

Graded Morita equivalences for generic Artin-Schelter regular algebras

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Abstract Let $A = \mathcal{A}(E, \sigma)$, $A' = \mathcal{A}(E', \sigma')$ be Noetherian Artin-Schelter regular geometric algebras with $\dim_k A_1 = \dim_k A'_1 = n$, and let ν, ν' be generalized Nakayama automorphisms of A, A' . In this paper, we study relationships between the conditions

(A) A is graded Morita equivalent to A' , and

(B) $\mathcal{A}(E, \nu^* \sigma^n)$ is isomorphic to $\mathcal{A}(E', (\nu')^* (\sigma')^n)$ as graded algebras.

It is proved that if A, A' are “generic” 3-dimensional quadratic Artin-Schelter regular algebras, then (A) is equivalent to (B), and if A, A' are n -dimensional skew polynomial algebras, then (A) implies (B).

1. Introduction

In noncommutative algebraic geometry, classification of Artin-Schelter (AS-) regular algebras has been one of the major projects since its beginning. In fact, AS-regular algebras up to dimension 3 were classified. Since classifying 4-dimensional AS-regular algebras up to isomorphism of graded algebras is difficult, it is natural to try to classify them up to something weaker than graded isomorphisms, such as graded Morita equivalences. In general, it is difficult to check whether two graded algebras are graded Morita equivalent. The motivation of this paper is to find a nice criterion of graded Morita equivalence for AS-regular algebras. In this paper, we associate to a geometric AS-regular algebra A a symmetric AS-regular algebra B , and we will prove that B is isomorphic to B' if (and only if) A is graded Morita equivalent to A' in some nice cases.

Throughout this paper, we fix an algebraically closed field k . Let A be a graded k -algebra. We denote by $\text{GrMod } A$ the category of graded right A -modules and right A -module homomorphisms preserving degree. We say that two graded algebras A and A' are graded Morita equivalent if there exists an equivalence of categories between $\text{GrMod } A$ and $\text{GrMod } A'$. For $M \in \text{GrMod } A$ and $n \in \mathbb{Z}$, the shift of M , denoted by $M(n)$, is the graded right A -module such that $M(n)_i = M_{i+n}$. For $M, N \in \text{GrMod } A$, we define the graded k -vector spaces $\underline{\text{Hom}}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_A(M, N(n))$ and $\underline{\text{Ext}}_A^i(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Ext}_A^i(M, N(n))$. We say that A is connected if $A_i = 0$ for all $i < 0$, and $A_0 = k$.

An AS-regular algebra defined below is one of the first classes of algebras studied in noncommutative algebraic geometry.

DEFINITION 1.1 ([1])

Let A be a connected graded k -algebra. Then A is called a d -dimensional AS-regular (resp., AS-Gorenstein) algebra if it satisfies the following conditions:

- $\text{gldim } A = d < \infty$ (resp., $\text{id}(A) = d < \infty$),
- $\text{GKdim } A < \infty$, and
- (Gorenstein condition)

$$\underline{\text{Ext}}_A^i(k, A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ k(l) \text{ for some } l \in \mathbb{Z} & \text{if } i = d. \end{cases}$$

A 1-dimensional AS-regular algebra is isomorphic to a polynomial algebra $k[x]$. A 2-dimensional AS-regular algebra generated in degree 1 is isomorphic to either of the forms

$$k\langle x, y \rangle / (-x^2 + xy - yx) \quad \text{or} \quad k\langle x, y \rangle / (xy - \lambda yx) \quad (\lambda \neq 0).$$

Moreover, every 2-dimensional AS-regular algebra generated in degree 1 is graded Morita equivalent to $k[x, y]$. Classification of 3-dimensional AS-regular algebras generated in degree 1 was started by Artin and Schelter in their paper [1]. Later Artin, Tate, and Van den Bergh [2] completed the classification of 3-dimensional AS-regular algebras generated in degree 1 by using geometric approach.

Let $T(V)$ be the tensor algebra on V over k , where V is a finite-dimensional vector space. We say that A is a quadratic algebra if A is a graded algebra of the form $T(V)/(R)$, where $R \subseteq V \otimes_k V$ is a subspace and (R) is the ideal of $T(V)$ generated by R . For a quadratic algebra $A = T(V)/(R)$, we define

$$\Gamma_2 := \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p, q) = 0 \text{ for all } f \in R\}.$$

DEFINITION 1.2 ([5, DEFINITION 4.3])

A quadratic algebra $A = T(V)/(R)$ is called geometric if there exists a geometric pair (E, σ) , where $E \subseteq \mathbb{P}(V^*)$ is a closed k -subscheme and σ is a k -automorphism of E such that

- (G1) $\Gamma_2 = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$, and
- (G2) $R = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0 \text{ for all } p \in E\}$.

Let $A = T(V)/(R)$ be a quadratic algebra. If A satisfies condition (G1), then A determines a geometric pair (E, σ) . If A satisfies condition (G2), then A is determined by a geometric pair (E, σ) , so we write $A = \mathcal{A}(E, \sigma)$.

If A is a 3-dimensional quadratic AS-regular algebra, then $A = \mathcal{A}(E, \sigma)$ is geometric, and E is either \mathbb{P}^2 or a cubic curve in \mathbb{P}^2 . Artin, Tate, and Van den

Bergh [2] described the “generic” classification of 3-dimensional quadratic AS-regular algebras in terms of their geometric pairs (E, σ) . In their classification, E is one of the following:

- (i) a triangle,
- (ii) a union of a line and a conic meeting at two points,
- (iii) an elliptic curve.

For example, a 3-dimensional Sklyanin algebra $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quadratic AS-regular algebra such that E is an elliptic curve and σ is given by the translation automorphism by a fixed point $p \in E$. As another example, an n -dimensional skew polynomial algebra A , namely,

$$A = k\langle x_1, \dots, x_n \rangle / (x_i x_j - \alpha_{ij} x_j x_i) \quad (\alpha_{ij} \alpha_{ji} = \alpha_{ii} = 1, \forall i, j \in \{1, \dots, n\}),$$

is an n -dimensional geometric AS-regular algebra. Note that skew polynomial algebras are Noetherian Koszul.

