \circ \sigma = \mathrm{id}_{\mathbb{S}^2}$. We define the self-intersection of σ to be the self-intersection number of the homology class $[\sigma(\mathbb{S}^2)]$ in $H_2(X; \mathbb{Z})$. Let $\delta_1, \delta_2, \ldots, \delta_n$ be n boundary curves of Σ_g^n. If there exists a lift of a positive relator $\varrho = t_{v_1} t_{v_2} \cdots t_{v_m} = 1$ in Γ_g to Γ_g^n as

$$t_{\tilde{v}_1} t_{\tilde{v}_2} \cdots t_{\tilde{v}_m} = t_{\delta_1} t_{\delta_2} \cdots t_{\delta_n},$$

then f_ϱ admits n disjoint sections of self-intersection -1. Here, $t_{\tilde{v}_i}$ is a Dehn twist mapped to t_{v_i} under $\Gamma_g^n \to \Gamma_g$. Conversely, if a genus g Lefschetz fibration admits n disjoint sections of self-intersection -1, then we obtain such a relation in Γ_g^n.

Next, let us recall the signature formula for hyperelliptic Lefschetz fibrations, which is due to Matsumoto and Endo. We will make use of this formula in Section 6, where we prove that all our Lefschetz fibrations obtained via daisy substitutions are nonhyperelliptic.

THEOREM 7 ([26], [27], [10])

Let $f : X \to \mathbb{S}^2$ be a genus g hyperelliptic Lefschetz fibration. Let s_0 and $s = \sum_{h=1}^{[g/2]} s_h$ be the number of nonseparating and separating vanishing cycles of f, where s_h denotes the number of separating vanishing cycles which separate the surface of genus g into two surfaces, one of which has genus h. Then, we have the following formula for the signature:

$$\sigma(X) = -\frac{g+1}{2g+1} s_0 + \sum_{h=1}^{\left[\frac{g}{2}\right]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$

3.1. Spinness criteria for Lefschetz fibrations

In this section, we recall a few results due to A. Stipsicz [34] concerning the nonspinness and spinness of the Lefschetz fibrations over $\mathbb{D}^2$ and $\mathbb{S}^2$. We will use them to verify that our families of Lefschetz fibrations in Theorem 31 all are nonspin. Since Rohlin's criteria [31] cannot be used to verify nonspinness when the signature of our families of Lefschetz fibrations is divisible by 16, Stipsicz's results will be more suitable for our purpose.

Let $f : X \to \mathbb{D}^2$ be a Lefschetz fibration over a disk, and let F denote the generic fiber of f. Denote the homology classes of the vanishing cycles of the given fibration by $v_1, \ldots, v_m \in H_1(F; \mathbb{Z}_2)$.

THEOREM 8 ([34])

The Lefschetz fibration $f : X \to \mathbb{D}^2$ is not spin if and only if there are l vanishing cycles $v_1, \ldots, v_l$ such that $v = \sum_{i=1}^l v_i$ is also a vanishing cycle and $l + \sum_{1 \le i < j \le l} v_i \cdot v_j \equiv 0 \pmod 2$.

Note that the above theorem implies that if the Lefschetz fibration has the separating vanishing cycle, then its total space is not spin. To see this, set $l = 0$, and take the empty sum to be 0.

THEOREM 9 ([34])

The Lefschetz fibration $f : X \to \mathbb{S}^2$ is spin if and only if $X \setminus \nu(F)$ is spin and for some dual σ of F we have $\sigma^2 \equiv 0 \pmod 2$.

3.2. Three families of hyperelliptic Lefschetz fibrations

In this section, we introduce three well-known families of hyperelliptic Lefschetz fibrations, which will serve as building blocks in our construction of new Lefschetz fibrations. Let $c_1, c_2, \ldots, c_{2g}, c_{2g+1}$ denote the collection of simple closed curves given in Figure 2, and let c_i denote the right-handed Dehn twists t_{c_i} along the curve c_i. It is well known that the following relations hold in the mapping class group Γ_g:

$$H(g) = (c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1}{}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1,$$

(1) $$I(g) = (c_1 c_2 \cdots c_{2g} c_{2g+1})^{2g+2} = 1,$$

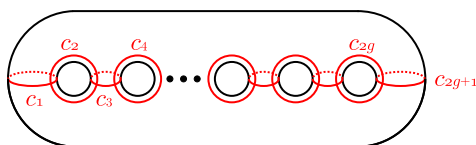$$G(g) = (c_1 c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)} = 1.$$

Figure 2. Vanishing cycles of the genus g Lefschetz fibration on $X(g)$, $Y(g)$, and $Z(g)$.

Let $X(g)$, $Y(g)$, and $Z(g)$ denote the total spaces of the above genus g hyperelliptic Lefschetz fibrations given by the monodromies $H(g) = 1$, $I(g) = 1$, and $J(g) = 1$, respectively, in the mapping class group Γ_g. For the first monodromy relation, the corresponding genus g Lefschetz fibrations over $\mathbb{S}^2$ have total space $X(g) = \mathbb{CP}^2 \# (4g + 5) \overline{\mathbb{CP}}^2$, the complex projective plane blown up at $4g + 5$ points. In the case of the second and third relations, the total spaces of the corresponding genus g Lefschetz fibrations over $\mathbb{S}^2$ are also well-known families of complex surfaces. For example, $Y(2) = K3 \# 2\overline{\mathbb{CP}}^2$ and $Z(2)$ is the *Horikawa surface*. In what follows, we recall the branched-cover description of the 4-manifolds $Y(g)$ and $Z(g)$, which we will use in the proofs of our main results. The branched-cover description of $X(g)$ is well known, and we refer the reader to [19, Remark 7.3.5, p. 257].

LEMMA 10

The genus g Lefschetz fibration on $Y(g)$ over $\mathbb{S}^2$ with the monodromy $(c_1 c_2 \cdots c_{2g+1})^{2g+2} = 1$ can be obtained as the double branched covering of $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ branched along a smooth algebraic curve B in the linear system $|2(g + 1)\tilde{L}|$, where $\tilde{L}$ is the proper transform of line L in $\mathbb{CP}^2$ avoiding the blown-up point. Furthermore, this Lefschetz fibration admits two disjoint (-1)-sphere sections.

Proof
We will follow the proof of [4, Lemma 3.1], where the $g = 2$ case has been considered (see also the discussion in [6]), and make necessary adjustments where needed. Let D denote an algebraic curve of degree d in $\mathbb{CP}^2$. We fix a generic projection map $\pi : \mathbb{CP}^2 \setminus pt \to \mathbb{CP}^1$ such that the pole of π does not belong to D. It was shown in [37] that the braid monodromy of D in $\mathbb{CP}^2$ is given via a braid factorization. More specifically, the braid monodromy around the point at infinity in $\mathbb{CP}^1$, which is given by the central element Δ^2 in B_d, can be written as the product of the monodromies about the critical points of π. Hence, the factorization $\Delta^2 = (\sigma_1 \cdots \sigma_{d-1})^d$ holds in the braid group B_d, where σ_i denotes a positive half-twist exchanging two points and fixing the remaining $d - 2$ points.

Now let us degenerate the smooth algebraic curve B in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ into a union of $2(g + 1)$ lines in a general position. By the discussion above, the braid group factorization corresponding to the configuration B is given by $\Delta^2 = (\sigma_1 \sigma_2 \cdots \sigma_{2g} \sigma_{2g+1})^{2g+2}$. Now, by lifting this braid factorization to the mapping class group of the genus g surface, we obtain the monodromy factorization $(c_1 c_2 \cdots c_{2g+1})^{2g+2} = 1$ for the corresponding double branched covering.

Moreover, observe that a regular fiber of the given fibration is a 2-fold cover of a sphere in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ with homology class $f = h - e_1$ branched over $2(g + 1)$ points, where h denotes the hyperplane class in $\mathbb{CP}^2$. Hence, a regular fiber is a surface of genus g. The exceptional sphere e_1 in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, which intersects $f = h - e_1$ once positively, lifts to two disjoint (-1)-sphere sections in $Y(g)$. $\square$

The proof of the following lemma can be extracted from [19, Example 7.3.27, p. 268]; we omit the proof.

LEMMA 11

The double branched cover $W(g)$ of $\mathbb{CP}^2$ along a smooth algebraic curve B in the linear system $|2(g+1)\tilde{L}|$ can be decomposed as the fiber sum of two copies of $\mathbb{CP}^2\#(g+1)^2\overline{\mathbb{CP}}^2$ along a complex curve of genus equal to $g(g-1)/2$. Moreover, $W(g)$ admits a genus g Lefschetz pencil with two base points, and $Y(g) = W(g)\#2\overline{\mathbb{CP}}^2$.

EXAMPLE 12

In this example, we study the topology of complex surfaces $W(g)$ in some detail. Recall that by Lemma 11 the complex surface $W(g)$ is the fiber sum of two copies of the rational surface $\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2$ along the complex curve Σ of genus $g(g-1)/2$ and self-intersection zero. Using the fiber sum decomposition, we compute the Euler characteristic and the signature of $W(g)$ as follows: $e(W(g)) = 2e(\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2) - 2e(\Sigma) = 4g^2+2g+4$, and $\sigma(W(g)) = 2\sigma(\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2) = -2(g^2+2g)$. Next, we recall from [17] that $\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2 = \Phi_{g(g-1)/2}(1) \cup N_{g(g-1)/2}(1)$, where $\Phi_{g(g-1)/2}(1)$ and $N_{g(g-1)/2}(1)$ are the Milnor fiber and generalized Gompf nucleus in $\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2$, respectively. Notice that this decomposition shows that the intersection form of $\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2$ splits as $N \oplus M(g)$, where $N = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ and $M(g)$ is a matrix whose entries are given by a negative definite plumbing tree in Figure 3. Consequently, we obtain the following decomposition of the intersection form of $W(g)$: $2M(g) \oplus H \oplus g(g-1)H$, where H is a hyperbolic pair. Let us choose the following basis which realizes the intersection matrix $M(g) \oplus N$ of $\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2$: $\langle f = (g+1)h - e_1 - \cdots - e_{(g+1)^2}, e_{(g+1)^2}, e_1 - e_2, e_2 - e_3, \ldots, e_{(g+1)^2-2} - e_{(g+1)^2-1}, h - e_{(g+1)^2-(g+1)} - \cdots - e_{(g+1)^2-2} - e_{(g+1)^2-1}\rangle$. Observe that the last $(g+1)^2 - 1$ classes can be represented by spheres and their self-intersections are given as in Figure 3. Here, f is the class of fibers of the genus $g(g-1)/2$ Lefschetz fibration on $\mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2$ and $e_{(g+1)^2}$ is a sphere section of self-intersection -1. By using the generalized fiber sum decomposition of $W(g)$, it is not hard to see the surfaces that generate the intersection matrix $2M(g) \oplus H \oplus g(g-1)H$. The two copies of the Milnor fiber $\Phi_{g(g-1)/2}(1) \subset \mathbb{CP}^2\#(g^2+2g+1)\overline{\mathbb{CP}}^2$ are in $W(g)$, providing $2((g+1)^2 - 1)$ spheres of self-intersections -2 and $-g$ (corresponding to the classes $\{e_1 - e_2, e_2 - e_3, \ldots, e_{(g+1)^2-2} - e_{(g+1)^2-1}, h - e_{(g+1)^2-(g+1)} - \cdots -$
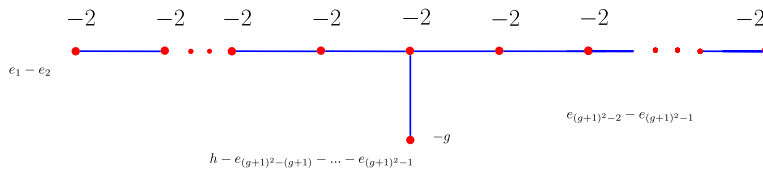

Figure 3. Plumbing tree for $\Phi_{g(g-1)/2}(1)$.

