Toward a geometric analogue of Dirichlet's unit theorem

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Abstract In this article, we propose a geometric analogue of Dirichlet's unit theorem on arithmetic varieties; that is, if X is a normal projective variety over a finite field and D is a pseudo-effective $\mathbb{Q}$-Cartier divisor on X, does it follow that D is $\mathbb{Q}$-effective? We also give affirmative answers on an abelian variety and a projective bundle over a curve.

Introduction

Let K be a number field, and let O_K be the ring of integers in K. Let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. For $\sigma \in K(\mathbb{C})$, the complex conjugation of σ is denoted by $\overline{\sigma}$; that is, $\overline{\sigma}(x) = \overline{\sigma(x)}$ $(x \in K)$. Here we define Ξ_K and Ξ_K^0 to be

$$\begin{cases} \Xi_K := \{\xi \in \mathbb{R}^{K(\mathbb{C})} \mid \xi(\sigma) = \xi(\overline{\sigma}) \ (\forall \sigma)\}, \\ \Xi_K^0 := \{\xi \in \Xi_K \mid \sum_{\sigma \in K(\mathbb{C})} \xi(\sigma) = 0\}. \end{cases}$$

The Dirichlet unit theorem asserts that the group $O_K^\times$ consisting of units in O_K is a finitely generated abelian group of rank $s := \dim_{\mathbb{R}} \Xi_K^0$.

Let us consider the homomorphism $L : K^\times \to \mathbb{R}^{K(\mathbb{C})}$ given by

$$L(x)(\sigma) := \log\bigl|\sigma(x)\bigr| \quad \bigl(x \in K^\times, \sigma \in K(\mathbb{C})\bigr).$$

It is easy to see the following.

(a) For a compact set B in $\mathbb{R}^{K(\mathbb{C})}$, the set $\{x \in O_K^\times \mid L(x) \in B\}$ is finite.
(b) $L : K^\times \to \mathbb{R}^{K(\mathbb{C})}$ extends to $L_{\mathbb{R}} : K^\times \otimes \mathbb{R} \to \mathbb{R}^{K(\mathbb{C})}$.
(c) $L_{\mathbb{R}} : O_K^\times \otimes \mathbb{R} \to \mathbb{R}^{K(\mathbb{C})}$ is injective.
(d) $L_{\mathbb{R}}(O_K^\times \otimes \mathbb{R}) \subseteq \Xi_K^0$.

By using (a) and (c), we can see that $O_K^\times$ is a finitely generated abelian group. The most essential part of the Dirichlet unit theorem is to show that $O_K^\times$ is of rank s, which is equivalent to seeing that, for any $\xi \in \Xi_K^0$, there is $x \in O_K^\times \otimes \mathbb{R}$ with $L_{\mathbb{R}}(x) = \xi$.

To understand the equality $L_{\mathbb{R}}(x) = \xi$ in terms of Arakelov geometry, let us introduce several notations for arithmetic divisors on the arithmetic curve $\mathrm{Spec}(O_K)$. An arithmetic $\mathbb{R}$-divisor on $\mathrm{Spec}(O_K)$ is a pair (D, ξ) consisting of

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an $\mathbb{R}$-divisor D on $\operatorname{Spec}(O_K)$ and $\xi \in \Xi_K$. We often denote the pair (D, ξ) by $\overline{D}$. The arithmetic principal $\mathbb{R}$-divisor $\widehat{(x)}_{\mathbb{R}}$ of $x \in K^{\times} \otimes \mathbb{R}$ is the arithmetic $\mathbb{R}$-divisor given by

$$\widehat{(x)}_{\mathbb{R}} := \Big(\sum_P \operatorname{ord}_P(x)[P], -2L_{\mathbb{R}}(x) \Big),$$

where P runs over the set of all maximal ideals of O_K and

$$\operatorname{ord}_P(x) := a_1 \operatorname{ord}_P(x_1) + \cdots + a_r \operatorname{ord}_P(x_r)$$

for $x = x_1^{a_1} \cdots x_r^{a_r}$ ($x_1, \dots, x_r \in K^{\times}$ and $a_1, \dots, a_r \in \mathbb{R}$). The arithmetic degree $\widehat{\deg}(\overline{D})$ of an arithmetic $\mathbb{R}$-divisor $\overline{D} = (\sum_P a_P[P], \xi)$ is defined to be

$$\widehat{\deg}(\overline{D}) := \sum_P a_P \log \#(O_K/P) + \frac{1}{2} \sum_{\sigma \in K(\mathbb{C})} \xi(\sigma).$$

Note that

$$\widehat{\deg}\big(\widehat{(x)}_{\mathbb{R}}\big) = 0 \quad (x \in K^{\times} \otimes \mathbb{R})$$

by virtue of the product formula. Further, $\overline{D} = (\sum_P a_P[P], \xi)$ is said to be effective if $a_P \geq 0$ for all P and $\xi(\sigma) \geq 0$ for all σ.

In [17, Section 3.4], we proved the following.

(0.1) "If $\widehat{\deg}(\overline{D}) \geq 0$, then $\overline{D} + \widehat{(x)}_{\mathbb{R}}$ is effective for some $x \in K^{\times} \otimes \mathbb{R}$."

This implies the essential part of the Dirichlet unit theorem. Indeed, we set $\overline{D} = (0, \xi)$ for $\xi \in \Xi_K^0$. As $\widehat{\deg}(\overline{D}) = 0$, by (0.1), $\overline{D} + \widehat{(y)}_{\mathbb{R}}$ is effective for some $y \in K^{\times} \otimes \mathbb{R}$, and hence $\overline{D} + \widehat{(y)}_{\mathbb{R}} = (0, 0)$ because $\widehat{\deg}(\overline{D} + \widehat{(y)}_{\mathbb{R}}) = 0$. Here we set $y = u_1^{a_1} \cdots u_r^{a_r}$ such that $u_1, \dots, u_r \in K^{\times}$, $a_1, \dots, a_r \in \mathbb{R}$, and $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. By using the linear independence of $a_1, \dots, a_r$ over $\mathbb{Q}$, $\operatorname{ord}_P(y) = 0$ implies that $\operatorname{ord}_P(u_i) = 0$ for all maximal ideals P of O_K and $i = 1, \dots, r$; that is, $u_i \in O_K^{\times}$ for $i = 1, \dots, r$. Therefore, $\xi = L_{\mathbb{R}}(y^2)$ and $y \in O_K^{\times} \otimes \mathbb{R}$, as required. In this sense, (0.1) is an Arakelov-theoretic interpretation of the classical Dirichlet unit theorem.

In [17] and [18], we considered a higher-dimensional analogue of (0.1). In the higher-dimensional case, the condition "$\widehat{\deg}(\overline{D}) \geq 0$" should be replaced by the pseudo-effectivity of $\overline{D}$. Of course, this analogue is not true in general (cf. [5]). It is, however, a very interesting problem to find a sufficient condition for the existence of an arithmetic small $\mathbb{R}$-section, that is, an element x such that

$$x = x_1^{a_1} \cdots x_r^{a_r} \quad (x_1, \dots, x_r \text{ are rational functions and } a_1, \dots, a_r \in \mathbb{R})$$

and $\overline{D} + \widehat{(x)}_{\mathbb{R}}$ is effective. For example, in [17] and [18], we proved that if D is numerically trivial and $\overline{D}$ is pseudo-effective, then $\overline{D}$ has an arithmetic small $\mathbb{R}$-section. In this article, we would like to consider a geometric analogue of the Dirichlet unit theorem in the above sense.