Let A be a Noetherian d -dimensional AS-Gorenstein algebra, and let $\mathfrak{m} := A_{\geq 1}$ be the unique maximal homogeneous ideal of A . We define the graded $(A-A)$ bimodule ω_A by

$$\omega_A := H_{\mathfrak{m}}^d(A)^* = \underline{\text{Hom}}_k \left(\lim_{n \rightarrow \infty} \underline{\text{Ext}}_A^d(A/A_{\geq n}, A), k \right).$$

It is known that $\omega_A \cong {}_{\nu}A(-l)$ as graded $(A-A)$ -bimodules for some graded k -algebra automorphism $\nu \in \text{Aut}_k A$, where ${}_{\nu}A$ is the graded $(A-A)$ -bimodule defined by ${}_{\nu}A = A$ as a graded k -vector space with a new action $a * x * b := \nu(a)xb$ (cf. [4, Theorem 1.2]).

DEFINITION 1.3 ([6])

Let A be a Noetherian d -dimensional AS-Gorenstein algebra. We call $\nu \in \text{Aut}_k A$ the generalized Nakayama automorphism of A , where $\omega_A \cong {}_{\nu}A(-l)$ as graded $(A-A)$ -bimodules. If the generalized Nakayama automorphism $\nu \in \text{Aut}_k A$ is id_A , then A is called symmetric.

A finite-dimensional algebra A is called graded Frobenius if $A^* \cong A(-l)$ as right- and left-graded A -modules. Let A be a graded Frobenius algebra. Then $A^* \cong {}_{\nu}A(-l)$ as graded $A-A$ bimodules, where ν is the usual Nakayama automorphism. Since A is a Noetherian AS-Gorenstein algebra of $\text{id}(A) = 0$ and $\omega_A = H_{\mathfrak{m}}^0(A)^* \cong A^* \cong {}_{\nu}A(-l)$, the generalized Nakayama automorphism of A is the usual Nakayama automorphism (see [6]).

The following theorem motivates this paper.

THEOREM 1.4 ([5, THEOREM 5.4])

Let $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ be 3-dimensional Sklyanin algebras. If $\sigma^9, \sigma'^9 \neq \text{id}$, then the following are equivalent:

- (1) $\text{GrMod } \mathcal{A}(E, \sigma) \cong \text{GrMod } \mathcal{A}(E', \sigma')$.
- (2) $\mathcal{A}(E, \sigma^3) \cong \mathcal{A}(E', \sigma'^3)$ as graded algebras.

So we consider relationships between the following conditions for Noetherian geometric AS-regular algebras $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ with $\dim_k A_1 = \dim_k A'_1 = n$:

- (A) $\text{GrMod } \mathcal{A}(E, \sigma) \cong \text{GrMod } \mathcal{A}(E', \sigma')$,
- (B) $\mathcal{A}(E, \nu^* \sigma^n) \cong \mathcal{A}(E', (\nu')^* (\sigma')^n)$ as graded algebras,

where $\nu \in \text{Aut}_k A, \nu' \in \text{Aut}_k A'$ are the generalized Nakayama automorphisms.

The generalized Nakayama automorphism of a 3-dimensional Sklyanin algebra A is id_A (cf. [8, Example 10.1]). Hence if $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ are 3-dimensional Sklyanin algebras with $\sigma^9, \sigma'^9 \neq \text{id}$, then (A) \Leftrightarrow (B) is true.

In this paper, we prove (A) \Leftrightarrow (B) for “generic” 3-dimensional quadratic AS-regular algebras A, A' whose geometric pairs are of the same type (see Theorem 3.1). Moreover, we show that if A, A' are n -dimensional skew polynomial algebras, then (A) \Rightarrow (B) is true (see Theorem 4.1). However, (B) \Rightarrow (A) is false for $n = 4$. In each situation, we also show that $\mathcal{A}(E, \nu^* \sigma^n)$ is symmetric (see Theorems 3.2, 4.3).

2. Preliminaries

For the purpose of this paper, we define the types of some geometric pairs (E, σ) of 3-dimensional quadratic AS-regular algebras as follows.

- Type \mathbb{P}^2 : E is \mathbb{P}^2 , and $\sigma \in \text{Aut}_k \mathbb{P}^2 = \text{PGL}_3(k)$.
- Type S_1 : E is a triangle, and σ stabilizes each component.
- Type S_2 : E is a triangle, and σ interchanges two components.
- Type S_3 : E is a triangle, and σ circulates three components.
- Type S'_1 : E is a union of a line and a conic meeting at two points, and σ stabilizes each component and two intersection points.
- Type S'_2 : E is a union of a line and a conic meeting at two points, and σ stabilizes each component and interchanges two intersection points.

REMARK 2.1

If E is a union of a line and a conic meeting at two points, and σ interchanges these two components, then $\mathcal{A}(E, \sigma)$ is not an AS-regular algebra (see [2, Proposition 4.11]). Thus the above types completely cover the “generic” singular cases and $E = \mathbb{P}^2$.

Recall that the Hilbert series of A is defined by

$$H_A(t) = \sum_{i=-\infty}^{\infty} (\dim A_i) t^i \in \mathbb{Z}[[t, t^{-1}]].$$

If A is a 3-dimensional quadratic AS-regular algebra, then A is a Noetherian Koszul domain and $H_A(t) = (1 - t)^{-3}$.

For geometric algebras, isomorphism and graded Morita equivalence can be characterized in terms of their geometric pairs.

THEOREM 2.2 ([5, THEOREM 4.7])

Let $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ be geometric algebras. Then the following hold.

(1) We have $A \cong A'$ as graded algebras if and only if there exists an isomorphism $\tau : E \rightarrow E'$ which extends to an automorphism $\bar{\tau} : \mathbb{P}(V^*) \rightarrow \mathbb{P}(V^*)$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E' \\ \sigma \downarrow & & \downarrow \sigma' \\ E & \xrightarrow{\tau} & E' \end{array}$$

commutes.