$e_{(g+1)^2-3} + e_{(g+1)^2-2} + e_{(g+1)^2-1}\}$ and $\{e'_1 - e'_2, e'_2 - e'_3, \ldots, e'_{(g+1)^2-2} - e'_{(g+1)^2-1}, h' - e_{(g+1)^2-(g+1)} - \cdots - e'_{(g+1)^2-3} - e'_{(g+1)^2-2} - e'_{(g+1)^2-1}\}$), and realize two copies of $M(g)$. One copy of the hyperbolic pair H comes from an identification of the fibers f and f', and a sphere section σ of self-intersection -2 obtained by sewing the sphere sections $e_{(g+1)^2}$ and $e'_{(g+1)^2}$. The remaining $g(g-1)$ copies of H come from $g(g-1)$ rim tori and their dual (-2)-spheres (see related discussion in [19, p. 73]). These $4g^2 + 2g + 2$ classes generate H_2 of $W(g)$. Furthermore, using the formula for the canonical class of the generalized symplectic sum and the adjunction inequality, we compute $K_{W(g)} = (g-2)(h+h')$. Also, the class of the genus g surface of square 2 of the genus g Lefschetz pencil on $W(g)$ is given by $h + h'$. As a consequence, the class of the genus g fiber in $W(g) \# 2\overline{\mathbb{CP}}^2$ is given by $h + h' - E_1 - E_2$, where E_1 and E_2 are the homology classes of the exceptional spheres of the blowups at the points p_1 and p_2, the base points of the pencil. We can also verify that the symplectic surface Σ, given by the class $h + h' - E_1 - E_2$, has genus g by applying the adjunction formula to $(W(g) \# 2\overline{\mathbb{CP}}^2, \Sigma)$: $g(\Sigma) = 1 + 1/2(K_{W(g)\#2\overline{\mathbb{CP}}^2} \cdot [\Sigma] + [\Sigma]^2) = 1 + ((g-2)(h+h') + E_1 + E_2) \cdot (h+h' - E_1 - E_2) + (h+h' - E_1 - E_2)^2)/2 = 1 + (2(g-2)+2)/2 = g$. We can notice from the intersection form of $W(g)$ that all rim tori can be chosen to have no intersections with the genus g surface in the pencil given by the homology class $h + h'$. Thus, the genus g fiber Σ can be chosen to be disjoint from the rim tori that descend to $W(g) \# 2\overline{\mathbb{CP}}^2$.

Let k be any nonnegative integer, and let $\mathbb{F}_k$ denote the kth Hirzebruch surface. Recall that $\mathbb{F}_k$ admits the structure of the holomorphic $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$ with two disjoint holomorphic sections Δ_{+k} and Δ_{-k} with $\Delta_{\pm k}^2 = \pm k$.

LEMMA 13

The genus g Lefschetz fibration on $Z(g)$ over $\mathbb{S}^2$ with the monodromy $(c_1 c_2 \cdots c_{2g})^{2(2g+1)} = 1$ can be obtained as the 2-fold cover of $\mathbb{F}_2$ branched over the disjoint union of a smooth curve C in the linear system $|(2g+1)\Delta_{+2}|$ and Δ_{-2}.

Proof

The Lefschetz fibration on $Z(g) \to \mathbb{CP}^1$ is obtained by composing the branched-cover map $Z(g) \to \mathbb{F}_2$ with the bundle map $\mathbb{F}_2 \to \mathbb{CP}^1$. A generic fiber is the double cover of a sphere fiber of $\mathbb{F}_2$ branched over $2g+2$ points. The monodromy of this Lefschetz fibration can be derived from the braid monodromy of the branch curve $C \cup \Delta_{-2}$. The fibration admits a holomorphic sphere section S with $S^2 = -1$, which is obtained by lifting Δ_{-2} to $Z(g)$. $\qquad\square$

3.3. Rational blowdown

In this section, we review the rational blowdown surgery introduced by Fintushel and Stern [14]. For details the reader is referred to [14] and [29].

Let $p \geq 2$, and let C_p be the smooth 4-manifold obtained by plumbing disk bundles over the 2-sphere according to the following linear diagram:

$$-(p+2) \quad -2 \qquad\qquad\qquad -2$$
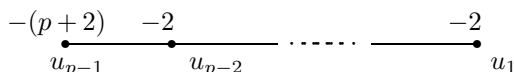
where each vertex u_i of the linear diagram represents a disk bundle over the 2-sphere with the given Euler number.

The boundary of C_p is the lens space $L(p^2, 1-p)$, which also bounds a rational ball B_p with $\pi_1(B_p) = \mathbb{Z}_p$ and $\pi_1(\partial B_p) \to \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X, then the rational blowdown manifold X_p is obtained by replacing C_p with B_p, that is, $X_p = (X \setminus C_p) \cup B_p$. If X and $X \setminus C_p$ are simply connected, then so is X_p. The following lemma is easy to check, so we omit the proof.

LEMMA 14

We have $b_2^+(X_p) = b_2^+(X)$, $e(X_p) = e(X) - (p-1)$, $\sigma(X_p) = \sigma(X) + (p-1)$, $c_1^2(X_p) = c_1^2(X) + (p-1)$, and $\chi_h(X_p) = \chi_h(X)$.

We now collect some theorems on rational blowdown for later use.

THEOREM 15 ([14], [29])

Suppose that X is a smooth 4-manifold with $b_2^+(X) > 1$ which contains a configuration C_p. If L is a Seiberg–Witten invariant basic class of X satisfying $L \cdot u_i = 0$ for any i with $1 \leq i \leq p-2$ and $L \cdot u_{p-1} = \pm p$, then L induces an SW basic class $\bar{L}$ of X_p such that $SW_{X_p}(\bar{L}) = SW_X(L)$.

THEOREM 16 ([14], [29])

If a simply connected smooth 4-manifold X contains a configuration C_p, then the Seiberg–Witten invariants of X_p are completely determined by those of X. That is, for any characteristic line bundle $\bar{L}$ on X_p with $SW_{X_p}(\bar{L}) \neq 0$, there exists a characteristic line bundle L on X such that $SW_X(L) = SW_{X_p}(\bar{L})$.

THEOREM 17 ([13], [11] $(P = 2)$, [12] $(P \geq 3)$)

Let ϱ, ϱ' be positive relators of Γ_g, and let X_ϱ, $X_{\varrho'}$ be the corresponding Lefschetz fibrations over $\mathbb{S}^2$, respectively. Suppose that ϱ' is obtained by applying a daisy substitution of type p to ϱ. Then, $X_{\varrho'}$ is a rational blowdown of X_ϱ along a configuration C_p. Therefore, we have

$$\sigma(X'_\varrho) = \sigma(X_\varrho) + (p-1) \qquad and \qquad e(X'_\varrho) = e(X_\varrho) - (p-1).$$

3.4. Knot surgery

In this section, we briefly review the knot surgery operation, which gives rise to mutually nondiffeomorphic manifolds. For details, the reader is referred to [15].

Let X be a 4-manifold with $b_2^+(X) > 1$ and containing a homologically essential torus T of self-intersection 0. Let $N(K)$ be a tubular neighborhood of K in $\mathbb{S}^3$, and let $T \times D^2$ be a tubular neighborhood of T in X. The knot surgery manifold

X_K is defined by $X_K = (X \setminus (T \times D^2)) \cup (\mathbb{S}^1 \times (\mathbb{S}^3 \setminus N(K)))$ where two pieces are glued in such a way that the homology class of $[pt \times \partial D^2]$ is identified with $[pt \times \lambda]$, where λ is the class of the longitude of knot K. Fintushel and Stern [15] proved the theorem that shows that Seiberg–Witten invariants of X_K can be completely determined by the Seiberg–Witten invariant of X and the Alexander polynomial of K. Moreover, if X and $X \setminus T$ are simply connected, then so is X_K.

THEOREM 18

Suppose that $\pi_1(X) = \pi_1(X \setminus T) = 1$, and suppose that T lies in a cusp neighborhood in X. Then X_K is homeomorphic to X and the Seiberg–Witten invariants of X_K are $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$, where $t = t_T$ (in the notation of [15]) and Δ_K is the symmetrized Alexander polynomial of K. If the Alexander polynomial $\Delta_K(t)$ of knot K is not monic, then X_K admits no symplectic structure. Moreover, if X is symplectic and K is a fibered knot, then X_K admits a symplectic structure.

4. Lemmas

In this section, we construct some relations by applying elementary transformations. These relations will be used to construct new relations obtained by daisy substitutions in Section 6.