Let X be a normal projective variety over an algebraically closed field k. Let $\operatorname{Div}(X)$ denote the group of Cartier divisors on X. Let $\mathbb{K}$ be either the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{R}$ of real numbers. We define $\operatorname{Div}(X)_{\mathbb{K}}$ to be

$\mathrm{Div}(X)_{\mathbb{K}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$, whose elements are called $\mathbb{K}$-*Cartier divisors* on X. For $\mathbb{K}$-Cartier divisors D_1 and D_2, we say that D_1 is $\mathbb{K}$-*linearly equivalent* to D_2, which is denoted by $D_1 \sim_{\mathbb{K}} D_2$, if there are nonzero rational functions $\phi_1, \dots, \phi_r$ on X and $a_1, \dots, a_r \in \mathbb{K}$ such that

$$D_1 - D_2 = a_1(\phi_1) + \cdots + a_r(\phi_r).$$

Let D be a $\mathbb{K}$-Cartier divisor on X. We say that D is *big* if there is an ample $\mathbb{Q}$-Cartier divisor A on X such that $D - A$ is $\mathbb{K}$-linearly equivalent to an effective $\mathbb{K}$-Cartier divisor. Further, D is said to be *pseudo-effective* if $D + B$ is big for any big $\mathbb{K}$-Cartier divisor B on X. Note that if D is $\mathbb{K}$-*effective* (i.e., D is $\mathbb{K}$-linearly equivalent to an effective $\mathbb{K}$-Cartier divisor), then D is pseudo-effective. The converse of the above statement holds on toric varieties (e.g., [4, Proposition 4.9]). However, it is not true in general. In the case where k is uncountable (e.g., $k = \mathbb{C}$), several examples are known such as nontorsion numerically trivial invertible sheaves and Mumford's example on a minimal ruled surface (cf. [8, Chapter 1, Example 10.6], [14]). Nevertheless, we would like to propose the following question.

QUESTION 0.2 ($\mathbb{K}$-VERSION)

We assume that k is an algebraic closure of a finite field. If a $\mathbb{K}$-Cartier divisor D on X is pseudo-effective, does it follow that D is $\mathbb{K}$-effective?

This question is a geometric analogue of the fundamental question introduced in [17]. In this sense, it turns out to be a geometric Dirichlet's unit theorem if it is true, so that we often say that a $\mathbb{K}$-Cartier divisor D has the *Dirichlet property* if D is $\mathbb{K}$-effective. Note that the $\mathbb{R}$-version implies the $\mathbb{Q}$-version (cf. Proposition 1.5). Moreover, the $\mathbb{R}$-version does not hold in general. In Example 3.2, we give an example, so that, for the $\mathbb{R}$-version, the question should be

"Under what conditions does it follow that D is $\mathbb{K}$-effective?"

Further, the $\mathbb{Q}$-version implies the following question due to Keel (cf. [10, Question 0.9], Remark 2.4). The similar arguments on an algebraic surface are discussed in the recent article by Langer [12, Conjectures 1.7–1.9 and Lemma 1.10].

QUESTION 0.3 (S. KEEL)

We assume that k is an algebraic closure of a finite field and that X is an algebraic surface over k. Let D be a Cartier divisor on X. If $(D \cdot C) > 0$ for all irreducible curves C on X, is D ample?

By virtue of the Zariski decomposition, Question 0.2 on an algebraic surface is equivalent to asking the following:

"If D is nef, then is D $\mathbb{K}$-effective?"

One might expect that D is semiample (cf. [10, Question 0.8.2]). However, Totaro [23, Theorem 6.1] found a Cartier divisor D on an algebraic surface over a finite field such that D is nef but not semiample. Totaro's example does not give a counterexample to our question because we assert only the $\mathbb{Q}$-effectivity in Question 0.2. Inspired by Biswas and Subramanian [3], we have the following partial answer to the above question.

THEOREM 0.4

We assume that k is an algebraic closure of a finite field. Let C be a smooth projective curve over k, and let E be a locally free sheaf of rank r on C. Let $\mathbb{P}(E)$ be the projective bundle of E; that is, $\mathbb{P}(E) := \operatorname{Proj}(\bigoplus_{m=0}^{\infty} \operatorname{Sym}^m(E))$. If D is a pseudo-effective $\mathbb{K}$-Cartier divisor on $\mathbb{P}(E)$, then D is $\mathbb{K}$-effective.

In addition to the above result, we can also give an affirmative answer to the $\mathbb{Q}$-version of Question 0.2 on abelian varieties.

PROPOSITION 0.5

We assume that k is an algebraic closure of a finite field. Let A be an abelian variety over k. If D is a pseudo-effective $\mathbb{Q}$-Cartier divisor on A, then D is $\mathbb{Q}$-effective.

1. Preliminaries

Let k be an algebraic closed field. Let C be a smooth projective curve over k, and let E be a locally free sheaf of rank r on C. The projective bundle $\mathbb{P}(E)$ of E is given by

$$\mathbb{P}(E) := \operatorname{Proj}\Big(\bigoplus_{m=0}^{\infty} \operatorname{Sym}^m(E)\Big).$$

The canonical morphism $\mathbb{P}(E) \to C$ is denoted by f_E. A tautological divisor Θ_E on $\mathbb{P}(E)$ is a Cartier divisor on $\mathbb{P}(E)$ such that $\mathcal{O}_{\mathbb{P}(E)}(\Theta_E)$ is isomorphic to the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. We say that E is *strongly semistable* if, for any surjective morphism $\pi : C' \to C$ of smooth projective curves, $\pi^*(E)$ is semistable. By definition, if E is strongly semistable and $\pi : C' \to C$ is a surjective morphism of smooth projective curves over k, then $\pi^*(E)$ is also strongly semistable. A filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

of E is called the *strong Harder–Narasimhan filtration* if

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_{s-1}/E_{s-2}) > \mu(E_s/E_{s-1})$$

and E_i/E_{i-1} is a strongly semistable locally free sheaf on C for each $i = 1, \ldots, s$. Recall the following well-known facts (F1)–(F5) on strong semistability.

(F1) A locally free sheaf E on C is strong semistable if and only if $\Theta_E - f_E^*(\xi_E/r)$ is nef, where ξ_E is a Cartier divisor on C with $\mathcal{O}_C(\xi_E) \simeq \det(E)$ (e.g., see [16, Proposition 7.1(3)]).