(2) We have $\text{GrMod } A \cong \text{GrMod } A'$ if and only if there exists a sequence isomorphism $\tau_n : E \rightarrow E'$ for $n \in \mathbb{Z}$ which extends to automorphisms $\bar{\tau}_n : \mathbb{P}(V^*) \rightarrow \mathbb{P}(V^*)$ such that the diagrams

$$\begin{array}{ccc} E & \xrightarrow{\tau_n} & E' \\ \sigma \downarrow & & \downarrow \sigma' \\ E & \xrightarrow{\tau_{n+1}} & E' \end{array}$$

commute for $n \in \mathbb{Z}$.

If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional skew polynomial algebra, then the geometric pair (E, σ) is of type \mathbb{P}^2 or S_1 . In either case, the following theorem holds.

THEOREM 2.3

Let

$$A = k\langle x_1, \dots, x_n \rangle / (x_i x_j - \alpha_{ij} x_j x_i), \quad A' = k\langle x_1, \dots, x_n \rangle / (x_i x_j - \alpha'_{ij} x_j x_i)$$

be n -dimensional skew polynomial algebras. Then the following hold.

(1) We have $A \cong A'$ as graded algebras if and only if there exists a permutation $\theta \in S_n$ such that $\alpha'_{ij} = \alpha_{\theta(i)\theta(j)}$ for any $1 \leq i, j \leq n$ (see [10, Lemma 2.1]).

(2) We have $\text{GrMod } A \cong \text{GrMod } A'$ if and only if there exists a permutation $\theta \in S_n$ such that $\alpha'_{ki} \alpha'_{ij} \alpha'_{jk} = \alpha_{\theta(k)\theta(i)} \alpha_{\theta(i)\theta(j)} \alpha_{\theta(j)\theta(k)}$ for any $1 \leq i, j, k \leq n$ (see [3, Theorem 5.1]).

The following lemma shows when two algebras of type S'_1 are isomorphic.

LEMMA 2.4

Let

$$A = k\langle x, y, z \rangle / (xy - \alpha yx, zx - \alpha xz, -x^2 + yz - \alpha zy),$$

$$A' = k\langle x, y, z \rangle / (xy - \alpha'yx, zx - \alpha'xz, -x^2 + yz - \alpha'zy),$$

where $\alpha, \alpha' \neq 0, \alpha^3, \alpha'^3 \neq 1$. Then $A \cong A'$ if and only if $\alpha' = \alpha^{\pm 1}$.

Proof

We prove (\Rightarrow) by using Theorem 2.2(1). Now $A = \mathcal{A}(E, \sigma)$ is a geometric algebra such that $E = C \cup l, C = \mathcal{V}(x^2 + \gamma yz), l = \mathcal{V}(x), \gamma = \alpha^2 - \alpha^{-1}$, and $\sigma \in \text{Aut}_k E$ is given by

$$\begin{aligned} \sigma|_l(0, b, c) &= (0, b, \alpha c), \\ \sigma|_C(a, b, c) &= (a, \alpha b, \alpha^{-1}c) \end{aligned}$$

(type S'_1). The same is true for $A' = \mathcal{A}(E', \sigma')$. If $\tau : E \rightarrow E'$ is an isomorphism which extends to an automorphism $\bar{\tau} : \mathbb{P}(V^*) \rightarrow \mathbb{P}(V^*)$, then τ sends two intersection points $\{(0, 1, 0), (0, 0, 1)\} = C \cap l \subset E$ to two intersection points $\{(0, 1, 0), (0, 0, 1)\} = C' \cap l' \subset E'$, so

$$\bar{\tau}(a, b, c) = (a, pa + qb, ra + sc) \quad (\text{fixing two intersection points})$$

or

$$\bar{\tau}(a, b, c) = (a, ra + sc, pa + qb) \quad (\text{interchanging two intersection points}),$$

where $p, q, r, s \in k$. It follows from $\tau\sigma = \sigma'\tau$ that $\alpha' = \alpha^{\pm 1}$. □

If $A = T(V)/(R)$ is a quadratic algebra, then we define the quadratic dual algebra of A by

$$A^! = T(V^*)/(R^\perp), \quad R^\perp = \{ \lambda \in V^* \otimes_k V^* \mid \lambda(r) = 0 \text{ for all } r \in R \}.$$

Let $A = T(V)/(R) = \mathcal{A}(E, \sigma)$ be a Noetherian AS-regular geometric algebra of $\dim A_1 = n$. If ν is the generalized Nakayama automorphism of A , then it restricts to an automorphism $\nu \in \text{Aut}_k V = \text{Aut}_k A_1$. So its dual induces an automorphism $\nu^* \in \text{Aut}_k \mathbb{P}(V^*)$, which induces an automorphism $\nu^* \in \text{Aut}_k E$ (see [6]). Moreover, ν^* extends to the unique graded algebra automorphism $\nu^* \in \text{Aut}_k A^!$, and $\nu^! = \varepsilon^{n+1}\nu^* \in \text{Aut}_k A^!$ is the Nakayama automorphism of $A^!$, where $\varepsilon \in \text{Aut}_k A^!$ is the multiplication by $(-1)^i$ on $A^!_i$ (see [9]), so $\nu^* = \nu^!$ in $\text{Aut}_k E$.

If $A = \mathcal{A}(E, \sigma)$ is a d -dimensional Koszul AS-regular algebra, then $A^!$ is graded Frobenius (see [8, Theorem 5.10]), so $\dim_k A^!_d = 1$, and the map $(-, -) : A^! \times A^! \rightarrow k \cong A^!_d$ defined by

$$(a, b) = \text{the component } ab \text{ in } A^!_d$$

is a nondegenerate bilinear pairing (Frobenius pairing) such that $(ab, c) = (a, bc)$ for all $a, b, c \in A^!$ (see [8, Lemma 3.2]). If $\nu^!$ is the Nakayama automorphism of $A^!$, then

$$(a, b) = (\nu^!(b), a)$$

for all $a, b \in A^!$ (see [8, Lemma 3.3]). Using these facts, we can calculate $\nu^* \in \text{Aut}_k E$.

3. Results for three-dimensional AS-regular algebras

In this section, we prove (A) \Leftrightarrow (B) for “generic” 3-dimensional quadratic AS-regular algebras.