Let $a_1, \ldots, a_k$ be a sequence of simple closed curves on an oriented surface such that a_i and a_j are disjoint if $|i - j| \geq 2$ and such that a_i and a_{i+1} intersect at one point. For simplicity of notation, we write a_i, $_f(a_i)$ instead of t_{a_i}, $t_{f(a_i)} = ft_{a_i}f^{-1}$, respectively. Moreover, write

$$b_i = {}_{a_{i+1}}(a_i) \qquad \text{and} \qquad \bar{b}_i = {}_{a_{i+1}^{-1}}(a_i).$$

Below we denote the arrangement using the conjugation (i.e., the cyclic permutation) and the arrangement using the relation (i) by $\xrightarrow{C}$ and $\xrightarrow{(i)}$, respectively. We recall the following relations:

$$a_{i+1} \cdot a_i \sim b_i \cdot a_{i+1} \qquad \text{and} \qquad a_i \cdot a_{i+1} \sim a_{i+1} \cdot \bar{b}_i.$$

In particular, we note that

$$a_i \cdot a_j \sim a_j \cdot a_i \quad \text{for } |i - j| > 1.$$

By drawing the curves, it is easy to verify that, for $m = 1, \ldots, k - 1$ and $i = m, \ldots, k - 1$,

$$a_k a_{k-1} \cdots a_{m+1} a_m (a_{i+1}) = a_i \qquad \text{and} \qquad a_m a_{m+1} \cdots a_{k-1} a_k (a_i) = a_{i+1}.$$

Using the relation $t_{f(c)} = ft_c f^{-1}$, we obtain the following:

$$(2) \qquad (a_k a_{k-1} \cdots a_{m+1} a_m) \cdot a_{i+1} \sim a_i \cdot (a_k a_{k-1} \cdots a_{m+1} a_m),$$

$$(3) \qquad (a_m a_{m+1} \cdots a_{k-1} a_k) \cdot a_i \sim a_{i+1} \cdot (a_m a_{m+1} \cdots a_{k-1} a_k).$$

LEMMA 19

For $2 \leq k$, we have the following relations:

(a) $(a_{k-1}a_{k-2}\cdots a_2 a_1) \cdot (a_k a_{k-1} \cdots a_2 a_1) \sim a_k^k \cdot \bar{b}_{k-1} \cdots \bar{b}_2 \bar{b}_1,$

(b) $(a_1 a_2 \cdots a_{k-1} a_k) \cdot (a_1 a_2 \cdots a_{k-2} a_{k-1}) \sim b_1 b_2 \cdots b_{k-1} \cdot a_k^k.$

Proof

The proof will be given by induction on k. Suppose that $k = 2$. Then, we have

$$a_1 a_2 a_1 \xrightarrow{(2)} a_2 a_1 a_2 \sim a_2^2 \cdot \bar{b}_1.$$

Hence, the conclusion of the lemma holds for $k = 2$.

   Let us assume inductively that the relation holds for $k = j$. Then,

$$(a_j a_{j-1} \cdots a_1) \cdot (a_{j+1} a_j \cdots a_1)$$

$$\sim a_j a_{j+1} \cdot (a_{j-1} \cdots a_1) \cdot (a_j \cdots a_1)$$

$$\sim a_j a_{j+1} \cdot a_j^{j+1} \cdot \bar{b}_{j-1} \cdots \bar{b}_1$$

$$\xrightarrow{(3)} a_{j+1}^{j+1} \cdot a_j a_{j+1} \cdot \bar{b}_{j-1} \cdots \bar{b}_1$$

$$\sim a_{j+1}^{j+1} \cdot a_{j+1} \cdot \bar{b}_j \cdot \bar{b}_{j-1} \cdots \bar{b}_1.$$

This proves part (a). The proof of (b) is similar and therefore omitted. □

LEMMA 20

Let $l \geq 0$. We define an element ϕ to be

$$\phi_l = a_{2l+1}^{l+1} a_{2l-1}^l \cdots a_5^3 a_3^2 a_1.$$

Let D and E be two products of right-handed Dehn twists, and write them as $D = d_1 \cdots d_{k_1}$ and $E = e_1 \cdots e_{k_2}$, respectively. If a word W_1 is obtained by applying a sequence of the conjugation and the elementary transformations to a word W_2, then we denote it by $\sim_C$. For $l \geq 1$, we have the following:

(a) $D \cdot a_{2l} \cdots a_2 a_1 \cdot a_{2l+1} \cdots a_2 a_1 \cdot E$

$$\sim_C {}_{\phi_l}(D) \cdot (a_{2l+1}^{l+1} \cdot a_{2l} \cdots a_2 a_1) \cdot (b_{2l} \cdots b_4 \cdot b_2) \cdot {}_{\phi_l}(E),$$

(b) $D \cdot a_1 a_2 \cdots a_{2l+1} \cdot a_1 a_2 \cdots a_{2l} \cdot E$

$$\sim_C {}_{\phi_l^{-1}}(D) \cdot (\bar{b}_2 \cdot \bar{b}_4 \cdots \bar{b}_{2l}) \cdot (a_1 a_2 \cdots a_{2l} \cdot a_{2l+1}^{l+1}) \cdot {}_{\phi_l^{-1}}(E),$$

where ${}_{\phi_l}(D) = {}_{\phi_l}(d_1) \cdots {}_{\phi_l}(d_{k_1})$ and ${}_{\phi_l}(E) = {}_{\phi_l}(e_1) \cdots {}_{\phi_l}(e_{k_2})$.

Proof

For $1 \leq m \leq 2l-1$ and $m \leq i \leq 2l-1$, we have the following equalities from (2) and (3):

(4) $\qquad (a_{2l} \cdots a_1 \cdot a_{2l+1} \cdots a_m) \cdot a_{i+2} \sim a_i \cdot (a_{2l} \cdots a_1 \cdot a_{2l+1} \cdots a_m),$

(5) $\qquad a_{i+2} \cdot (a_m \cdots a_{2l+1} \cdot a_1 \cdots a_{2l}) \sim (a_m \cdots a_{2l+1} \cdot a_1 \cdots a_{2l}) \cdot a_i.$

We first show (a). Since

$$_{\phi_{l-1}}(D) \cdot (a_{2l} a_{2l-1} \cdots a_1) \cdot (b_{2l} b_{2l-2} \cdots b_2) \cdot a_{2l+1}^{l+1} \cdot {}_{\phi_{l-1}}(E)$$

$$\xrightarrow{C} {}_{a_{2l+1}^{l+1} \phi_{l-1}}(D) \cdot a_{2l+1}^{l+1} \cdot (a_{2l} a_{2l-1} \cdots a_1) \cdot (b_{2l} b_{2l-2} \cdots b_2) \cdot {}_{a_{2l+1}^{l+1} \phi_{l-1}}(E)$$

$$= {}_{\phi_l}(D) \cdot a_{2l+1}^{l+1} \cdot (a_{2l} a_{2l-1} \cdots a_1) \cdot (b_{2l} b_{2l-2} \cdots b_2) \cdot {}_{\phi_l}(E),$$

it is sufficient to prove

$$D \cdot (a_{2l} a_{2l-1} \cdots a_1) \cdot (a_{2l+1} a_{2l} \cdots a_1) \cdot E$$

$$\sim {}_{\phi_{l-1}}(D) \cdot (a_{2l} a_{2l-1} \cdots a_1) \cdot (b_{2l} b_{2l-2} \cdots b_2) \cdot a_{2l+1}^{l+1} \cdot {}_{\phi_{l-1}}(E).$$

The proof is by induction on l. Suppose that $l = 0$. Then, we have

$$D \cdot a_2 a_1 \cdot a_3 a_2 a_1 \cdot E$$

$$\xrightarrow{C} {}_{a_1}(D) \cdot a_1 \cdot a_2 a_1 \cdot a_3 a_2 \cdot {}_{a_1}(E)$$

$$\xrightarrow{(4)} {}_{a_1}(D) \cdot a_2 a_1 \cdot a_3 a_2 \cdot a_3 \cdot {}_{a_1}(E)$$

$$\sim {}_{a_1}(D) \cdot a_2 a_1 \cdot b_2 \cdot a_3 \cdot a_3 \cdot {}_{a_1}(E).$$

Since $\phi_0 = a_1$, the conclusion of the lemma holds for $l = 0$.

Let us assume, inductively, that the relation holds for $l = j$. Note that since $\phi_{j-1}(a_{2j+2}) = a_{2j+2}$ and $\phi_{j-1}(a_{2j+3}) = a_{2j+3}$, we have

$$_{\phi_{j-1}}(D a_{2j+2} a_{2j+1} a_{2j+3} a_{2j+2}) = {}_{\phi_{j-1}}(D) a_{2j+2} a_{2j+1} a_{2j+3} a_{2j+2}.$$

Since a_{2j+3} is disjoint from $b_2, b_4, \ldots, b_{2j}$ and $\phi_j = a_{2j+1}^{j+1} \phi_{j-1}$, we have

$$D \cdot (a_{2j+2} a_{2j+1} a_{2j} \cdots a_1) \cdot (a_{2j+3} a_{2j+2} a_{2j+1} \cdots a_1) \cdot E$$

$$\sim D \cdot a_{2j+2} a_{2j+1} a_{2j+3} a_{2j+2} \cdot (a_{2j} \cdots a_1) \cdot (a_{2j+1} \cdots a_1) \cdot E$$

$$\sim {}_{\phi_{j-1}}(D) \cdot a_{2j+2} a_{2j+1} a_{2j+3} a_{2j+2} \cdot (a_{2j} \cdots a_1) \cdot (b_{2j} \cdots b_2) \cdot a_{2j+1}^{j+1} \cdot {}_{\phi_{j-1}}(E)$$

$$\xrightarrow{C} {}_{\phi_j}(D) \cdot a_{2j+1}^{j+1} \cdot a_{2j+2} a_{2j+1} a_{2j+3} a_{2j+2} \cdot (a_{2j} \cdots a_1) \cdot (b_{2j} \cdots b_2) \cdot {}_{\phi_j}(E)$$

$$\sim {}_{\phi_j}(D) \cdot a_{2j+1}^{j+1} \cdot (a_{2j+2} a_{2j+1} a_{2j} \cdots a_1) \cdot a_{2j+3} a_{2j+2} \cdot (b_{2j} \cdots b_2) \cdot {}_{\phi_j}(E)$$

$$\xrightarrow{(4)} {}_{\phi_j}(D) \cdot (a_{2j+2} a_{2j+1} a_{2j} \cdots a_1) \cdot a_{2j+3} a_{2j+2} \cdot a_{2j+3}^{j+1} \cdot (b_{2j} \cdots b_2) \cdot {}_{\phi_j}(E)$$

$$\sim {}_{\phi_j}(D) \cdot (a_{2j+2} a_{2j+1} a_{2j} \cdots a_1) \cdot b_{2j+2} \cdot a_{2j+3} \cdot a_{2j+3}^{j+1} \cdot (b_{2j} \cdots b_2) \cdot {}_{\phi_j}(E)$$

$$\sim {}_{\phi_j}(D) \cdot (a_{2j+2} a_{2j+1} a_{2j} \cdots a_1) \cdot (b_{2j+2} b_{2j} \cdots b_2) \cdot a_{2j+3}^{j+2} \cdot {}_{\phi_j}(E).$$

This proves part (a) of the lemma. The proof of part (b) is similar and left to the reader. $\qquad\square$

5. A lift of hyperelliptic relations

In this section, we construct a relation which gives a lift of a relation (and which is Hurwitz equivalent to $I(g)$) from Γ_g to Γ_g^1. This relation will be used in Section 6.

Suppose that $g \geq 2$. Let Σ_g^n be the surface of genus g with b boundary components $\delta_1, \delta_2, \ldots, \delta_n$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{2g}, \alpha_{2g+1}, \alpha'_{2g+1}$ and $\delta, \zeta_1, \ldots, \zeta_n$ be the simple closed curves as shown in Figure 4.