(F2) Let $\pi : C' \to C$ be a surjective morphism of smooth projective curves over k such that the function field of C' is a separable extension field over the function field of C. If E is semistable, then $\pi^*(E)$ is also semistable (e.g., see [16, Proposition 7.1(1)]). In particular, if $\mathrm{char}(k) = 0$, then E is strongly semistable if and only if E is semistable. Moreover, in the case where $\mathrm{char}(k) > 0$, E is strongly semistable if and only if $(F^m)^*(E)$ is semistable for all $m \geq 0$, where $F : C \to C$ is the absolute Frobenius map and

$$F^m = \overbrace{F \circ \cdots \circ F}^{m}.$$

(F3) If E and G are strongly semistable locally free sheaves on C, then $\mathrm{Sym}^m(E)$ and $E \otimes G$ are also strongly semistable for all $m \geq 1$ (e.g., see [16, Theorem 7.2, Corollary 7.3]).

(F4) There is a surjective morphism $\pi : C' \to C$ of smooth projective curves over k such that $\pi^*(E)$ has the strong Harder–Narasimhan filtration (cf. [11, Theorem 7.2]).

(F5) We assume that k is an algebraic closure of a finite field. If E is a strongly semistable locally free sheaf on C with $\det(E) \simeq \mathcal{O}_C$, then there is a surjective morphism $\pi : C' \to C$ of smooth projective curves over k such that $\pi^*(E) \simeq \mathcal{O}_{C'}^{\oplus \operatorname{rk} E}$ (cf. [1, p. 557], [22, Theorem 3.2], [3]).

The purpose of this section is to prove the following characterizations of pseudo-effective $\mathbb{R}$-Cartier divisors and nef $\mathbb{R}$-Cartier divisors on $\mathbb{P}(E)$. This result is essentially due to Nakayama [21, Lemma 3.7] in which he works over the complex number field.

PROPOSITION 1.1

We assume that E has the strong Harder–Narasimhan filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E.$$

Then, for an $\mathbb{R}$-divisor A on C, we have the following:

(a) $\Theta_E - f^*(A)$ *is pseudo-effective if and only if* $\deg(A) \leq \mu(E_1)$.
(b) $\Theta_E - f^*(A)$ *is nef if and only if* $\deg(A) \leq \mu(E/E_{s-1})$.

Let us begin with the following lemma.

LEMMA 1.2

We assume that E has a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

such that E_i/E_{i-1} is a strongly semistable locally free sheaf on C and $\deg(E_i/E_{i-1}) < 0$ *for all $i = 1, \dots, s$. Then, $H^0(C, \mathrm{Sym}^m(E) \otimes G) = 0$ for $m \geq 1$ and a strongly semistable locally free sheaf G on C with* $\deg(G) \leq 0$.

Proof
We prove it by induction on s. In the case where $s = 1$, E is strongly semistable and $\deg(E) < 0$, so that $\mathrm{Sym}^m(E) \otimes G$ is also strongly semistable by (F3) and

$$\deg\bigl(\mathrm{Sym}^m(E) \otimes G\bigr) < 0.$$

Therefore, $H^0(C, \mathrm{Sym}^m(E) \otimes G) = 0$.

Here we assume that $s > 1$. Let us consider an exact sequence

$$0 \to E_{s-1} \to E \to E/E_{s-1} \to 0.$$

By [9, Chapter II, Exercise 5.16(c)], there is a filtration

$$\mathrm{Sym}^m(E) = F^0 \supsetneq F^1 \supsetneq \cdots \supsetneq F^m \supsetneq F^{m+1} = 0$$

such that

$$F^j/F^{j+1} \simeq \mathrm{Sym}^j(E_{s-1}) \otimes \mathrm{Sym}^{m-j}(E/E_{s-1})$$

for each $j = 0, \dots, m$. By using the hypothesis of induction,

$$H^0\bigl(C, (F^j/F^{j+1}) \otimes G\bigr) = 0$$

for $j = 1, \dots, m$ because $\mathrm{Sym}^{m-j}(E/E_{s-1}) \otimes G$ is strongly semistable by (F3) and

$$\deg\bigl(\mathrm{Sym}^{m-j}(E/E_{s-1}) \otimes G\bigr) \leq 0.$$

Moreover, since $\mathrm{Sym}^m(E/E_{s-1}) \otimes G$ is strongly semistable by (F3) and

$$\deg\bigl(\mathrm{Sym}^m(E/E_{s-1}) \otimes G\bigr) < 0,$$

we have that

$$H^0\bigl(C, (F^0/F^1) \otimes G\bigr) = H^0\bigl(C, \mathrm{Sym}^m(E/E_{s-1}) \otimes G\bigr) = 0.$$

Therefore, by using an exact sequence

$$0 \to F^{j+1} \otimes G \to F^j \otimes G \to (F^j/F^{j+1}) \otimes G \to 0,$$

we have that

$$H^0(C, F^{j+1} \otimes G) \xrightarrow{\sim} H^0(C, F^j \otimes G)$$

for $j = 0, \dots, m$, which implies that $H^0(C, \mathrm{Sym}^m(E) \otimes G) = 0$, as required. □

Proof of Proposition 1.1
It is sufficient to show the following.

(a) If A is a $\mathbb{Q}$-Cartier divisor and $\deg(A) < \mu(E_1)$, then $\Theta_E - f^*(A)$ is $\mathbb{Q}$-effective.

(b) If A is a $\mathbb{Q}$-Cartier divisor and $\deg(A) > \mu(E_1)$, then $\Theta_E - f^*(A)$ is not pseudo-effective.

(c) If $\Theta_E - f^*(A)$ is nef, then $\deg(A) \leq \mu(E/E_{s-1})$.

(d) If $\Theta_E - f^*(A)$ is not nef, then $\deg(A) > \mu(E/E_{s-1})$.

(a) Let θ be a divisor on C with $\deg(\theta) = 1$. As E_1 is strongly semistable, by (F1), $\Theta_{E_1} - \mu(E_1) f_{E_1}^*(\theta)$ is nef, so that we can see that $\Theta_{E_1} - f_{E_1}^*(A)$ is nef and big because

$$\Theta_{E_1} - \deg(A) f_{E_1}^*(\theta) = \Theta_{E_1} - \mu(E_1) f_{E_1}^*(\theta) + \bigl(\mu(E_1) - \deg(A)\bigr) f_{E_1}^*(\theta).$$

Therefore, there is a positive integer m_1 such that $m_1 A$ is a divisor on C and

$$H^0\bigl(\mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}\bigl(m_1\Theta_{E_1} - f_{E_1}^*(m_1 A)\bigr)\bigr) \neq 0.$$

In addition,

$$\begin{aligned} &H^0\bigl(\mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}\bigl(m_1\Theta_{E_1} - f_{E_1}^*(m_1 A)\bigr)\bigr) \\ &\quad = H^0\bigl(C, \operatorname{Sym}^{m_1}(E_1) \otimes \mathcal{O}_C(-m_1 A)\bigr) \\ &\quad \subseteq H^0\bigl(C, \operatorname{Sym}^{m_1}(E) \otimes \mathcal{O}_C(-m_1 A)\bigr) \\ &\quad = H^0\bigl(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}\bigl(m_1\Theta_E - f_E^*(m_1 A)\bigr)\bigr), \end{aligned}$$

so that $\Theta_E - f_E^*(A)$ is $\mathbb{Q}$-effective.