THEOREM 3.1

Let $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ be 3-dimensional quadratic AS-regular algebras, and let $\nu \in \text{Aut}_k A, \nu' \in \text{Aut}_k A'$ be the generalized Nakayama automorphisms. If (E, σ) and (E', σ') are of the following same type: $\mathbb{P}^2, S_1, S_2, S_3, S'_1$ or S'_2 , then

$$\text{GrMod } \mathcal{A}(E, \sigma) \cong \text{GrMod } \mathcal{A}(E', \sigma') \Leftrightarrow \mathcal{A}(E, \nu^* \sigma^3) \cong \mathcal{A}(E', (\nu')^*(\sigma')^3).$$

Proof

We give proofs for types \mathbb{P}^2, S_1, S_2 , and S'_1 . Using [5, proof of Theorem 5.2], the proofs for the other types are analogous.

Case (I). Assume that both (E, σ) and (E', σ') are of type \mathbb{P}^2 . In this case, applying Theorem 2.2(2), we can show that

$$\text{GrMod } \mathcal{A}(\mathbb{P}^2, \sigma) \cong \text{GrMod } \mathcal{A}(\mathbb{P}^2, \text{id}) \cong \text{GrMod } k[x, y, z]$$

for any $\sigma \in \text{PGL}_3(k)$, so it is enough to show

$$\mathcal{A}(\mathbb{P}^2, \nu^* \sigma^3) \cong k[x, y, z]$$

for any $\sigma \in \text{PGL}_3(k)$. Since $\mathcal{A}(\mathbb{P}^2, \sigma) \cong \mathcal{A}(\mathbb{P}^2, \tau\sigma\tau^{-1})$ for any $\tau \in \text{PGL}_3(k)$ by Theorem 2.2(1), we may assume that $\sigma \in \text{PGL}_3(k)$ is one of the following three cases:

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 1 & \alpha \end{pmatrix}$$

by Jordan canonical form.

If $\sigma = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$, then $A = \mathcal{A}(\mathbb{P}^2, \sigma)$ is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \beta\alpha^{-1}yx, \quad yz - \gamma\beta^{-1}zy, \quad zx - \alpha\gamma^{-1}xz.$$

Now $A^!$ is $k\langle x, y, z \rangle$ with the defining relations

$$\beta\alpha^{-1}xy + yx, \quad x^2, \quad \gamma\beta^{-1}yz + zy, \quad y^2, \quad \alpha\gamma^{-1}zx + xz, \quad z^2.$$

Since

$$\begin{aligned} (yz, x) &= (yz)x = \beta\gamma\alpha^{-2}x(yz) = (\beta\gamma\alpha^{-2}x, yz), \\ (zx, y) &= (zx)y = \alpha\gamma\beta^{-2}y(zx) = (\alpha\gamma\beta^{-2}y, zx), \\ (xy, z) &= (xy)z = \alpha\beta\gamma^{-2}z(xy) = (\alpha\beta\gamma^{-2}z, xy) \end{aligned}$$

in A^1 , the automorphisms $\nu^* = (-1)^{3+1}\nu^* = \nu^1 \in \text{Aut}_k V^*$ and $\nu^* \in \text{Aut}_k \mathbb{P}^2$ are given by

$$(3.1) \quad \nu^* = \begin{pmatrix} \beta\gamma\alpha^{-2} & 0 & 0 \\ 0 & \alpha\gamma\beta^{-2} & 0 \\ 0 & 0 & \alpha\beta\gamma^{-2} \end{pmatrix}$$

when viewed as an element in $\text{GL}_3(k)$ and in $\text{PGL}_3(k)$, respectively. Then $\nu^* \sigma^3$ is id; so $B = \mathcal{A}(\mathbb{P}^2, \nu^* \sigma^3) \cong k[x, y, z]$.

If $\sigma = \begin{pmatrix} \alpha & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$, then $A = \mathcal{A}(\mathbb{P}^2, \sigma)$ is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha^{-1}x^2 - yx, \quad xz + \alpha yz - \beta zy, \quad zx - \alpha\beta^{-1}xz.$$

Now A^1 is $k\langle x, y, z \rangle$ with the defining relations

$$\begin{aligned} xy + \alpha x^2, & \quad xy + yx, & \quad \beta^{-1}zy + xz + \alpha\beta^{-1}zx, \\ xy + \alpha\beta^{-1}zy, & \quad y^2, & \quad z^2. \end{aligned}$$

Since

$$\begin{aligned} (byz + czx)x &= (\beta\alpha^{-1}x - 3\beta\alpha^{-2}y)(byz + czx), \\ (zx)y &= \beta\alpha^{-1}y(zx), \\ (xy)z &= \alpha^2\beta^{-2}z(xy) \end{aligned}$$

in A^1 , the automorphisms $\nu^* \in \text{Aut}_k V^*$ and $\nu^* \in \text{Aut}_k \mathbb{P}^2$ are given by

$$\nu^* = \begin{pmatrix} \beta\alpha^{-1} & 0 & 0 \\ -3\beta\alpha^{-2} & \beta\alpha^{-1} & 0 \\ 0 & 0 & \alpha^2\beta^{-2} \end{pmatrix}.$$

Then $\nu^* \sigma^3$ is id, so $A = \mathcal{A}(\mathbb{P}^2, \nu^* \sigma^3) \cong k[x, y, z]$.

The proof for $\sigma = \begin{pmatrix} \alpha & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$ is similar to the above and is left to the readers.