LEMMA 21
The following relation holds in Γ_g^n:

$$(\alpha_1 \alpha_2 \cdots \alpha_{2g})^{2g+1} = (\alpha_1 \alpha_2 \cdots \alpha_{2g-1})^{2g} \cdot \alpha_{2g} \cdots \alpha_2 \alpha_1 \alpha_1 \alpha_2 \cdots \alpha_{2g}.$$

Proof
The proof follows from the braid relations, $\alpha_i \cdot \alpha_{i+1} \cdot \alpha_i = \alpha_{i+1} \cdot \alpha_i \cdot \alpha_{i+1}$ and $\alpha_i \cdot \alpha_j = \alpha_j \cdot \alpha_i$ for $|i - j| > 1$. $\square$

LEMMA 22
The following relation holds in Γ_g^n:

$$\delta = (\alpha_{2g+1} \alpha_{2g} \cdots \alpha_2 \alpha_1 \alpha_1 \alpha_2 \cdots \alpha_{2g} \alpha'_{2g+1})^2.$$

In particular, since $\delta_1 = \delta$ for $n = 1$, this relation is a lift of $H(g)$ from Γ_g to Γ_g^1.

Proof
A regular neighborhood of $\alpha_1 \cup \alpha_2 \cup \cdots \cup a_{2g-1}$ is a subsurface of genus $g - 1$ with two boundary components, α_{2g+1} and α'_{2g+1}. Moreover, a regular neighborhood of $\alpha_1 \cup \alpha_2 \cup \cdots \cup a_{2g}$ is a subsurface of genus g with one boundary component δ. Then, it is well known that the following relations, called the *chain relations*, hold:
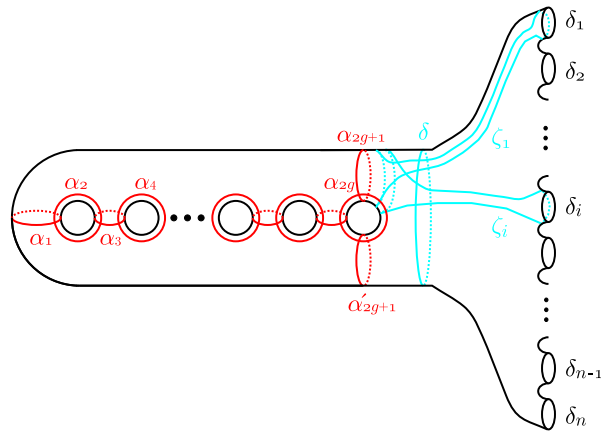

Figure 4. The curves $\alpha_1, \ldots, \alpha_{2g+1}, \alpha'_{2g+1}, \delta, \zeta_1, \ldots, \zeta_n$ on Σ_g^n and the boundary components $\delta_1, \ldots, \delta_n$ of Σ_g^n.

$$\alpha_{2g+1}\alpha'_{2g+1} = (\alpha_1 \cdots \alpha_{2g-1})^{2g}, \qquad \delta = (\alpha_1\alpha_2\alpha_3\cdots\alpha_{2g})^{4g+2}.$$

By applying the relations above and Lemma 21, we obtain the following relation:

$$\delta = (\alpha_{2g+1}\alpha'_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g})^2.$$

Since α'_{2g+1} is disjoint from α_{2g+1}, by the conjugation of α'_{2g+1}, we obtain the claim. If $n = 1$, then it is easily seen that this is a lift of $H(g)$ from Γ_g to Γ_g^1. This completes the proof. $\qquad\square$

LEMMA 23

The following relation holds in Γ_g^n:

$$(\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1})^2$$
$$\sim (\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1)^2 \cdot (\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1}) \cdot (\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1}).$$

Proof
By drawing a picture, we find that, for each $i = 1, 2, \ldots, 2g$,

$$\alpha_{2g+1}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g+1}(\alpha_i) = \alpha_i,$$
$$\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}(\alpha'_{2g+1}) = \alpha_{2g+1}.$$

These give the following relations:

(6) $\quad \alpha_{2g+1}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g+1} \cdot \alpha_i \sim \alpha_i \cdot \alpha_{2g+1}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g+1},$

(7) $\quad \alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g} \cdot \alpha'_{2g+1} \sim \alpha_{2g+1} \cdot \alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}.$

From these relations, we have

$$\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1}\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1}$$

$$\xrightarrow{(7)} \alpha_{2g+1} \cdot \alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1} \cdot \alpha_{2g}\cdots\alpha_2\alpha_1$$
$$\cdot \alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1}$$

$$\xrightarrow{(6)} (\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1) \cdot (\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1})$$
$$\cdot (\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1}).$$

This completes the proof. $\qquad\square$

6. New words in the mapping class group via daisy relation

We define ϕ in Γ_g^n to be

$$\phi = \alpha_{2g+1}^{g+1}\alpha_{2g-1}^g\cdots\alpha_5^3\alpha_3^2\alpha_1.$$

Note that $\phi(\alpha_{2i-1}) = \alpha_{2i-1}$ for each $i = 1, \ldots, g+1$ and $\phi(\alpha'_{2g+1}) = \alpha'_{2g+1}$. For simplicity of notation, we write

$$\beta_i = {}_{\alpha_{i+1}}(\alpha_i), \qquad \bar\beta_i = {}_{\alpha_{i+1}^{-1}}(\alpha_i), \qquad \gamma_{i+1} = {}_{\alpha_i}(\alpha_{i+1}), \qquad \bar\gamma_{i+1} = {}_{\alpha_i^{-1}}(\alpha_{i+1}).$$
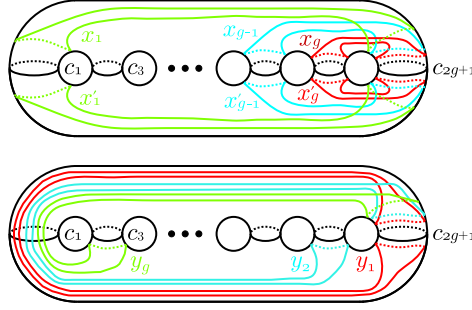
Figure 5. The curves $x_1, \ldots, x_g$, $x'_1, \ldots, x'_g$, and $y_1, \ldots, y_g$ on Σ_g.

We denote by φ, d_i, $\bar{d}_i$, e_{i+1}, and $\bar{e}_{i+1}$ the images of ϕ, β_i, $\bar{\beta}_i$, γ_{i+1} and $\bar{\gamma}_{i+1}$ under the map $\Gamma_g^n \to \Gamma_g$; that is,

$$\varphi = c_{2g+1}^{g+1} c_{2g-1}^g \cdots c_5^3 c_3^2 c_1,$$

$$d_i = {}_{c_{i+1}}(c_i), \qquad \bar{d}_i = {}_{c_{i+1}^{-1}}(c_i), \qquad e_{i+1} = {}_{c_i}(c_{i+1}), \qquad \text{and} \qquad \bar{e}_{i+1} = {}_{c_i^{-1}}(c_{i+1}).$$

Then, note that $\varphi(c_{2i-1}) = c_{2i-1}$ for each $i = 1, \ldots, g+1$. If a word W_1 is obtained by applying simultaneous conjugations by ψ to a word W_2, then we denote it by $\xrightarrow{\psi}$.

Let $x_1, \ldots, x_g$, $x'_1, \ldots, x'_g$, and $y_1, \ldots, y_g$ be the simple closed curves on Σ_g given in Figure 5. Moreover, we define $y_{g+1}, \ldots, y_{2g-1}$ to be $x_2, \ldots, x_g$, respectively. We take the following two daisy relators of type $g-1$ in Γ_g:

$$D_{g-1} := c_1^{-1} c_3^{-1} \cdots c_{2g-1}^{-1} \cdot c_{2g+1}^{-(g-2)} \cdot x_1 x_2 \cdots x_g,$$

$$D'_{g-1} := c_1^{-1} c_3^{-1} \cdots c_{2g-1}^{-1} \cdot c_{2g+1}^{-(g-2)} \cdot x'_1 x'_2 \cdots x'_g,$$

and the following daisy relator of type $2(g-1)$:

$$D_{2(g-1)} := c_{2g+1}^{-1} c_{2g-1}^{-1} \cdots c_5^{-1} c_3^{-1} \cdot c_3^{-1} c_5^{-1} \cdots c_{2g-1}^{-1} \cdot c_{2g+1}^{-2g+3} \cdot y_1 y_2 \cdots y_{2g-1}$$

$$= c_3^{-2} c_5^{-2} \cdots c_{2g-1}^{-2} \cdot c_{2g+1}^{-2g+2} \cdot y_1 y_2 \cdots y_{2g-1}.$$

Let $\chi_1, \ldots, \chi_g$ be the simple closed curves on Σ_g^n given in Figure 6. We denote by $\mathcal{D}_{g-1}$ the following daisy relator of type $g-1$ in Γ_g^n:

$$\mathcal{D}_{g-1} = \alpha_1^{-1} \alpha_3^{-1} \cdots \alpha_{2g-1}^{-1} \cdot \alpha_{2g+1}^{-(g-2)} \cdot \chi_1 \chi_2 \cdots \chi_g.$$
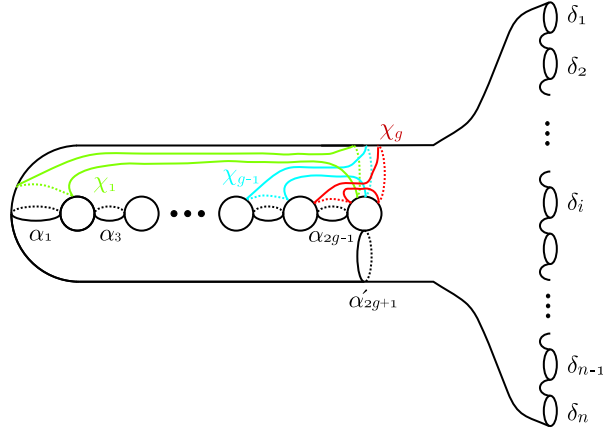
Since it is easily seen that α_i and χ_i are mapped to c_i and x_i under the map $\Gamma_g^n \to \Gamma_g$, we see that the image of this map of $\mathcal{D}_{g-1}$ is D_{g-1}.

THEOREM 24

There is a positive relator

$$H(g,1) = {}_{\varphi^{-1}}(\bar{d}_{2g}) \cdots {}_{\varphi^{-1}}(\bar{d}_2) {}_{\varphi^{-1}}(\bar{d}_1) \bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g} e_2 e_4 \cdots e_{2g} x_1 x_2 \cdots x_g c_{2g+1}^{2g+6},$$

which is obtained by applying D_{g-1}-substitution once to $H(g)$. Moreover, the Lefschetz fibration $f_{H(g,1)}$ has $2g+6$ disjoint sections of self-intersection -1.