(b) Let B be an ample $\mathbb{Q}$-divisor on C with $\deg(B) < \deg(A) - \mu(E_1)$. Let $\pi : C' \to C$ be a surjective morphism of smooth projective curves over k such that $\pi^*(-A + B)$ is a Cartier divisor on C'. Note that

$$\mu\bigl(\pi^*(E_i/E_{i-1}) \otimes \mathcal{O}_{C'}\bigl(\pi^*(-A + B)\bigr)\bigr) < 0$$

for $i = 1, \ldots, s$, and hence, by Lemma 1.2,

$$H^0\bigl(C', \operatorname{Sym}^m\bigl(\pi^*(E)\bigr) \otimes \mathcal{O}_{C'}\bigl(m\pi^*(-A + B)\bigr)\bigr) = 0$$

for all $m \geq 1$. In particular, if b is a positive integer such that $b(-A + B)$ is a Cartier divisor, then

$$H^0\bigl(C, \operatorname{Sym}^{mb}(E) \otimes \mathcal{O}_C\bigl(mb(-A + B)\bigr)\bigr) = 0$$

for $m \geq 1$. Here we assume that $\Theta_E - f_E^*(A)$ is pseudo-effective. Let a be a positive integer such that $\Theta_E - f_E^*(A) + a f_E^*(B)$ is ample. Then

$$(a-1)\bigl(\Theta_E - f_E^*(A)\bigr) + \Theta_E - f_E^*(A) + a f_E^*(B) = a\bigl(\Theta_E + f_E^*(-A + B)\bigr)$$

is big, so that we can find a positive integer m_1 such that

$$\begin{aligned} &H^0\bigl(C, \operatorname{Sym}^{m_1 ab}(E) \otimes \mathcal{O}_C\bigl(m_1 ab(-A + B)\bigr)\bigr) \\ &\quad = H^0\bigl(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}\bigl(m_1 ab\bigl(\Theta_E + f_E^*(-A + B)\bigr)\bigr)\bigr) \neq 0, \end{aligned}$$

which is a contradiction.

(c) Note that

$$\mathbb{P}(E/E_{s-1}) \subseteq \mathbb{P}(E), \quad \Theta_{E/E_{s-1}} \sim \Theta_E|_{\mathbb{P}(E/E_{s-1})}, \qquad \text{and}$$
$$f_{E/E_{s-1}} = f_E|_{\mathbb{P}(E/E_{s-1})},$$

so that $\Theta_{E/E_{s-1}} - f^*_{E/E_{s-1}}(A)$ is nef on $\mathbb{P}(E/E_{s-1})$. Let $\xi_{E/E_{s-1}}$ be a Cartier divisor on C with $\mathcal{O}_C(\xi_{E/E_{s-1}}) \simeq \det(E/E_{s-1})$. If we set $e = \operatorname{rk} E/E_{s-1}$ and $G = \xi_{E/E_{s-1}}/e - A$, then

$$\Theta_{E/E_{s-1}} - f^*_{E/E_{s-1}}(A) = \Theta_{E/E_{s-1}} - f^*_{E/E_{s-1}}(\xi_{E/E_{s-1}}/e) + f^*_{E/E_{s-1}}(G).$$

Since $\Theta_{E/E_{s-1}} - f^*_{E/E_{s-1}}(\xi_{E/E_{s-1}}/e)$ is nef by (F1) and

$$\bigl(\Theta_{E/E_{s-1}} - f^*_{E/E_{s-1}}(\xi_{E/E_{s-1}}/e)\bigr)^e = 0,$$

we have that

$$0 \leq \bigl(\Theta_{E/E_{s-1}} - f^*_{E/E_{s-1}}(A)\bigr)^e = e\deg(G).$$

Therefore, $\deg(G) \geq 0$, and hence $\deg(A) \leq \mu(E/E_{s-1})$.

(d) We can find an irreducible curve C_0 of X such that $(\Theta_E - f^*_E(A) \cdot C_0) < 0$. Clearly C_0 is flat over C. Let C_1 be the normalization of C_0, and let $h : C_1 \to C$ be the induced morphism. Let us consider the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}(E) \arrow[d, "f_E"'] & \mathbb{P}\bigl(h^*(E)\bigr) \arrow[l, "\mathbb{P}(h)"'] \arrow[d, "f_{h^*(E)}"] \\ C & C_1 \arrow[l, "h"'] \end{tikzcd}$$

Note that $\mathbb{P}(h)^*(\Theta_E - f^*_E(A)) \sim_{\mathbb{R}} \Theta_{h^*(E)} - f^*_{h^*(E)}(h^*(A))$. Further, there is a section S of $f_{h^*(E)}$ such that $\mathbb{P}(h)_*(S) = C_0$. Let Q be the quotient line bundle of $h^*(E)$ corresponding to the section S. As

$$0 = h^*(E_0) \subsetneq h^*(E_1) \subsetneq h^*(E_2) \subsetneq \cdots \subsetneq h^*(E_{s-1}) \subsetneq h^*(E_s) = h^*(E)$$

is the Harder–Narasimhan filtration of $h^*(E)$, we can easily see that

$$\deg(Q) \geq \mu\bigl(h^*(E/E_{s-1})\bigr) = \deg(h)\mu(E/E_{s-1}).$$

On the other hand,

$$\begin{aligned} \deg(Q) - \deg(h)\deg(A) &= \bigl(\Theta_{h^*(E)} - f^*_{h^*(E)}\bigl(h^*(A)\bigr) \cdot S\bigr) \\ &= \bigl(\Theta_E - f^*_E(A) \cdot C_0\bigr) < 0, \end{aligned}$$

and hence $\mu(E/E_{s-1}) < \deg(A)$. □

Finally let us consider the following three results.

LEMMA 1.3

Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $\mu : X' \to X$ be a generically finite morphism of normal *projective varieties over k. For a $\mathbb{K}$-Cartier divisor D on X, D is $\mathbb{K}$-effective if and only if $\mu^*(D)$ is $\mathbb{K}$-effective.*

Proof
Clearly, if D is $\mathbb{K}$-effective, then $\mu^*(D)$ is $\mathbb{K}$-effective. Let K and K' be the function fields of X and X', respectively. Here we assume that $\mu^*(D)$ is $\mathbb{K}$-effective; that is, there are $\phi'_1, \ldots, \phi'_r \in K'^{\times}$ and $a_1, \ldots, a_r \in \mathbb{K}$ such that $\mu^*(D) + a_1(\phi'_1) + \cdots + a_r(\phi'_r)$ is effective, so that

$$\mu_*\big(\mu^*(D) + a_1(\phi'_1) + \cdots + a_r(\phi'_r)\big) = \deg(\mu)D + a_1\mu_*\big((\phi'_1)\big) + \cdots + a_r\mu_*\big((\phi'_r)\big)$$

is effective. Note that $\mu_*(\phi'_i) = (N_{K'/K}(\phi'_i))$ (cf. [7, Proposition 1.4]), where $N_{K'/K}$ is the norm map of K' over K, and hence

$$D + \big(a_1/\deg(\mu)\big)\big(N_{K'/K}(\phi'_1)\big) + \cdots + \big(a_r/\deg(\mu)\big)\big(N_{K'/K}(\phi'_r)\big)$$

is effective. Therefore, D is $\mathbb{K}$-effective. □

LEMMA 1.4
Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. We assume that k is an algebraic closure of a finite field. Let X be a normal projective variety over k, and let D be a $\mathbb{K}$-Cartier divisor on X. If D is numerically trivial, then D is $\mathbb{K}$-linearly equivalent to the zero divisor.