Case (II). Assume that both (E, σ) and (E', σ') are of type S_1 (cf. [5, Example 4.10], [7, Example 2.15]). We may assume that $E = E' = l_1 \cup l_2 \cup l_3$, where $l_1 = \mathcal{V}(x), l_2 = \mathcal{V}(y), l_3 = \mathcal{V}(z)$, and $\sigma \in \text{Aut}_k E$ is given by

$$\begin{aligned} \sigma|_{l_1}(0, b, c) &= (0, b, \alpha c), \\ \sigma|_{l_2}(a, 0, c) &= (\beta a, 0, c), \\ \sigma|_{l_3}(a, b, 0) &= (a, \gamma b, 0), \end{aligned}$$

and $\sigma' \in \text{Aut}_k E'$ is given by

$$\begin{aligned} \sigma'|_{l_1}(0, b, c) &= (0, b, \alpha' c), \\ \sigma'|_{l_2}(a, 0, c) &= (\beta' a, 0, c), \\ \sigma'|_{l_3}(a, b, 0) &= (a, \gamma' b, 0), \end{aligned}$$

where $\alpha\beta\gamma, \alpha'\beta'\gamma' \neq 0, 1$. Then $A = \mathcal{A}(E, \sigma)$ is $k\langle x, y, z \rangle$ with the defining relations

$$yz - \alpha zy, \quad zx - \beta xz, \quad xy - \gamma yx,$$

and $A' = \mathcal{A}(E', \sigma')$ is $k\langle x, y, z \rangle$ with the defining relations

$$yz - \alpha' zy, \quad zx - \beta' xz, \quad xy - \gamma' yx.$$

Now A^1 is $k\langle x, y, z \rangle$ with the defining relations

$$\alpha yz + zy, \quad x^2, \quad \beta zx + xz, \quad y^2, \quad \gamma xy + yx, \quad z^2.$$

Since

$$\begin{aligned} (zy)x &= \gamma\beta^{-1}x(zy), \\ (xz)y &= \alpha\gamma^{-1}y(xz), \\ (yx)z &= \beta\alpha^{-1}z(yx) \end{aligned}$$

in A^1 , the automorphisms $\nu^* \in \text{Aut}_k V^*$ and $\nu^* \in \text{Aut}_k E$ are given by

$$(3.2) \quad \nu^* = \begin{pmatrix} \gamma\beta^{-1} & 0 & 0 \\ 0 & \alpha\gamma^{-1} & 0 \\ 0 & 0 & \beta\alpha^{-1} \end{pmatrix}.$$

Then $\nu^*\sigma^3$ is given by

$$\begin{aligned} \nu^*\sigma^3|_{l_1}(0, b, c) &= (0, b, \alpha\beta\gamma c), \\ \nu^*\sigma^3|_{l_2}(a, 0, c) &= (\alpha\beta\gamma a, 0, c), \\ \nu^*\sigma^3|_{l_3}(a, b, 0) &= (a, \alpha\beta\gamma b, 0), \end{aligned}$$

so $B = \mathcal{A}(E, \nu^*\sigma^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$yz - \alpha\beta\gamma zy, \quad zx - \alpha\beta\gamma xz, \quad xy - \alpha\beta\gamma yx.$$

Similarly, $B' = \mathcal{A}(E', (\nu')^*(\sigma')^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$yz - \alpha'\beta'\gamma' zy, \quad zx - \alpha'\beta'\gamma' xz, \quad xy - \alpha'\beta'\gamma' yx.$$

It follows that if both (E, σ) and (E', σ') are of type S_1 , then

$$\mathcal{A}(E, \nu^*\sigma^3) \cong \mathcal{A}(E', (\nu')^*(\sigma')^3) \iff \alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$$

by Theorem 2.3(1). Furthermore, [5, Example 4.10] says that

$$\text{GrMod } \mathcal{A}(E, \sigma) \cong \text{GrMod } \mathcal{A}(E', \sigma') \iff \alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}.$$

The assertion is proved.

Case (III). Assume that both (E, σ) and (E', σ') are of type S_2 . We may assume that $E = E' = l_1 \cup l_2 \cup l_3$, where $l_1 = \mathcal{V}(x), l_2 = \mathcal{V}(y), l_3 = \mathcal{V}(z)$, and $\sigma \in \text{Aut}_k E$ is given by

$$\sigma|_{l_1}(0, b, c) = (\alpha b, 0, c),$$

$$\sigma|_{l_2}(a, 0, c) = (0, a, \beta c),$$

$$\sigma|_{l_3}(a, b, 0) = (b, \gamma a, 0),$$

and $\sigma' \in \text{Aut}_k E'$ is given by

$$\sigma'|_{l_1}(0, b, c) = (\alpha' b, 0, c),$$

$$\sigma'|_{l_2}(a, 0, c) = (0, a, \beta' c),$$

$$\sigma'|_{l_3}(a, b, 0) = (b, \gamma' a, 0),$$

where $\alpha\beta\gamma, \alpha'\beta'\gamma' \neq 0, 1$. Then A is $k\langle x, y, z \rangle$ with the defining relations

$$zx - \alpha yz, \quad xz - \beta zy, \quad y^2 - \gamma x^2,$$

and A' is $k\langle x, y, z \rangle$ with the defining relations

$$zx - \alpha' yz, \quad xz - \beta' zy, \quad y^2 - \gamma' x^2.$$

Now A^1 is $k\langle x, y, z \rangle$ with the defining relations

$$\alpha zx + yz, \quad xy, \quad \beta xz + zy, \quad yx, \quad \gamma y^2 + x^2, \quad z^2.$$

Since

$$(zx)x = -\alpha^{-1}y(zx),$$

$$(xz)y = -\beta x(xz),$$

$$(y^2)z = -\alpha^2\gamma z(y^2) = -\gamma^{-1}\beta^{-2}z(y^2)$$

in A^1 , we must have

$$-\alpha^2\gamma = -\gamma^{-1}\beta^{-2}.$$

(Suppose $z(y^2) = 0$; then $H_{A^1}(t) \neq (1+t)^3$, which contradicts the fact that A is a 3-dimensional quadratic AS-regular algebra.) It follows from $\alpha\beta\gamma \neq 1$ that $\alpha\beta\gamma = -1$ is a necessary condition for A to be a 3-dimensional AS-regular algebra. Furthermore, the automorphisms $\nu^* \in \text{Aut}_k V^*$ and $\nu^* \in \text{Aut}_k E$ are given by

$$\nu^* = \begin{pmatrix} 0 & \beta & 0 \\ \alpha^{-1} & 0 & 0 \\ 0 & 0 & \alpha^2\gamma \end{pmatrix}.$$

Then $\nu^*\sigma^3$ is given by

$$\nu^*\sigma^3|_{l_1}(0, b, c) = (0, b, -c),$$

$$\nu^*\sigma^3|_{l_2}(a, 0, c) = (-a, 0, c),$$

$$\nu^*\sigma^3|_{l_3}(a, b, 0) = (a, -b, 0),$$

so $B = \mathcal{A}(E, \nu^*\sigma^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$yz + zy, \quad zx + xz, \quad xy + yx.$$