Figure 6. The curves $\chi_1, \ldots, \chi_g$ on Σ_g^n.

Proof

Note that by Lemma 19(a), we have

$$(8) \qquad \begin{aligned} (\alpha_{2g+1}\alpha_{2g}\cdots\alpha_1)^2 &= \alpha_{2g+1}\cdot(\alpha_{2g}\cdots\alpha_1)\cdot(\alpha_{2g+1}\alpha_{2g}\cdots\alpha_1) \\ &\sim \alpha_{2g+1}^{2g+2}\bar{\beta}_{2g}\cdots\bar{\beta}_1. \end{aligned}$$

Moreover, one can check that

$$(9) \qquad \alpha_1\alpha_2\alpha_3\cdots\alpha_{2g} \sim (\gamma_2\gamma_4\gamma_6\cdots\gamma_{2g})\cdot(\alpha_1\alpha_3\alpha_5\cdots\alpha_{2g-1}).$$

Therefore, by Lemma 20(b), we have

$$(\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1)^2\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1}')$$

$$\xrightarrow{(8)} \alpha_{2g+1}^{2g+2}\bar{\beta}_{2g}\cdots\bar{\beta}_1\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g})\alpha_{2g+1}'$$

$$\sim_C {}_{\phi^{-1}}\{\alpha_{2g+1}^{2g+2}\cdot\bar{\beta}_{2g}\cdots\bar{\beta}_1\}\cdot(\bar{\beta}_2\bar{\beta}_4\cdots\bar{\beta}_{2g})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\cdot\alpha_{2g+1}^{g+1})\cdot\alpha_{2g+1}'$$

$$= \alpha_{2g+1}^{2g+2}\cdot{}_{\phi^{-1}}(\bar{\beta}_{2g})\cdots{}_{\phi^{-1}}(\bar{\beta}_1)\cdot(\bar{\beta}_2\bar{\beta}_4\cdots\bar{\beta}_{2g})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\cdot\alpha_{2g+1}^{g+1})\cdot\alpha_{2g+1}'$$

$$\xrightarrow{C} {}_{\phi^{-1}}(\bar{\beta}_{2g})\cdots{}_{\phi^{-1}}(\bar{\beta}_1)\cdot(\bar{\beta}_2\bar{\beta}_4\cdots\bar{\beta}_{2g})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g})\cdot\alpha_{2g+1}^{3g+3}\cdot\alpha_{2g+1}'$$

$$\xrightarrow{(9)} {}_{\phi^{-1}}(\bar{\beta}_{2g})\cdots{}_{\phi^{-1}}(\bar{\beta}_1)\cdot(\bar{\beta}_2\bar{\beta}_4\cdots\bar{\beta}_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot(\alpha_1\alpha_3\cdots\alpha_{2g-1})$$

$$\cdot\alpha_{2g+1}^{3g+3}\cdot\alpha_{2g+1}'.$$

Since δ is a central element of the group generated by $\alpha_1, \ldots, \alpha_{2g+1}, \alpha_{2g+1}'$, by Lemma 23, the operation $\sim_C$ is Hurwitz equivalent (e.g., see [5, Lemma 6]). We have the following relation in Γ_g^{2g+6} which is Hurwitz equivalent to the relation $\delta = (\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1}')^2$:

$$\delta = {}_{\phi^{-1}}(\bar{\beta}_{2g})\cdots{}_{\phi^{-1}}(\bar{\beta}_1)\cdot(\bar{\beta}_2\bar{\beta}_4\cdots\bar{\beta}_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot(\alpha_1\alpha_3\cdots\alpha_{2g-1})$$

$$\cdot\alpha_{2g+1}^{3g+3}\cdot\alpha_{2g+1}'.$$

By applying $\mathcal{D}_{g-1}$-substitution once to this relation, we have the following relation:

$$\delta = {}_{\phi^{-1}}(\bar{\beta}_{2g}) \cdots {}_{\phi^{-1}}(\bar{\beta}_1) \cdot (\bar{\beta}_2 \bar{\beta}_4 \cdots \bar{\beta}_{2g}) \cdot (\gamma_2 \gamma_4 \cdots \gamma_{2g}) \cdot (\chi_1 \chi_2 \cdots \chi_g)$$
$$\cdot \alpha_{2g+1}^{2g+5} \cdot \alpha_{2g+1}'.$$

Moreover, by $\alpha_{2g+1}^{2g+5} \cdot \alpha_{2g+1}' \cdot \delta_1 \delta_2 \cdots \delta_{2g+6} = \alpha_{2g+1}^{2g+5} \cdot \delta_1 \delta_2 \cdots \delta_{2g+6} \cdot \alpha_{2g+1}'$ and the daisy relation $\alpha_{2g+1}^{2g+5} \cdot \delta_1 \delta_2 \cdots \delta_{2g+6} \cdot \alpha_{2g+1}' = \zeta_1 \zeta_2 \cdots \zeta_{2g+6} \cdot \delta$, we obtain

$$\delta \cdot \delta_1 \delta_2 \cdots \delta_{2g+6}$$
$$= {}_{\phi^{-1}}(\bar{\beta}_{2g}) \cdots {}_{\phi^{-1}}(\bar{\beta}_1) \cdot (\bar{\beta}_2 \bar{\beta}_4 \cdots \bar{\beta}_{2g}) \cdot (\gamma_2 \gamma_4 \cdots \gamma_{2g}) \cdot (\chi_1 \chi_2 \cdots \chi_g)$$
$$\cdot \alpha_{2g+1}^{2g+5} \cdot \alpha_{2g+1}' \cdot \delta_1 \delta_2 \cdots \delta_{2g+6}$$
$$= {}_{\phi^{-1}}(\bar{\beta}_{2g}) \cdots {}_{\phi^{-1}}(\bar{\beta}_1) \cdot (\bar{\beta}_2 \bar{\beta}_4 \cdots \bar{\beta}_{2g}) \cdot (\gamma_2 \gamma_4 \cdots \gamma_{2g}) \cdot (\chi_1 \chi_2 \cdots \chi_g)$$
$$\cdot \zeta_1 \zeta_2 \cdots \zeta_{2g+6} \cdot \delta.$$

Therefore, we have the following relation in Γ_g^{2g+6}:

$$\delta_1 \cdot \delta_2 \cdots \delta_{2g+6}$$
$$= {}_{\phi^{-1}}(\bar{\beta}_{2g}) \cdots {}_{\phi^{-1}}(\bar{\beta}_1) \cdot (\bar{\beta}_2 \bar{\beta}_4 \cdots \bar{\beta}_{2g}) \cdot (\gamma_2 \gamma_4 \cdots \gamma_{2g}) \cdot (\chi_1 \chi_2 \cdots \chi_g)$$
$$\cdot \zeta_1 \zeta_2 \cdots \zeta_{2g+6}.$$

It is easily seen that $\zeta_1, \ldots, \zeta_{2g+6}$ are mapped to c_{2g+1} under the map $\Gamma_g^{2g+6} \to \Gamma_g$. This completes the proof. $\qquad\square$

THEOREM 25

There is a positive relator

$$H(g,2) = {}_{\varphi^{-2}}(\bar{e}_{2g}) \cdots {}_{\varphi^{-2}}(\bar{e}_4) {}_{\varphi^{-2}}(\bar{e}_2) {}_{\varphi^{-2}}(d_{2g}) \cdots {}_{\varphi^{-2}}(d_4) \cdot {}_{\varphi^{-2}}(d_2)$$
$$\cdot \bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g} e_2 e_4 \cdots e_{2g} (x_1 x_2 \cdots x_g)^2 c_{2g+1}^8$$

which is obtained by applying D_{g-1}-substitutions twice to $H(g)$. Moreover, the Lefschetz fibration $f_{H(g,2)}$ has eight disjoint sections of self-intersection -1.

Proof

Let D be a product of Dehn twists. Then, we note that by Lemma 20(a) we have

$$(10) \quad \begin{aligned} (\alpha_{2g+1} \alpha_{2g} \cdots \alpha_2 \alpha_1)^2 \cdot D &= \alpha_{2g+1} \cdot (\alpha_{2g} \cdots \alpha_1) \cdot (\alpha_{2g+1} \alpha_{2g} \cdots \alpha_1) \cdot D \\ &\sim_C \alpha_{2g+1}^{g+2} \cdot (\alpha_{2g} \cdots \alpha_2 \alpha_1) \cdot (\beta_{2g} \cdots \beta_4 \beta_2) \cdot {}_{\phi}(D). \end{aligned}$$

Moreover, we have

$$(11) \qquad \alpha_{2g} \cdots \alpha_3 \alpha_2 \alpha_1 \sim (\alpha_{2g-1} \cdots \alpha_5 \alpha_3 \alpha_1) \cdot (\bar{\gamma}_{2g} \cdots \bar{\gamma}_6 \bar{\gamma}_4 \bar{\gamma}_2).$$

Then, by Lemma 20(b) we have

$$(\alpha_{2g+1} \alpha_{2g} \cdots \alpha_2 \alpha_1)^2 \cdot (\alpha_1 \alpha_2 \cdots \alpha_{2g} \alpha_{2g+1}) \cdot (\alpha_1 \alpha_2 \cdots \alpha_{2g} \alpha_{2g+1}')$$
$$\xrightarrow{(10)} \alpha_{2g+1}^{g+2} \cdot (\alpha_{2g} \cdots \alpha_2 \alpha_1) \cdot (\beta_{2g} \cdots \beta_4 \beta_2)$$

$$\cdot{}_\phi\big\{(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1})\big\}$$

$$\xrightarrow{(11)} \alpha_{2g+1}^{g+2}\cdot(\alpha_{2g-1}\cdots\alpha_3\alpha_1)\cdot(\bar\gamma_{2g}\cdots\bar\gamma_4\bar\gamma_2)\cdot(\beta_{2g}\cdots\beta_4\beta_2)$$

$$\cdot{}_\phi\big\{(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1})\big\}$$

$$\xrightarrow{\phi^{-1}} {}_{\phi^{-1}}\big\{\alpha_{2g+1}^{g+2}\cdot(\alpha_{2g-1}\cdots\alpha_3\alpha_1)\cdot(\bar\gamma_{2g}\cdots\bar\gamma_4\bar\gamma_2)\cdot(\beta_{2g}\cdots\beta_4\beta_2)\big\}$$

$$\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha_{2g+1})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1})$$

$$\sim_C {}_{\phi^{-2}}\big\{\alpha_{2g+1}^{g+2}\cdot(\alpha_{2g-1}\cdots\alpha_3\alpha_1)\cdot(\bar\gamma_{2g}\cdots\bar\gamma_4\bar\gamma_2)\cdot(\beta_{2g}\cdots\beta_4\beta_2)\big\}$$

$$\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\cdot\alpha_{2g+1}^{g+1})\cdot\alpha'_{2g+1}$$