Proof
If $\mathbb{K} = \mathbb{Q}$, then the assertion is well known, so that we assume that $\mathbb{K} = \mathbb{R}$. We set $D = a_1D_1 + \cdots + a_rD_r$, where $D_1, \ldots, D_r$ are Cartier divisors on X and $a_1, \ldots, a_r \in \mathbb{R}$. Considering a $\mathbb{Q}$-basis of $\mathbb{Q}a_1 + \cdots + \mathbb{Q}a_r$ in $\mathbb{R}$, we may assume that $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Let C be an irreducible curve on X. Note that

$$0 = (D \cdot C) = a_1(D_1 \cdot C) + \cdots + a_r(D_r \cdot C)$$

and $(D_1 \cdot C), \ldots, (D_r \cdot C) \in \mathbb{Z}$, and hence $(D_1 \cdot C) = \cdots = (D_r \cdot C) = 0$ because $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Thus, $D_1, \ldots, D_r$ are numerically equivalent to zero, so that $D_1, \ldots, D_r$ are $\mathbb{Q}$-linearly equivalent to the zero divisor. Therefore, the assertion follows. □

PROPOSITION 1.5
Let X be a normal projective variety over k, and let D be a $\mathbb{Q}$-Cartier divisor on X. If D is $\mathbb{R}$-effective, then D is $\mathbb{Q}$-effective.

Proof
As D is $\mathbb{R}$-effective, there are nonzero rational functions $\psi_1, \ldots, \psi_l$ on X and $b_1, \ldots, b_l \in \mathbb{R}$ such that $D + b_1(\psi_1) + \cdots + b_l(\psi_l)$ is effective. We set $V = \mathbb{Q}b_1 + \cdots + \mathbb{Q}b_l \subseteq \mathbb{R}$. If $V \subseteq \mathbb{Q}$, then $b_1, \ldots, b_l \in \mathbb{Q}$, so that we may assume that $V \not\subseteq \mathbb{Q}$.

CLAIM 1.5.1

There are nonzero rational functions $\phi_1, \ldots, \phi_r$ on X, $a_1, \ldots, a_r \in \mathbb{R}$, and a $\mathbb{Q}$-Cartier divisor D' on X such that $D \sim_{\mathbb{Q}} D'$, $D' + a_1(\phi_1) + \cdots + a_r(\phi_r)$ is effective, and $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$.

Proof

We can find a basis $a_1, \ldots, a_r$ of V over $\mathbb{Q}$ with the following properties:

(i) If we set $b_i = \sum_{j=1}^{r} c_{ij} a_j$, then $c_{ij} \in \mathbb{Z}$ for all i, j.

(ii) If $V \cap \mathbb{Q} \neq \{0\}$, then $a_1 \in \mathbb{Q}^{\times}$.

We put $\phi_j = \prod_{i=1}^{l} \psi_i^{c_{ij}}$. Note that $\sum_{i=1}^{l} b_i(\psi_i) = \sum_{j=1}^{r} a_j(\phi_j)$. Therefore, in the case where $V \cap \mathbb{Q} = \{0\}$, $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$ and $D + \sum_{j=1}^{r} a_j(\phi_j)$ is effective. Otherwise, $1, a_2, \ldots, a_r$ are linearly independent over $\mathbb{Q}$ and $(D + a_1(\phi_1)) + \sum_{j=2}^{r} a_j(\phi_j)$ is effective. $\square$

We set $L = D' + a_1(\phi_1) + \cdots + a_r(\phi_r)$. Let Γ be a prime divisor with $\Gamma \not\subseteq \operatorname{Supp}(L)$. Then

$$0 = \operatorname{mult}_\Gamma(L) = \operatorname{mult}_\Gamma(D') + a_1 \operatorname{ord}_\Gamma(\phi_1) + \cdots + a_r \operatorname{ord}_\Gamma(\phi_r),$$

so that $\operatorname{mult}_\Gamma(D') = \operatorname{ord}_\Gamma(\phi_1) = \cdots = \operatorname{ord}_\Gamma(\phi_r) = 0$ because $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Thus,

$$\operatorname{Supp}(D'), \operatorname{Supp}\big((\phi_1)\big), \ldots, \operatorname{Supp}\big((\phi_r)\big) \subseteq \operatorname{Supp}(L).$$

Therefore, we can find $a'_1, \ldots, a'_r \in \mathbb{Q}$ such that $D' + a'_1(\phi_1) + \cdots + a'_r(\phi_r)$ is effective, and hence D is $\mathbb{Q}$-effective. $\square$

2. Proof of Theorem 0.4

Let k be an algebraic closure of a finite field. Let C be a smooth projective curve over k. Let us begin with the following lemma.

LEMMA 2.1

Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let A be a $\mathbb{K}$-Cartier divisor on C. If $\deg(A) \geq 0$, then A is $\mathbb{K}$-effective.

Proof

If $\mathbb{K} = \mathbb{Q}$, then the assertion is obvious. We assume that $\mathbb{K} = \mathbb{R}$. If $\deg(A) = 0$, then the assertion follows from Lemma 1.4. Next we consider the case where $\deg(A) > 0$. We can find a $\mathbb{Q}$-Cartier divisor A' such that $A' \leq A$ and $\deg(A') > 0$. Thus, the previous observation implies the assertion. $\square$

As a consequence of (F3), (F4), and (F5), we have the following splitting theorem, which was obtained by Biswas and Parameswaran [2, Proposition 2.1].

THEOREM 2.2

For a locally free sheaf E on C, there are a surjective morphism $\pi : C' \to C$ of smooth projective curves over k and invertible sheaves $L_1, \dots, L_r$ on C' such that $\pi^*(E) \simeq L_1 \oplus \cdots \oplus L_r$.

Proof

For the reader's convenience, we give a sketch of the proof. First we assume that E is strongly semistable. Let ξ_E be a Cartier divisor on C with $\mathcal{O}_C(\xi_E) \simeq \det(E)$. Let $h : B \to C$ be a surjective morphism of smooth projective curves over k such that $h^*(\xi_E)$ is divisible by $\mathrm{rk}(E)$. We set $E' = h^*(E) \otimes \mathcal{O}_B(-h^*(\xi_E)/\mathrm{rk}(E))$. As $\det(E') \simeq \mathcal{O}_B$, the assertion follows from (F5).