Similarly, $B' = \mathcal{A}(E', (\nu')^*(\sigma')^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$yz + zy, \quad zx + xz, \quad xy + yx.$$

It follows that if both (E, σ) and (E', σ') are of type S_2 such that $\mathcal{A}(E, \sigma)$ and $\mathcal{A}(E', \sigma')$ are AS-regular algebras, then

$$\mathcal{A}(E, \nu^* \sigma^3) \cong \mathcal{A}(E', (\nu')^* (\sigma')^3).$$

Furthermore, since $\alpha\beta\gamma = \alpha'\beta'\gamma' = -1$,

$$\text{GrMod } \mathcal{A}(E, \sigma) \cong \text{GrMod } \mathcal{A}(E', \sigma')$$

by Theorem 2.2(2). The assertion is proved.

Case (IV). Assume that both (E, σ) and (E', σ') are of type S'_1 . We may assume that $E = C \cup l$, where $C = \mathcal{V}(x^2 + \gamma yz)$, $l = \mathcal{V}(x)$, $\gamma = \alpha\beta - \beta^{-1}$, and $\sigma \in \text{Aut}_k E$ is given by

$$\begin{aligned} \sigma|_l(0, b, c) &= (0, b, \alpha c), \\ \sigma|_C(a, b, c) &= (a, \beta b, \beta^{-1}c). \end{aligned}$$

Similarry, we assume that $E' = C' \cup l'$, where $C' = \mathcal{V}(x^2 + \gamma' yz)$, $l' = \mathcal{V}(x)$, $\gamma' = \alpha'\beta' - \beta'^{-1}$, and $\sigma' \in \text{Aut}_k E'$ is given by

$$\begin{aligned} \sigma'|_{l'}(0, b, c) &= (0, b, \alpha' c), \\ \sigma'|_{C'}(a, b, c) &= (a, \beta' b, \beta'^{-1}c), \end{aligned}$$

where $\alpha\beta^2, \alpha'\beta'^2 \neq 0, 1$. Then A is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \beta yx, \quad zx - \beta xz, \quad -x^2 + yz - \alpha zy,$$

and A' is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \beta' yx, \quad zx - \beta' xz, \quad -x^2 + yz - \alpha' zy.$$

Now $A^!$ is $k\langle x, y, z \rangle$ with the defining relations

$$\beta xy + yx, \quad y^2, \quad \beta zx + xz, \quad z^2, \quad x^2 + yz, \quad \alpha yz + zy.$$

Since

$$\begin{aligned} (zy)x &= x(zy), \\ (xz)y &= \alpha\beta^{-1}y(xz), \\ (yx)z &= \beta\alpha^{-1}z(yx) \end{aligned}$$

in $A^!$, the automorphisms $\nu^* \in \text{Aut}_k V^*$ and $\nu^* \in \text{Aut}_k E$ are given by

$$(3.3) \quad \nu^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha\beta^{-1} & 0 \\ 0 & 0 & \beta\alpha^{-1} \end{pmatrix}.$$

Then $\nu^* \sigma^3$ is given by

$$\begin{aligned} \nu^* \sigma^3|_l(0, b, c) &= (0, b, \alpha\beta^2 c), \\ \nu^* \sigma^3|_C(a, b, c) &= (a, \alpha\beta^2 b, \alpha^{-1}\beta^{-2}c). \end{aligned}$$

(1) If $(\alpha\beta^2)^3 \neq 1$, then $(E, \nu^*\sigma^3)$ is of type S'_1 , so $B = \mathcal{A}(E, \nu^*\sigma^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha\beta^2yx, \quad zx - \alpha\beta^2xz, \quad -\delta\gamma^{-1}x^2 + yz - \alpha\beta^2zy,$$

where $\delta = (\alpha\beta^2)^2 - (\alpha\beta^2)^{-1}$. If we define the new graded algebra $\overline{B} = k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha\beta^2yx, \quad zx - \alpha\beta^2xz, \quad -x^2 + yz - \alpha\beta^2zy,$$

then $B \cong \overline{B}$ by change of generators $x \mapsto \sqrt{\gamma\delta^{-1}}x$.

Similarly, if $(\alpha'\beta'^2)^3 \neq 1$, then $B' = \mathcal{A}(E', (\nu')^*(\sigma')^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha'\beta'^2yx, \quad zx - \alpha'\beta'^2xz, \quad -\delta'\gamma'^{-1}x^2 + yz - \alpha'\beta'^2zy,$$

where $\delta' = (\alpha'\beta'^2)^2 - (\alpha'\beta'^2)^{-1}$. We define $\overline{B}' = k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha'\beta'^2yx, \quad zx - \alpha'\beta'^2xz, \quad -x^2 + yz - \alpha'\beta'^2zy;$$

then $B' \cong \overline{B}'$. Thus, by Lemma 2.4,

$$B \cong B' \iff \overline{B} \cong \overline{B}' \iff \alpha'\beta'^2 = (\alpha\beta^2)^{\pm 1}.$$

(2) If $(\alpha\beta^2)^3 = 1$, then $(E, \nu^*\sigma^3)$ is of type \mathbb{P}^2 , so $B = \mathcal{A}(E, \nu^*\sigma^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha\beta^2yx, \quad zx - \alpha\beta^2xz, \quad yz - \alpha\beta^2zy.$$

Similarly, if $(\alpha'\beta'^2)^3 = 1$, then $B' = \mathcal{A}(E', (\nu')^*(\sigma')^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha'\beta'^2yx, \quad zx - \alpha'\beta'^2xz, \quad yz - \alpha'\beta'^2zy.$$

Thus, by Theorem 2.3(1),

$$B \cong B' \iff \alpha'\beta'^2 = (\alpha\beta^2)^{\pm 1}.$$

It follows that if both (E, σ) and (E', σ') are of type S'_1 , then

$$\mathcal{A}(E, \nu^*\sigma^3) \cong \mathcal{A}(E', (\nu')^*(\sigma')^3) \iff \alpha'\beta'^2 = (\alpha\beta^2)^{\pm 1}.$$

Applying Theorem 2.2(2), we can show that

$$\text{GrMod } \mathcal{A}(E, \sigma) \cong \text{GrMod } \mathcal{A}(E', \sigma') \iff \alpha'\beta'^2 = (\alpha\beta^2)^{\pm 1}$$

by Theorem 2.2(2). The assertion is proved. □

Next, we show that $\mathcal{A}(E, \nu^*\sigma^3)$ in the above theorem are symmetric.