$$= \alpha_{2g+1}^{g+2}\cdot(\alpha_{2g-1}\cdots\alpha_3\alpha_1)\cdot{}_{\phi^{-2}}\big\{(\bar\gamma_{2g}\cdots\bar\gamma_4\bar\gamma_2)\cdot(\beta_{2g}\cdots\beta_4\beta_2)\big\}$$

$$\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\alpha_1\alpha_2\cdots\alpha_{2g}\cdot\alpha_{2g+1}^{g+1})\cdot\alpha'_{2g+1}$$

$$\xrightarrow{(9)} \alpha_{2g+1}^{g+2}\cdot(\alpha_{2g-1}\cdots\alpha_3\alpha_1)\cdot{}_{\phi^{-2}}\big\{(\bar\gamma_{2g}\cdots\bar\gamma_4\bar\gamma_2)\cdot(\beta_{2g}\cdots\beta_4\beta_2)\big\}$$

$$\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot(\alpha_1\alpha_3\cdots\alpha_{2g-1})\cdot\alpha_{2g+1}^{g+1}\cdot\alpha'_{2g+1}$$

$$\xrightarrow{C} {}_{\phi^{-2}}\big\{(\bar\gamma_{2g}\cdots\bar\gamma_4\bar\gamma_2)\cdot(\beta_{2g}\cdots\beta_4\beta_2)\big\}$$

$$\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot\alpha_1^2\alpha_3^2\cdots\alpha_{2g-1}^2\cdot\alpha_{2g+1}^{2g+3}\cdot\alpha'_{2g+1}$$

$$= \big({}_{\phi^{-2}}(\bar\gamma_{2g})\cdots{}_{\phi^{-2}}(\bar\gamma_4)\cdot{}_{\phi^{-2}}(\bar\gamma_2)\big)\cdot\big({}_{\phi^{-2}}(\beta_{2g})\cdots{}_{\phi^{-2}}(\beta_4)\cdot{}_{\phi^{-2}}(\beta_2)\big)$$

$$\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot\alpha_1^2\alpha_3^2\cdots\alpha_{2g-1}^2\cdot\alpha_{2g+1}^{2g+3}\cdot\alpha'_{2g+1}.$$

Note that, by Lemma 23, the operation $\xrightarrow{\phi^{-1}}$ is also Hurwitz equivalent from [5, Lemma 6]. We have the following relation in Γ_g^8, which is Hurwitz equivalent to the relation $\delta = (\alpha_{2g+1}\alpha_{2g}\cdots\alpha_2\alpha_1\alpha_1\alpha_2\cdots\alpha_{2g}\alpha'_{2g+1})^2$:

$$(12)\quad \begin{aligned}\delta &= \big({}_{\phi^{-2}}(\bar\gamma_{2g})\cdots{}_{\phi^{-2}}(\bar\gamma_4)\cdot{}_{\phi^{-2}}(\bar\gamma_2)\big)\cdot\big({}_{\phi^{-2}}(\beta_{2g})\cdots{}_{\phi^{-2}}(\beta_4)\cdot{}_{\phi^{-2}}(\beta_2)\big)\\ &\quad\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot\alpha_1^2\alpha_3^2\cdots\alpha_{2g-1}^2\cdot\alpha_{2g+1}^{2g+3}\cdot\alpha'_{2g+1}.\end{aligned}$$

By applying $\mathcal{D}_{g-1}$-substitutions twice to this relation, we have the following relation:

$$\begin{aligned}\delta &= \big({}_{\phi^{-2}}(\bar\gamma_{2g})\cdots{}_{\phi^{-2}}(\bar\gamma_4)\cdot{}_{\phi^{-2}}(\bar\gamma_2)\big)\cdot\big({}_{\phi^{-2}}(\beta_{2g})\cdots{}_{\phi^{-2}}(\beta_4)\cdot{}_{\phi^{-2}}(\beta_2)\big)\\ &\quad\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot(\chi_1\chi_2\cdots\chi_g)^2\cdot\alpha_{2g+1}^7\cdot\alpha'_{2g+1}.\end{aligned}$$

Moreover, by $\alpha_{2g+1}^7\cdot\alpha'_{2g+1}\cdot\delta_1\delta_2\cdots\delta_8 = \alpha_{2g+1}^7\cdot\delta_1\delta_2\cdots\delta_8\cdot\alpha'_{2g+1}$ and the daisy relation $\alpha_{2g+1}^7\cdot\delta_1\delta_2\cdots\delta_8\cdot\alpha'_{2g+1} = \zeta_1\zeta_2\cdots\zeta_8\cdot\delta$, we obtain

$$\begin{aligned}\delta\cdot\delta_1\delta_2\cdots\delta_8 &\\ =\; &\big({}_{\phi^{-2}}(\bar\gamma_{2g})\cdots{}_{\phi^{-2}}(\bar\gamma_4)\cdot{}_{\phi^{-2}}(\bar\gamma_2)\big)\cdot\big({}_{\phi^{-2}}(\beta_{2g})\cdots{}_{\phi^{-2}}(\beta_4)\cdot{}_{\phi^{-2}}(\beta_2)\big)\\ &\cdot(\bar\beta_2\bar\beta_4\cdots\bar\beta_{2g})\cdot(\gamma_2\gamma_4\cdots\gamma_{2g})\cdot(\chi_1\chi_2\cdots\chi_g)^2\\ &\cdot\alpha_{2g+1}^7\cdot\alpha'_{2g+1}\cdot\delta_1\delta_2\cdots\delta_8\end{aligned}$$

$$= \left({}_{\phi^{-2}}(\bar\gamma_{2g}) \cdots {}_{\phi^{-2}}(\bar\gamma_4) \cdot {}_{\phi^{-2}}(\bar\gamma_2) \right) \cdot \left({}_{\phi^{-2}}(\beta_{2g}) \cdots {}_{\phi^{-2}}(\beta_4) \cdot {}_{\phi^{-2}}(\beta_2) \right)$$

$$\cdot (\bar\beta_2 \bar\beta_4 \cdots \bar\beta_{2g}) \cdot (\gamma_2 \gamma_4 \cdots \gamma_{2g}) \cdot (\chi_1 \chi_2 \cdots \chi_g)^2 \cdot \zeta_1 \zeta_2 \cdots \zeta_8 \cdot \delta.$$

Therefore, we have the following relation in Γ_g^8:

$$\delta_1 \delta_2 \cdots \delta_8 = \left({}_{\phi^{-2}}(\bar\gamma_{2g}) \cdots {}_{\phi^{-2}}(\bar\gamma_4) \cdot {}_{\phi^{-2}}(\bar\gamma_2) \right) \cdot \left({}_{\phi^{-2}}(\beta_{2g}) \cdots {}_{\phi^{-2}}(\beta_4) \cdot {}_{\phi^{-2}}(\beta_2) \right)$$

$$\cdot (\bar\beta_2 \bar\beta_4 \cdots \bar\beta_{2g}) \cdot (\gamma_2 \gamma_4 \cdots \gamma_{2g}) \cdot (\chi_1 \chi_2 \cdots \chi_g)^2 \cdot \zeta_1 \zeta_2 \cdots \zeta_8.$$

It is easily seen that α_i and χ_i are mapped to c_i and x_i, respectively, and $\zeta_1, \ldots, \zeta_8$ are mapped to c_{2g+1} under the map $\Gamma_g^8 \to \Gamma_g$. This completes the proof. □

For $j = 1, \ldots, 2g$, we write

$$f_j = {}_{(c_{j-1}c_{j+1})^{-1}}(\bar d_j),$$

where $c_0 = 1$.

THEOREM 26

Let $g \geq 3$. Then, the monodromy of the hyperelliptic Lefschetz fibration given by the word $H(g) = 1$ can be conjugated to contain a daisy relations of type $2(g-1)$.

Proof

Since α_i and δ are mapped to c_i and 1 under the map $\Gamma_g^n \to \Gamma_g$, respectively, by (12) acquired in Theorem 25, we obtain the following relator:

$$1 = \left({}_{\varphi^{-2}}(\bar e_{2g}) \cdots {}_{\varphi^{-2}}(\bar e_4) \cdot {}_{\varphi^{-2}}(\bar e_2) \right) \cdot \left({}_{\varphi^{-2}}(d_{2g}) \cdots {}_{\varphi^{-2}}(d_4) \cdot {}_{\varphi^{-2}}(d_2) \right)$$

$$\cdot (\bar d_2 \bar d_4 \cdots \bar d_{2g}) \cdot (e_2 e_4 \cdots e_{2g}) \cdot c_1^2 c_3^2 \cdots c_{2g-1}^2 \cdot c_{2g+1}^{2g+4}.$$

This relator contains the $D_{2(g-1)}$-relator. This completes the proof. □

THEOREM 27

Let $g \geq 3$. Then, the monodromy of the Lefschetz fibration given by the word $I(g) = 1$ can be conjugated to contain

(i) *$g + 1$ daisy relations of type $g - 1$, and*

(ii) *$(g+1)/2$ (resp., $g/2$) daisy relations of type $2(g-1)$ for odd (resp., even) g.*

Proof

Let us first prove (i). Since $\varphi(c_{2i-1}) = c_{2i-1}$ for each $i = 1, \ldots, g + 1$, by Lemma 20(b) we have

$$(13) \qquad (c_1 c_2 \cdots c_{2g} c_{2g+1})^2 = (c_1 c_2 \cdots c_{2g} c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g}) \cdot c_{2g+1}$$

$$\sim_C (\bar d_2 \bar d_4 \cdots \bar d_{2g}) \cdot (c_1 c_2 \cdots c_{2g} \cdot c_{2g+1}^{g+1}) \cdot c_{2g+1}.$$

Moreover, we have

$$(14) \qquad c_1 c_2 c_3 \cdots c_{2g} \sim (e_2 e_4 e_6 \cdots e_{2g}) \cdot (c_1 c_3 c_5 \cdots c_{2g-1}).$$

Therefore, we have

$$(c_1 c_2 \cdots c_{2g+1})^{2g+2} \xrightarrow{(13)} (\bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g} \cdot c_1 c_2 \cdots c_{2g} \cdot c_{2g+1}^{g+2})^{g+1}$$

$$\xrightarrow{(14)} (\bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g} \cdot e_2 e_4 \cdots e_{2g} \cdot c_1 c_3 \cdots c_{2g-1} \cdot c_{2g+1}^{g+2})^{g+1}.$$

Then, we see that we can apply D'_{g-1}-substitution once and D_{g-1}-substitutions g times to $(\bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g} \cdot e_2 e_4 \cdots e_{2g} \cdot c_1 c_3 \cdots c_{2g-1} c_{2g+1}^{g+2})^{g+1}$. The reason that we apply D'_{g-1}-substitution once is to construct a minimal symplectic manifold $Y(g,k)$ in Theorem 31 (see the proof of Theorem 31 and Remark 36). This completes the proof of (i).