By the above observation, it is sufficient to find a surjective morphism $\pi : C' \to C$ of smooth projective curves over k and strongly semistable locally free sheaves $Q_1, \dots, Q_n$ on C' such that

$$\pi^*(E) = Q_1 \oplus \cdots \oplus Q_n.$$

Moreover, by (F4), we may assume that E has the strong Harder–Narasimhan filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = E.$$

Clearly we may further assume that $n \geq 2$. For a nonnegative integer m, we set

$$C_m := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k),$$

where the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is given by $x \mapsto x^{1/p^m}$. Let $F_k^m : C_m \to C$ be the relative mth Frobenius morphism over k. Put

$$G_{i,j}^m := (F_k^m)^*\big((E_j/E_i) \otimes (E_i/E_{i-1})^{\vee}\big) \otimes \omega_{C_m}$$

for $i = 1, \dots, n-1$ and $j = i, \dots, n$. We can find a positive integer m such that

$$\mu(G_{i,i+1}^m) = p^m\big(\mu(E_{i+1}/E_i) - \mu(E_i/E_{i-1})\big) + \deg(\omega_C) < 0$$

for all $i = 1, \dots, n-1$. By using (F3), we can see that

$$0 = G_{i,i}^m \subsetneq G_{i,i+1}^m \subsetneq G_{i,i+2}^m \subsetneq \cdots \subsetneq G_{i,n-1}^m \subsetneq G_{i,n}^m$$

is the strong Harder–Narasimhan filtration of $G_{i,n}^m$, so that $H^0(C_m, G_{i,n}^m) = \{0\}$, which yields

$$\mathrm{Ext}^1\big((F_k^m)^*(E/E_i), (F_k^m)^*(E_i/E_{i-1})\big) = 0$$

because of Serre's duality theorem. Therefore, an exact sequence

$$0 \to (F_k^m)^*(E_i/E_{i-1}) \to (F_k^m)^*(E/E_{i-1}) \to (F_k^m)^*(E/E_i) \to 0$$

splits; that is, $(F_k^m)^*(E/E_{i-1}) \simeq (F_k^m)^*(E_i/E_{i-1}) \oplus (F_k^m)^*(E/E_i)$ for $i = 1, \dots, n-1$, and hence

$$(F_k^m)^*(E) \simeq \bigoplus_{i=1}^{n} (F_k^m)^*(E_i/E_{i-1}),$$

as required. □

Proof of Theorem 0.4

By virtue of Theorem 2.2 and Lemma 1.3, we may assume that

$$E \simeq L_1 \oplus \cdots \oplus L_r$$

for some invertible sheaves $L_1, \ldots, L_r$ on C. We set

$$d = \max\{\deg(L_1), \ldots, \deg(L_r)\} \qquad \text{and} \qquad I = \{i \mid \deg(L_i) = d\}.$$

There is a $\mathbb{K}$-Cartier divisor A on C such that $D \sim_{\mathbb{K}} \lambda\Theta_E - f_E^*(A)$ for some $\lambda \in \mathbb{K}$. Let M be an ample divisor on C such that $T := \Theta_E + f_E^*(M)$ is ample. As D is pseudo-effective, we have that

$$0 \le \left(D \cdot T^{r-2} \cdot f_E^*(M)\right) = \left(\left(\lambda T - f_E^*(A + \lambda M)\right) \cdot T^{r-2} \cdot f_E^*(M)\right) = \lambda \deg(M),$$

and hence $\lambda \ge 0$. If $\lambda = 0$, then $0 \le (D \cdot T^{r-1}) = \deg(-A)$. Thus, by Lemma 2.1, $-A$ is $\mathbb{K}$-effective, so that the assertion follows.

We assume that $\lambda > 0$. Replacing D by D/λ, we may assume that $\lambda = 1$. Let ξ be a Cartier divisor on C such that $\mathcal{O}_C(\xi) \simeq L_{i_0}$ for some $i_0 \in I$. Note that the first part E_1 of the strong Harder–Narasimhan filtration of E is $\bigoplus_{i \in I} L_i$, so that, by Proposition 1.1, $\deg(A) \le \deg(\xi)$. If we set $B = \xi - A$, then, by Lemma 2.1, B is $\mathbb{K}$-effective because $\deg(B) \ge 0$. Moreover, as

$$\Theta_E - f_E^*(A) = \Theta_E - f_E^*(\xi) + f_E^*(B),$$

it is sufficient to consider the case where $D = \Theta_E - f_E^*(\xi)$. In this case, the assertion is obvious because

$$\begin{aligned} H^0\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(D)\right) &= H^0\left(C, E \otimes \mathcal{O}_C(-\xi)\right) \\ &= H^0\left(C, \bigoplus_{i=1}^{r} L_i \otimes \mathcal{O}_C(-\xi)\right) \ne \{0\}. \end{aligned}$$

$\square$

As a consequence of Theorem 0.4, we can recover a result due to [3].

COROLLARY 2.3

Let k, C, and E be the same as in Theorem 0.4. We assume that $r = 2$. Let D be a Cartier divisor on $\mathbb{P}(E)$ such that $(D \cdot Y) > 0$ for all irreducible curves Y on $\mathbb{P}(E)$. Then D is ample.

Proof

As D is nef, D is pseudo-effective, so that, by Theorem 0.4, there is an effective $\mathbb{Q}$-Cartier divisor E on X such that $D \sim_{\mathbb{Q}} E$. As $E \ne 0$, we have that $(D \cdot D) = (D \cdot E) > 0$. Therefore, D is ample by the Nakai–Moishezon criterion. $\square$

REMARK 2.4

The argument in the proof of Corollary 2.3 actually shows that the $\mathbb{Q}$-version of Question 0.2 on algebraic surfaces implies Question 0.3.

3. Numerical effectivity on abelian varieties

The purpose of this section is to give an affirmative answer for the $\mathbb{Q}$-version of Question 0.2 on abelian varieties. Let A be an abelian variety over an algebraically closed field k. A key observation is the following proposition.

PROPOSITION 3.1

If a $\mathbb{Q}$-Cartier divisor D on A is nef, then D is numerically equivalent to a $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor.

Proof

We prove it by induction on $\dim A$. If $\dim A \leq 1$, then the assertion is obvious. Clearly we may assume that D is a Cartier divisor, so that we set $L = \mathcal{O}_A(D)$. As $L \otimes [-1]^*(L)$ is numerically equivalent to $L^{\otimes 2}$ (cf. [20, p. 75, (iv)]), we may assume that L is symmetric; that is, $L \simeq [-1]^*(L)$. Let $K(L)$ be the closed subgroup of A given by $K(L) = \{x \in A \mid T_x^*(L) \simeq L\}$ (cf. [20, p. 60, Definition]). If $K(L)$ is finite, then L is nef and big by virtue of [20, p. 150, Riemann–Roch theorem], so that D is $\mathbb{Q}$-effective. Otherwise, let B be the connected component of $K(L)$ containing 0.

CLAIM 3.1.1

(a) $T_x^*(L)|_B \simeq L|_B$ *for all* $x \in A$.
(b) $L^{\otimes 2}|_{B+x} \simeq \mathcal{O}_{B+x}$ *for* $x \in A$.