THEOREM 3.2

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quadratic AS-regular algebra, and let $\nu \in \text{Aut}_k A$ be the generalized Nakayama automorphism. If (E, σ) is of the following types: $\mathbb{P}^2, S_1, S_2, S_3, S'_1$ or S'_2 , then $B = \mathcal{A}(E, \nu^\sigma^3)$ is symmetric.*

Proof

We will give proofs for types \mathbb{P}^2 , S_2 , and S'_1 . The proofs for the other types are analogous. Since

$$\nu_B = \text{id}_B \in \text{Aut}_k B \Leftrightarrow \nu_B = \text{id} \in \text{Aut}_k V \Leftrightarrow \nu_B^* = \text{id} \in \text{Aut}_k V^*,$$

it is enough to show $\nu_B^* = \text{id} \in \text{Aut}_k V^*$, where ν_B is the generalized Nakayama automorphism of B .

If (E, σ) is of type \mathbb{P}^2 , then $B = \mathcal{A}(E, \nu^* \sigma^3)$ is $k\langle x, y, z \rangle$, so B is symmetric.

Assume that (E, σ) is of type S_2 . Then $B = \mathcal{A}(E, \nu^* \sigma^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$yz + zy, \quad zx + xz, \quad xy + yx,$$

so $(E, \nu^* \sigma^3)$ is of type S_1 . By (3.2), $\nu_B^* = \text{id} \in \text{Aut}_k V^*$, so the generalized Nakayama automorphism ν_B is id_B .

Next, we assume that (E, σ) is of type S'_1 . If $(\alpha\beta^2)^3 \neq 1$, then $(E, \nu^* \sigma^3)$ is of type S'_1 . Since $B \cong \overline{B}$, we consider $\overline{B} = k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha\beta^2yx, \quad zx - \alpha\beta^2xz, \quad -x^2 + yz - \alpha\beta^2zy.$$

By (3.3), $\nu_{\overline{B}}^* \in \text{Aut}_k V^*$ is given by

$$\nu_{\overline{B}}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\alpha\beta^2)(\alpha\beta^2)^{-1} & 0 \\ 0 & 0 & (\alpha\beta^2)(\alpha\beta^2)^{-1} \end{pmatrix} = \text{id}.$$

Thus the generalized Nakayama automorphism ν_B is id_B .

If $(\alpha\beta^2)^3 = 1$, then $(E, \nu^* \sigma^3)$ is of type \mathbb{P}^2 , so $B = \mathcal{A}(E, \nu^* \sigma^3)$ is $k\langle x, y, z \rangle$ with the defining relations

$$xy - \alpha\beta^2yx, \quad zx - \alpha\beta^2xz, \quad yz - \alpha\beta^2zy,$$

and $\nu^* \sigma^3$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha\beta^2 & 0 \\ 0 & 0 & (\alpha\beta^2)^2 \end{pmatrix},$$

so by (3.1), $\nu_B^* = \text{id} \in \text{Aut}_k V^*$. Thus the generalized Nakayama automorphism ν is id_B . □

4. Results for skew polynomial algebras

We prove **(A)** \Rightarrow **(B)** for skew polynomial algebras.

THEOREM 4.1

If $A = \mathcal{A}(E, \sigma)$, $A' = \mathcal{A}(E', \sigma')$ are n -dimensional skew polynomial algebras and $\nu \in \text{Aut}_k A$, $\nu' \in \text{Aut}_k A'$ are the generalized Nakayama automorphisms, then

$$\text{GrMod } \mathcal{A}(E, \sigma) \cong \text{GrMod } \mathcal{A}(E', \sigma') \Rightarrow \mathcal{A}(E, \nu^* \sigma^n) \cong \mathcal{A}(E', (\nu')^*(\sigma')^n).$$

Proof

Let $A = k\langle x_1, \dots, x_n \rangle / (x_i x_j - \alpha_{ij} x_j x_i)$ be an n -dimensional skew polynomial algebra. Then $A = \mathcal{A}(E, \sigma)$ is geometric, where

$$E = \bigcap_{\alpha_{ij}\alpha_{jk}\alpha_{ki} \neq 1} \mathcal{V}(x_i x_j x_k)$$

by [3, Proposition 3.1] and [10, Proposition 3.1], and $\sigma \in \text{Aut}_k E$ is given by

$$\sigma(0, \dots, 0, a_i, 0, \dots, 0, a_j, 0, \dots, 0) = (0, \dots, 0, a_i, 0, \dots, 0, \alpha_{ij} a_j, 0, \dots, 0)$$

for any $i < j$ by [3, Remark 3.4].

Now A^1 is $k\langle x_1, \dots, x_n \rangle / (\alpha_{ij} x_i x_j + x_j x_i, x_i^2)$. Since $a_{ii} = 1$ and

$$\begin{aligned} & (x_1 \cdots x_{i-1} x_{i+1} \cdots x_n) x_i \\ &= (-1)^{n-1} \alpha_{in} \cdots \alpha_{i,i+1} \alpha_{i,i-1} \cdots \alpha_{i1} x_i (x_1 \cdots x_{i-1} x_{i+1} \cdots x_n) \end{aligned}$$

in A^1 for any i , the automorphism $\nu^1 \in \text{Aut } V^*$ induced by the Nakayama automorphism of A^1 is given by

$$(4.1) \quad \nu^1(x_i)_{1 \leq i \leq n} = \left(\prod_{1 \leq s \leq n} (-1)^{n-1} \alpha_{is} x_i \right)_{1 \leq i \leq n}.$$