Next we prove (ii). By Lemma 19(b) we have

$$(15) \qquad \begin{aligned} (c_1 c_2 \cdots c_{2g} c_{2g+1})^2 &= (c_1 c_2 \cdots c_{2g} c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g}) \cdot c_{2g+1} \\ &\sim (d_1 d_2 \cdots d_{2g} \cdot c_{2g+1}^{2g+1}) \cdot c_{2g+1}. \end{aligned}$$

Moreover, we have

$$(16) \qquad (c_1 c_2 \cdots c_{2g+1})^2 \sim (c_1^2 c_3^2 \cdots c_{2g+1}^2) \cdot (f_2 f_4 \cdots f_{2g}) \cdot (\bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g}).$$

From the above relations, we have

$$(c_1 c_2 \cdots c_{2g} c_{2g+1})^4$$

$$\xrightarrow{(15)} (d_1 d_2 \cdots d_{2g} \cdot c_{2g+1}^{2g+2}) \cdot (c_1 c_2 \cdots c_{2g} c_{2g+1})^2$$

$$\xrightarrow{(16)} (d_1 d_2 \cdots d_{2g} \cdot c_{2g+1}^{2g+2}) \cdot (c_1^2 c_3^2 \cdots c_{2g+1}^2) \cdot (f_2 f_4 \cdots f_{2g}) \cdot (\bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g})$$

$$\sim (d_1 d_2 \cdots d_{2g}) \cdot (c_1^2 c_3^2 \cdots c_{2g-1}^2 c_{2g+1}^{2g+4}) \cdot (f_2 f_4 \cdots f_{2g}) \cdot (\bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g}).$$

From this, we see that $(c_1 c_2 \cdots c_{2g} c_{2g+1})^4$ can be conjugated to contain daisy relations of type $2(g-1)$. Therefore,

$$I(g) = \begin{cases} (c_1 c_2 \cdots c_{2g+1})^{4k} \cdot (c_1 c_2 \cdots c_{2g+1})^2 & g = 2k, \\ (c_1 c_2 \cdots c_{2g+1})^{4(k+1)} & g = 2k+1 \end{cases}$$

gives the proof of (ii). □

THEOREM 28

Let $g \geq 3$. Then, the monodromy of the Lefschetz fibration given by the word $G(g) = 1$ can be conjugated to contain g daisy relations of type $2(g-1)$.

Proof
By a similar argument to the proof of Theorem 27, we have

$$(c_1 \cdots c_{2g-1} c_{2g})^4$$

$$\sim (d_1 d_2 \cdots d_{2g-1}) \cdot (c_2^2 c_4^2 \cdots c_{2g-2}^2 c_{2g}^{2g+3}) \cdot (f_1 f_3 \cdots f_{2g-1}) \cdot (\bar{d}_1 \bar{d}_3 \cdots \bar{d}_{2g-1}).$$

Let $h := (a_1 \cdots a_{2g} a_{2g+1})$. Note that $h(a_i) = a_{i+1}$ for $i = 1, \ldots, 2g$. Then, we have

$$(c_1 \cdots c_{2g-1} c_{2g})^4$$

$$\sim (d_1 d_2 \cdots d_{2g-1}) \cdot (c_2^2 c_4^2 \cdots c_{2g-2}^2 c_{2g}^{2g+3}) \cdot (f_1 f_3 \cdots f_{2g-1}) \cdot (\bar{d}_1 \bar{d}_3 \cdots \bar{d}_{2g-1})$$

$$\xrightarrow{h} (d_2 d_3 \cdots d_{2g}) \cdot (c_3^2 c_5^2 \cdots c_{2g-1}^2 c_{2g+1}^{2g+3}) \cdot (f_2 f_3 \cdots f_{2g}) \cdot (\bar{d}_2 \bar{d}_4 \cdots \bar{d}_{2g}).$$

From this, we see that $(c_1 c_2 \cdots c_{2g} c_{2g})^4$ can be conjugated to contain daisy relations of type $2(g-1)$. Then $G(g) = (c_1 \cdots c_{2g-1} c_{2g})^{4g} \cdot (c_1 \cdots c_{2g-1} c_{2g})^2$ gives the proof. □

6.1. Nonhyperellipticity of our Lefschetz fibrations

The purpose of this section is to prove that all the Lefschetz fibrations obtained in this article via daisy substitutions are nonhyperelliptic. The proof will be obtained as a corollary of the more general theorem given below.

THEOREM 29

Let $g \geq 3$. Let $f_{\varrho_1} : X_{\varrho_1} \to \mathbb{S}^2$ be a genus g hyperelliptic Lefschetz fibration with only nonseparating vanishing cycles, and let ϱ_1 be a positive relator corresponding to f. Let k_1 and k_2 be nonnegative integers such that $k_1 + k_2 > 0$, and let k be a positive integer. Then we have the following.

(i) *If we can obtain a positive relator, denoted by ϱ_2, by applying D_{g-1}-substitutions k_1 times and D'_{g-1}-substitutions k_2 times to ϱ, then the genus g Lefschetz fibration $f_{\varrho_2} : X_{\varrho_2} \to \mathbb{S}^2$ is nonhyperelliptic.*

(ii) *If we can obtain a positive relator, denoted by ϱ_3, by applying $D_{2(g-1)}$-substitutions k times to ϱ, then the genus g Lefschetz fibration $f_{\varrho_3} : X_{\varrho_3} \to \mathbb{S}^2$ is nonhyperelliptic.*

COROLLARY 30

All our Lefschetz fibrations are nonhyperelliptic.

Proof of Theorem 29

Let s_0 be the number of nonseparating vanishing cycles of f_{ϱ_1}. Note that, by Theorem 7, we have $\sigma(X_{\varrho_1}) = -\frac{g+1}{2g+1} s_0$.

First, we assume that f_{ϱ_2} is a hyperelliptic Lefschetz fibration. The relators D_{g-1} and D'_{g-1} consist of only Dehn twists about nonseparating simple closed curves $c_1, c_3, \ldots, c_{2g+1}$ and $x_1, x'_1, \ldots, x_g, x'_g$ as in Figure 5. Therefore, we see that ϱ_2 consists of only right-handed Dehn twists about nonseparating simple closed curves, so f_{ϱ_2} has only nonseparating vanishing cycles. In particular, the number of nonseparating vanishing cycles of f_{ϱ_2} is $s_0 - \{(g-1) - 1\}(k_1 + k_2)$. By Theorem 7, we have

$$\sigma(X_{\varrho_2}) = -\frac{g+1}{2g+1}\big\{s_0 - (g-2)(k_1 + k_2)\big\}.$$

On the other hand, since the relators D_{g-1} and D'_{g-1} are daisy relators of type $g-1$, by Theorem 17, we have

$$\sigma(X_{\varrho_2}) = \sigma(X_{\varrho_1}) + (g-2)(k_1 + k_2) = -\frac{g+1}{2g+1}s_0 + (g-2)(k_1 + k_2).$$

We get a contradiction since the above equality does not hold for $g \geq 3$ and $k_1 + k_2 > 0$.

Next, we assume that f_{ϱ_3} is a hyperelliptic Lefschetz fibration. The relator $D_{2(g-1)}$ consists of Dehn twists about nonseparating simple closed curves $c_3, c_5, \ldots, c_{2g+1}$ and $y_2, y_3, \ldots, y_{2g-1}$ in Figure 5 and a Dehn twist about the separating simple closed curve y_1 in Figure 5. Note that $y_{g+1} = x_2, y_{g+2} = x_3, \ldots,$ $y_{2g-1} = x_g$, and note that y_1 separates Σ_g into two surfaces, one of which has genus 1. Therefore, f_{ϱ_3} has $s_0 - k\{2(g-1)\}$ nonseparating vanishing cycles and k separating vanishing cycles which are y_1. By Theorem 7, we have

$$\sigma(X_{\varrho_3}) = -\frac{g+1}{2g+1}\big\{s_0 - 2k(g-1)\big\} + \Big(\frac{4(g-1)}{2g+1} - 1\Big)k$$

$$= -\frac{g+1}{2g+1}s_0 + \frac{2g^2 + 2g - 7}{2g+1}k.$$

On the other hand, since the relator $D_{2(g-1)}$ is a daisy relator of type $2(g-1)$, by Theorem 17, we have

$$\sigma(X_{\varrho_3}) = \sigma(X_{\varrho_1}) + (2g-3)k = -\frac{g+1}{2g+1}s_0 + (2g-3)k.$$

Since $g \geq 3$, we have

$$\Big(-\frac{g+1}{2g+1}s_0 + (2g-3)k\Big) - \Big(-\frac{g+1}{2g+1}s_0 + \frac{2g^2 + 2g - 7}{2g+1}k\Big)$$

$$= \frac{2(g-1)(g-2)}{2g+1}k > 0.$$

This is a contradiction to the above equality. □

7. Constructing exotic 4-manifolds

The purpose of this section is to show that the symplectic 4-manifolds obtained in Theorem 27(i) are irreducible. Moreover, by performing the knot surgery operation along a homologically essential torus on these symplectic 4-manifolds, we obtain infinite families of mutually nondiffeomorphic irreducible smooth structures.

THEOREM 31

Let $g \geq 3$, and let M be one of the following 4-manifolds: $(g^2 - g + 1)\mathbb{CP}^2 \# (3g^2 + 3g + 3 - (g-2)k)\overline{\mathbb{CP}}^2$ for $k = 2, \ldots, g+1$. There exists an irreducible symplectic 4-manifold $Y(g,k)$ homeomorphic but not diffeomorphic to M that can be

obtained from the genus g Lefschetz fibration on $Y(g)$ over $\mathbb{S}^2$ with the monodromy $(c_1 c_2 \cdots c_{2g+1})^{2g+2} = 1$ in the mapping class group Γ_g by applying k daisy substitutions of type $g - 1$.

Proof
Let $Y(g,k)$ denote the symplectic 4-manifold obtained from $Y(g) = W(g) \# 2\overline{\mathbb{CP}}^2$ by applying $k - 1$ D_{g-1}-substitutions and one D'_{g-1}-substitution as in Theorem 27. Applying Lemma 14, we compute the topological invariants of $Y(g,k)$ as

$$e\big(Y(g,k)\big) = e\big(W(g)\#2\overline{\mathbb{CP}}^2\big) - k(g-2) = 2(2g^2 + g + 3) - k(g-2),$$

$$\sigma\big(Y(g,k)\big) = \sigma\big(W(g)\#2\overline{\mathbb{CP}}^2\big) + k(g-2) = -2(g+1)^2 + k(g-2).$$

Using the factorization of the global monodromy in terms of right-handed Dehn twists of the genus g Lefschetz fibration on $Y(g,k)$ (see Theorem 27(i)), it is easy to check that $\pi_1(Y(g,k)) = 1$.