Proof

(a) Let N be an invertible sheaf on $A \times A$ given by

$$N = m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1}),$$

where $p_i : A \times A \to A$ is the projection to the ith factor $(i = 1, 2)$ and m is the addition morphism. Note that $N|_{B\times A} \simeq \mathcal{O}_{B\times A}$ (cf. [20, Section 13, p. 123]). Fixing $x \in A$, let us consider a morphism $\alpha : B \to B \times A$ given by $\alpha(y) = (y, x)$. Then

$$\mathcal{O}_B \simeq \alpha^*\big(m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1})|_{B\times A}\big) \simeq T_x^*(L)|_B \otimes L^{-1}|_B,$$

as required.

(b) First we consider the case where $x = 0$. As $N|_{B\times A} \simeq \mathcal{O}_{B\times A}$, we have that $N|_{B\times B} \simeq \mathcal{O}_{B\times B}$. Using a morphism $\beta : B \to B \times B$ given by $\beta(y) = (y, -y)$, we have that

$$\mathcal{O}_B \simeq \beta^*(N|_{B\times B}) = L^{-1}|_B \otimes [-1]^*(L^{-1})|_B \simeq L^{\otimes -2}|_B,$$

as required.

In general, for $x \in A$, by (a) and the previous observation together with the following commutative diagram

$$\begin{array}{ccc} B+x & \longrightarrow & A \\ {\scriptstyle T_{-x}}\big\downarrow & & \big\downarrow{\scriptstyle T_{-x}} \\ B & \longrightarrow & A, \end{array}$$

we can see that

$$\begin{aligned} \mathcal{O}_{B+x} &= T_{-x}^*(\mathcal{O}_B) \simeq T_{-x}^*(L^{\otimes 2}|_B) \simeq T_{-x}^*\big(T_x^*(L)^{\otimes 2}|_B\big) \\ &= T_{-x}^*\big(T_x^*(L^{\otimes 2})|_B\big) = T_{-x}^*\big(T_x^*(L^{\otimes 2})\big)|_{B+x} = L^{\otimes 2}|_{B+x}. \end{aligned}$$ □

Let $\pi : A \to A/B$ be the canonical homomorphism. By Claim 3.1.1(b),

$$\dim_{k(y)} H^0\big(\pi^{-1}(y), L^{\otimes 2}\big) = 1$$

for all $y \in A/B$, so that, by [20, p. 51, Corollary 2], $\pi_*(L^{\otimes 2})$ is an invertible sheaf on A/B and $\pi_*(L^{\otimes 2}) \otimes k(y) \xrightarrow{\sim} H^0(\pi^{-1}(y), L^{\otimes 2})$. Therefore, the natural homomorphism $\pi^*(\pi_*(L^{\otimes 2})) \to L^{\otimes 2}$ is an isomorphism; that is, there is a $\mathbb{Q}$-Cartier divisor D' on A/B such that $\pi^*(D') \sim_{\mathbb{Q}} D$. Note that D' is also nef, so that, by the hypothesis of induction, D' is numerically equivalent to a $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor, and hence the assertion follows. □

Proof of Proposition 0.5
Proposition 0.5 is a consequence of Lemma 1.4 and Proposition 3.1 because a pseudo-effective $\mathbb{Q}$-Cartier divisor on an abelian variety is nef. □

EXAMPLE 3.2
Here we show that the $\mathbb{R}$-version of Question 0.2 does not hold in general. Let k be an algebraically closed field. (Note that k is not necessarily an algebraic closure of a finite field.) Let C be an elliptic curve over k, and let $A := C \times C$. Let $\mathrm{NS}(A)$ be the Néron–Severi group of A. Note that $\rho := \mathrm{rk}\,\mathrm{NS}(A) \geq 3$. By using the Hodge index theorem, we can find a basis $e_1, \ldots, e_\rho$ of $\mathrm{NS}(A)_{\mathbb{Q}} := \mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the following properties:

(a) e_1 is the class of the divisor $\{0\} \times C + C \times \{0\}$. In particular, $(e_1 \cdot e_1) = 2$.
(b) $(e_i \cdot e_i) < 0$ for all $i = 2, \ldots, \rho$.
(c) $(e_i \cdot e_j) = 0$ for all $1 \leq i \neq j \leq \rho$.

We set $\lambda_i := -(e_i \cdot e_i)$ for $i = 2, \ldots, \rho$. Let $\overline{\mathrm{Amp}}(A)$ be the closed cone in $\mathrm{NS}(A)_{\mathbb{R}} := \mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by ample $\mathbb{Q}$-Cartier divisors on A. It is well known that

$$\begin{aligned} \overline{\mathrm{Amp}}(A) &= \big\{\xi \in \mathrm{NS}(A)_{\mathbb{R}} \;\big|\; (\xi^2) \geq 0, (\xi \cdot e_1) \geq 0\big\} \\ &= \{x_1 e_1 + x_2 e_2 + \cdots + x_\rho e_\rho \mid \lambda_2 x_2^2 + \cdots + \lambda_\rho x_\rho^2 \leq 2x_1^2, x_1 \geq 0\}. \end{aligned}$$

We choose $(a_2, \ldots, a_\rho) \in \mathbb{R}^{\rho-1}$ such that

$$(a_2, \ldots, a_\rho) \notin \mathbb{Q}^{\rho-1} \qquad \text{and} \qquad \lambda_2 a_2^2 + \cdots + \lambda_\rho a_\rho^2 = 2.$$

Let E_i be a $\mathbb{Q}$-Cartier divisor on A such that the class of E_i in $\mathrm{NS}(A)_{\mathbb{Q}}$ is equal to e_i for $i = 1, \dots, \rho$. If we set $D := E_1 + a_2 E_2 + \cdots + a_\rho E_\rho$, then we have the following claim, which is sufficient for our purpose.

CLAIM 3.2.1

We have that D is nef and D is not numerically equivalent to an effective $\mathbb{R}$-Cartier divisor.

Proof

Clearly D is nef. If we set $e'_1 = e_1/\sqrt{2}$ and $e'_i = e_i/\sqrt{\lambda_i}$ for $i = 2, \dots, \rho$, then

$$\overline{\mathrm{Amp}}(A) = \{y_1 e'_1 + y_2 e'_2 + \cdots + y_\rho e'_\rho \mid {y_2}^2 + \cdots + {y_\rho}^2 \le {y_1}^2, y_1 \ge 0\}.$$

Therefore, as $[D] \in \partial(\overline{\mathrm{Amp}}(A)_{\mathbb{R}})$, we can choose

$$H \in \mathrm{Hom}_{\mathbb{R}}\big(\mathrm{NS}(A)_{\mathbb{R}}, \mathbb{R}\big)$$

such that

$$H \ge 0 \text{ on } \overline{\mathrm{Amp}}(A) \qquad \text{and} \qquad \{H = 0\} \cap \overline{\mathrm{Amp}}(A) = \mathbb{R}_{\ge 0}[D],$$

where $[D]$ is the class of D in $\mathrm{NS}(A)_{\mathbb{R}}$. We assume that D is numerically equivalent to an effective $\mathbb{R}$-Cartier divisor $c_1\Gamma_1 + \cdots + c_r\Gamma_r$, where $c_1, \dots, c_r \in \mathbb{R}_{\ge 0}$ and $\Gamma_1, \dots, \Gamma_r$ are prime divisors on A. As $[D] \ne 0$, we may assume that $c_1, \dots, c_r \in \mathbb{R}_{>0}$. Note that $[\Gamma_1], \dots, [\Gamma_r] \in \overline{\mathrm{Amp}}(A)$ and

$$0 = H\big([D]\big) = c_1 H\big([\Gamma_1]\big) + \cdots + c_r H\big([\Gamma_r]\big),$$

so that $H([\Gamma_1]) = \cdots = H([\Gamma_r]) = 0$, and hence $[\Gamma_1], \dots, [\Gamma_r] \in \mathbb{R}_{\ge 0}[D]$. In particular, there is $t \in \mathbb{R}_{\ge 0}$ with $[\Gamma_1] = t[D]$. Here we can set

$$[\Gamma_1] = b_1 e_1 + \cdots + b_\rho e_\rho \quad (b_1, \dots, b_\rho \in \mathbb{Q}).$$