Next, we show that $Y(g,k)$ is a nonspin 4-manifold. Let $c_1, \ldots, c_{2g+1}$ be the curves in Figure 2, and let $x_1, \ldots, x_g, x'_1, \ldots, x'_g$ be the curves in Figure 5. Note that $\bar{d}_i = {}_{c_{i+1}^{-1}}(c_i)$ and $e_{i+1} = {}_{c_i}(c_{i+1})$. The vanishing cycles of Lefschetz fibrations in Theorem 27 include the curves $\bar{d}_{2i}$, e_{2i} for $i = 1, \ldots, g$, x_j, x'_j for $j = 1, \ldots, g$, and c_{2g+1}. In $H_1(\Sigma_g; \mathbb{Z}_2)$, we find that $\bar{d}_{2g} = c_{2g+1} + c_{2g}$, $e_{2g} = c_{2g-1} + c_{2g}$, and $x_g = c_{2g-1} + c_{2g+1}$. Therefore, we have $\bar{d}_{2g} + e_{2g} = x_g$. In the notation of Theorem 8, we have $l = 2$ and $\bar{d}_{2g} \cdot e_{2g} = 0$. Therefore, $2 + \bar{d}_{2g} \cdot e_{2g} \equiv 0$ (mod 2).

By Theorem 9, $Y(g,k)$ is nonspin, and thus, it has an odd intersection form. By Freedman's classification (see [16]), we see that $Y(g,k)$ is homeomorphic to $(g^2 - g + 1)\mathbb{CP}^2 \# (3g^2 + 3g + 3 - (g-2)k)\overline{\mathbb{CP}}^2$.

Next, using the fact that $W(g)$ is a minimal complex surface of general type with $b_2^+ > 1$ and the blow-up formula for the Seiberg–Witten function (see [15]), we compute $SW_{W(g)\#2\overline{\mathbb{CP}}^2} = SW_{W(g)} \cdot \prod_{j=1}^2 (e^{E_i} + e^{-E_i}) = (e^{K_{W(g)}} + e^{-K_{W(g)}})(e^{E_1} + e^{-E_1})(e^{E_2} + e^{-E_2})$, where E_i denotes the exceptional class of the ith blowup. By the above formula, the SW basic classes of $W(g)\#2\overline{\mathbb{CP}}^2$ are given by $\pm K_{W(g)} \pm E_1 \pm E_2$, and the values of the Seiberg–Witten invariants on these classes are ± 1. Notice that, by [14, Corollary 8.6], $Y(g,k)$ has Seiberg–Witten simple type. Furthermore, by applying Theorems 15 and 16, we completely determine the Seiberg–Witten invariants of $Y(g,k)$ by using the basic classes and invariants of $W(g)\#2\overline{\mathbb{CP}}^2$. Up to sign, the symplectic manifold $Y(g,k)$ has only one basic class which descends from the $\pm$ canonical class of $Y(g)$ (see the detailed explanation below). By Theorem 16 or Taubes's work [36], the values of the Seiberg–Witten function on these basic classes, $\pm K_{Y(g,k)}$, are ± 1.

In what follows, we spell out the details of the above discussion. By Theorems 15 and 16, we can determine the Seiberg–Witten invariants of $Y(g,k)$ by computing the algebraic intersection number of the basic classes $\pm K_{W(g)} \pm E_1 \pm E_2$ of $W(g)\#2\overline{\mathbb{CP}}^2$, with the classes of spheres of k disjoint C_{g-1} configurations in $Y(g)$. Notice that the leading spheres of the configurations C_{g-1} are the components

of the singular fibers of $Y(g)$. By looking at the regions on the genus g surface, where the rational blowdowns along C_{g-1} are performed, and the location of the base points of the genus g pencil, we compute the algebraic intersection numbers as follows. Let S_1^j denote the homology class of the $-(g+1)$-sphere of the jth configurations C_{g-1}, and let $S_2^j, \ldots, S_{g-2}^j$ be the homology classes of (-2)-spheres of C_{g-1} in $W(g)\#2\overline{\mathbb{CP}}^2$, where $j = 1, 2$. These two rational blowdowns along C_{g-1} are chosen such that they correspond to one D_{g-1}-substitution and one D'_{g-1}-substitution as in Theorem 27(i).

We have $S_1^1 \cdot E_1 = 1$, $S_1^1 \cdot E_2 = 0$, $S_1^2 \cdot E_2 = 1$, $S_1^2 \cdot E_1 = 0$, $S_1^j \cdot K_{Wg} = g - 2$, and the canonical divisor does not intersect with S_i^j for $2 \le i \le g - 1$. Consequently, $S_1^j \cdot \pm(K_{Wg} + E_1 + E_2) = \pm(g-1)$ for $j = 1, 2$, and $S_1^j \cdot (\pm K_{Wg} \pm E_1 \mp E_2) \ne \pm(g-1)$ for one j. Observe that among the eight basic classes $\pm K_{Wg} \pm E_1 \pm E_2$, only $K_{Wg} + E_1 + E_2$ and $-(K_{Wg} + E_1 + E_2)$ have algebraic intersection $\pm(g-1)$ with $-(g+1)$-spheres of C_{g-1}. Thus, Theorem 15 implies that these are the only two basic classes that descend to $Y(g, 2)$ and, consequently, to $Y(g, k)$ from $W(g)\#2\overline{\mathbb{CP}}^2$.

By invoking the connected sum theorem for Seiberg–Witten invariants, we see that the SW function is trivial for $(g^2 - g + 1)\mathbb{CP}^2\#(3g^2 + 3g + 3 - (g-2)k)\overline{\mathbb{CP}}^2$. Since the Seiberg–Witten invariants are diffeomorphism invariants, $Y(g, k)$ is not diffeomorphic to $(g^2 - g + 1)\mathbb{CP}^2\#(3g^2 + 3g + 3 - (g-2)k)\overline{\mathbb{CP}}^2$.

The minimality of $Y(g, k)$ is a consequence of the fact that $Y(g, k)$ has no two Seiberg–Witten basic classes K and K' such that $(K - K')^2 = -4$. Notice that $\pm K_{Y(g,k)}$ are the only basic classes of $Y(g, k)$, and note that $(K_{Y(g,k)}^2 - (-K_{Y(g,k)}))^2 = 4(K_{Y(g,k)}^2) \ge 0$. Thus, we conclude that $Y(g, k)$ is symplectically minimal. Furthermore, since symplectic minimality implies irreducibility for simply connected 4-manifolds (see [21]), we deduce that $Y(g, k)$ is also smoothly irreducible. □

The analogous for $g = 2$, using lantern substitution, was proved in [3] (see Theorem 17).

THEOREM 32

There exists an infinite family of irreducible symplectic and an infinite family of irreducible nonsymplectic pairwise nondiffeomorphic 4-manifolds all homeomorphic to $Y(g, k)$.

Proof

We have that $Y(g, k)$ contains $g(g-1)$ Lagrangian tori which are disjoint from the singular fibers of the genus g Lefschetz fibration on $Y(g, k)$. These tori descend from $W(g)$ (see Example 12) and survive in $Y(g, k)$ after the rational blowdowns along C_{g-1}. These tori are Lagrangian, but we can perturb the symplectic form on $Y(g, k)$ so that one of these tori, say, T, becomes symplectic. Moreover, $\pi_1(Y(g, k) \setminus T) = 1$, which follows from Van Kampen's theorem by using the facts that $\pi_1(Y(g, k)) = 1$ and any rim torus has a dual (-2)-sphere (see [20,

Proposition 1.2] or Gompf [18, p. 564]). Hence, we have a symplectic torus T in $Y(g,k)$ of self-intersection 0 such that $\pi_1(Y(g,k) \setminus T) = 1$. By performing a knot surgery on T, inside $Y(g,k)$, we obtain an irreducible 4-manifold $Y(g,k)_K$ that is homeomorphic to $Y(g,k)$. By varying our choice of the knot K, we can realize infinitely many pairwise nondiffeomorphic 4-manifolds, either symplectic or nonsymplectic.   □

REMARK 33

We obtain analogous results to Theorem 31 for the genus g Lefschetz fibrations obtained in Theorem 28. The total space $Z(g)$ of the Lefschetz fibration with the monodromy $(c_1 c_2 \cdots c_{2g+1})^{2(2g+1)} = 1$ in the mapping class group Γ_g is the complex surface of general type with $b_2^+ > 1$ and the single blowup of a minimal complex surface (see [19, Section 8.4, pp. 320–322]). The computation of the Seiberg–Witten invariants follows the same line of argument as that of Theorem 31. In fact, the SW computation is simpler, since $Z(g)$ only admits one pair of basic classes. Also, the results of our article can be easily extendable to the Lefschetz fibrations with monodromies $(c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^{2n} = 1$, $(c_1 c_2 \cdots c_{2g} c_{2g+1})^{(2g+2)n} = 1$, and $(c_1 c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)n} = 1$ for $n \geq 2$. Since the computations are lengthy, we will not present them here.

REMARK 34

Note that all the Lefschetz fibrations constructed in our article are nonspin. The fibrations obtained in Theorems 24 and 25 admit a section of self-intersection -1. The fibrations in Theorem 26 and 28 contain separating vanishing cycles. (The curve y_1 in Figure 5 is separating.) Thus, the total spaces are nonspin. In general, if we apply the daisy substitution of type $2(g-1)$ to a positive relator in Γ_g, then the resulting Lefschetz fibration always contains separating vanishing cycles. The fibrations in Theorem 27 do not contain any separating vanishing cycles, but they are nonspin due to Stipsicz's criteria (see proof of Theorem 31).

REMARK 35

It would be interesting to know if an analogue of Theorem 31 holds for the Lefschetz fibrations of Theorem 25. Their corresponding monodromies are obtained by applying D_{g-1}-substitutions twice to $H(g)$. In the opposite direction, we can prove that the total spaces of the Lefschetz fibrations of Theorem 24, whose monodromies are obtained by applying one D_{g-1}-substitution to $H(g)$, are blowups of the complex projective plane. Notice that by Theorem 24 they admit at least $(2g+6)$-sphere sections of self-intersection -1. By using a result of Y. Sato [32, Theorem 1.2, p. 194], we see that the total spaces of these Lefschetz fibrations are diffeomorphic to $\mathbb{CP}^2 \# (3g+5)\overline{\mathbb{CP}^2}$.

REMARK 36

In the proof of Theorem 27(i), if we apply k D_{g-1}-substitutions for $k = 1, \ldots, g+1$ without applying a D'_{g-1}-substitution, then the Lefschetz fibrations over $\mathbb{S}^2$ given

by the resulting relations admit a section of self-intersection −1. (That is, the total spaces of the fibrations are nonminimal.)

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