Thus, $b_1 = t$, $b_2 = t a_2$, $\dots$, $b_\rho = t a_\rho$. As $[\Gamma_1] \ne 0$, $t \in \mathbb{Q}^\times$, and hence $(a_2, \dots, a_\rho) = t^{-1}(b_2, \dots, b_\rho) \in \mathbb{Q}^{\rho - 1}$. This is a contradiction. $\square$

REMARK 3.3

Let k be an algebraic closure of a finite field, and let X be a normal projective variety over k. Let $\mathrm{NS}(X)$ be the Néron–Severi group of X, and let $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\overline{\mathrm{Eff}}(X)$ be the closed cone in $\mathrm{NS}(X)_{\mathbb{R}}$ generated by pseudo-effective $\mathbb{R}$-Cartier divisors on X. We assume that $\overline{\mathrm{Eff}}(X)$ is a rational polyhedral cone; that is, there are pseudo-effective $\mathbb{Q}$-Cartier divisors $D_1, \dots, D_n$ on X such that $\overline{\mathrm{Eff}}(X)$ is generated by the classes of $D_1, \dots, D_n$. Then the $\mathbb{Q}$-version of Question 0.2 implies the $\mathbb{R}$-version of Question 0.2.

EXAMPLE 3.4

This is an example due to Yuan [24]. Let us fix an algebraically closed field k and an integer $g \ge 2$. Let C be a smooth projective curve over k, and let $f : X \to C$ be an abelian scheme over C of relative dimension g. Let L be an f-ample invertible sheaf on X such that $[-1]^*(L) \simeq L$ and L is trivial along the zero section of $f: X \to C$.

CLAIM 3.4.1

(a) $[2]^*(L) \simeq L^{\otimes 4}$.

(b) *L is nef.*

Proof

(a) As $[2]^*(L)|_{f^{-1}(x)} \simeq L^{\otimes 4}|_{f^{-1}(x)}$ for all $x \in C$, there is an invertible sheaf M on C such that $[2]^*(L) \simeq L^{\otimes 4} \otimes f^*(M)$. Let Z_0 be the zero section of $f: X \to C$. Then

$$\mathcal{O}_{Z_0} \simeq [2]^*(L|_{Z_0}) = [2]^*(L)|_{Z_0} \simeq L^{\otimes 4} \otimes f^*(M)|_{Z_0} \simeq M,$$

so that we have the assertion.

(b) Let A be an ample invertible sheaf on C such that $L \otimes f^*(A)$ is ample. Let Δ be a horizontal curve on X. As $f \circ [2^n] = f$ and $[2^n]^*(L) \simeq L^{\otimes 4^n}$, by using (a),

$$0 \le \big(L \otimes f^*(A) \cdot [2^n]_*(\Delta)\big) = \big([2^n]^*\big(L \otimes f^*(A)\big) \cdot \Delta\big) = \big(L^{\otimes 4^n} \otimes f^*(A) \cdot \Delta\big),$$

so that $(L \cdot \Delta) \ge -4^{-n}(f^*(A) \cdot \Delta)$ for all $n > 0$. Thus, $(L \cdot \Delta) \ge 0$. □

CLAIM 3.4.2

If the characteristic of k is zero and f is nonisotrivial, then L does not have the Dirichlet property (i.e., L is not $\mathbb{Q}$-effective).

Proof

The following proof is due to Yuan [24]. An alternative proof can be found in [6, Theorem 4.3]. We need to see that $H^0(X, L^{\otimes n}) = 0$ for all $n > 0$. We set $d_n = \operatorname{rk} f_*(L^{\otimes n})$. By changing the base C if necessary, we may assume that all $(d_n)^2$-torsion points on the generic fiber X_η of $f: X \to C$ are defined over the function field of C. By using the algebraic theta theory due to Mumford (especially [19, last line on p. 81]), there is an invertible sheaf M on C such that $f_*(L^{\otimes n}) = M^{\oplus d_n}$. On the other hand, by [13],

$$\deg\big(\det\big(f_*(L^{\otimes n})\big)^{\otimes 2} \otimes f_*(\omega_{X/C})^{\otimes d_n}\big) = 0;$$

that is, $2\deg(M) + \deg(f_*(\omega_{X/C})) = 0$. As f is nonisotrivial, we can see that $\deg(f_*(\omega_{X/C})) > 0$, so that $\deg(M) < 0$, and hence the assertion follows. □

When the characteristic of k is positive, we do not know the $\mathbb{Q}$-effectivity of L in general. In [15], there is an example with the following properties:

(a) $g = 2$ and $C = \mathbb{P}^1_k$.

(b) There are an abelian surface A over k and an isogeny $h: A \times \mathbb{P}^1_k \to X$ over $\mathbb{P}^1_k$.

CLAIM 3.4.3
In the above example, L has the Dirichlet property.

Proof
Replacing L by $L^{\otimes n}$, we may assume that $d := \operatorname{rk} f_*(L) > 0$. Let

$$p_1 : A \times \mathbb{P}^1_k \to A \qquad \text{and} \qquad p_2 : A \times \mathbb{P}^1_k \to \mathbb{P}^1_k$$

be the projections to A and $\mathbb{P}^1_k$, respectively. Note that $h^*(L)$ is symmetric and $h^*(L)$ is trivial along the zero section of p_2. Since $\omega_{A\times\mathbb{P}^1_k/P^1_k} \simeq p_1^*(\omega_A)$, we have that $(p_2)_*(\omega_{A\times\mathbb{P}^1_k/P^1_k}) \simeq \mathcal{O}_{\mathbb{P}^1_k}$, so that, by [13], $\deg(\det((p_2)_*(h^*(L)))) = 0$; that is, if we set

$$(p_2)_*\big(h^*(L)\big) = \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_d),$$

then $a_1 + \cdots + a_d = 0$. Thus, $a_i \geq 0$ for some i, and hence

$$H^0\big(A \times \mathbb{P}^1_k, h^*(L)\big) \neq 0.$$

Therefore, L is $\mathbb{Q}$-effective by Lemma 1.3. □

The above claim suggests that the set of preperiodic points of the map $[2] : X \to X$ is not dense in the analytification X_v^{an} at any place v of $\mathbb{P}^1_k$ with respect to the analytic topology (cf. [5]).

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