So the automorphism $\nu^* : E \rightarrow E$ is given by

$$\nu^*(a_i)_{1 \leq i \leq n} = \left(\prod_{1 \leq s \leq n} \alpha_{is} a_i \right)_{1 \leq i \leq n}.$$

Thus

$$\begin{aligned} & \nu^* \sigma^n(0, \dots, 0, a_i, 0, \dots, 0, a_j, 0, \dots, 0) \\ &= \nu^*(0, \dots, 0, a_i, 0, \dots, 0, \alpha_{ij}^n a_j, 0, \dots, 0) \\ &= \left(0, \dots, 0, \prod_{1 \leq s \leq n} \alpha_{is} a_i, 0, \dots, 0, \alpha_{ij}^n \prod_{1 \leq s \leq n} \alpha_{js} a_j, 0, \dots, 0 \right) \\ &= \left(0, \dots, 0, a_i, 0, \dots, 0, \alpha_{ij}^n \frac{\prod_{1 \leq s \leq n} \alpha_{js}}{\prod_{1 \leq s \leq n} \alpha_{is}} a_j, 0, \dots, 0 \right) \\ &= \left(0, \dots, 0, a_i, 0, \dots, 0, \alpha_{ij}^n \prod_{1 \leq s \leq n} \alpha_{si} \alpha_{js} a_j, 0, \dots, 0 \right) \\ &= \left(0, \dots, 0, a_i, 0, \dots, 0, \prod_{1 \leq s \leq n} (\alpha_{si} \alpha_{ij} \alpha_{js}) a_j, 0, \dots, 0 \right). \end{aligned}$$

Hence

$$\begin{aligned} B = \mathcal{A}(E, \nu^* \sigma^n) &= k\langle x_1, \dots, x_n \rangle / \left(x_i x_j - \prod_{1 \leq s \leq n} (\alpha_{si} \alpha_{ij} \alpha_{js}) x_j x_i \right) \\ &= k\langle x_1, \dots, x_n \rangle / (x_i x_j - \beta_{ij} x_j x_i), \end{aligned}$$

where $\beta_{ij} := \prod_{1 \leq s \leq n} (\alpha_{si} \alpha_{ij} \alpha_{js})$.

Let $A' = \mathcal{A}(E', \sigma') = k\langle x_1, \dots, x_n \rangle / (x_i x_j - \alpha'_{ij} x_j x_i)$ be another skew polynomial algebra. Then

$$\begin{aligned} B' &= \mathcal{A}(E, \nu^* \sigma^n) = k\langle x_1, \dots, x_n \rangle / \left(x_i x_j - \prod_{1 \leq s \leq n} (\alpha'_{si} \alpha'_{ij} \alpha'_{js}) x_j x_i \right) \\ &= k\langle x_1, \dots, x_n \rangle / (x_i x_j - \beta'_{ij} x_j x_i), \end{aligned}$$

where $\beta'_{ij} := \prod_{1 \leq s \leq n} (\alpha'_{si} \alpha'_{ij} \alpha'_{js})$.

By Theorem 2.3(1), (2),

$$\text{GrMod } A \cong \text{GrMod } A'$$

$$\iff \exists \theta \in S_n \text{ such that } \alpha'_{si} \alpha'_{ij} \alpha'_{js} = \alpha_{\theta(s)\theta(i)} \alpha_{\theta(i)\theta(j)} \alpha_{\theta(j)\theta(s)}$$

$$\implies \exists \theta \in S_n \text{ such that } \prod_{1 \leq s \leq n} \alpha'_{si} \alpha'_{ij} \alpha'_{js} = \prod_{1 \leq s \leq n} \alpha_{\theta(s)\theta(i)} \alpha_{\theta(i)\theta(j)} \alpha_{\theta(j)\theta(s)}$$

$$\iff \exists \theta \in S_n \text{ such that } \prod_{1 \leq s \leq n} \alpha'_{si} \alpha'_{ij} \alpha'_{js} = \prod_{1 \leq s \leq n} \alpha_{s\theta(i)} \alpha_{\theta(i)\theta(j)} \alpha_{\theta(j)s}$$

$$\iff \exists \theta \in S_n \text{ such that } \beta'_{ij} = \beta_{\theta(i)\theta(j)}$$

$$\iff B \cong B'.$$

Hence this is the result. □

Unfortunately, the converse of Theorem 4.1 is false for $n = 4$. We construct a counterexample.

EXAMPLE 4.2

Let $A = \mathcal{A}(E, \sigma) = \mathbb{C}\langle x_1, x_2, x_3, x_4 \rangle$ be a 4-dimensional skew polynomial algebra with defining relations

$$\begin{aligned} x_1 x_2 - 2x_2 x_1, & & x_1 x_3 - 2x_3 x_1, & & x_1 x_4 - 2x_4 x_1, \\ x_2 x_3 - 2x_3 x_2, & & x_2 x_4 - 2x_4 x_2, & & x_3 x_4 - 2x_4 x_3, \end{aligned}$$

that is,

$$\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = \alpha_{34} = 2.$$

By the definition of $B = \mathcal{A}(E, \nu^* \sigma^n)$ in the proof of Theorem 4.1,

$$\beta_{12} = \alpha_{31} \alpha_{12} \alpha_{23} \alpha_{41} \alpha_{12} \alpha_{24} = 4, \quad \beta_{13} = \alpha_{21} \alpha_{13} \alpha_{32} \alpha_{41} \alpha_{13} \alpha_{34} = 1,$$

$$\beta_{14} = \alpha_{21} \alpha_{14} \alpha_{42} \alpha_{31} \alpha_{14} \alpha_{43} = 4^{-1}, \quad \beta_{23} = \alpha_{12} \alpha_{23} \alpha_{31} \alpha_{42} \alpha_{23} \alpha_{34} = 4,$$

$$\beta_{24} = \alpha_{12} \alpha_{24} \alpha_{41} \alpha_{32} \alpha_{24} \alpha_{43} = 1, \quad \beta_{34} = \alpha_{13} \alpha_{34} \alpha_{41} \alpha_{23} \alpha_{34} \alpha_{42} = 4,$$

so B is $\mathbb{C}\langle x_1, x_2, x_3, x_4 \rangle$ with defining relations

$$\begin{aligned} x_1 x_2 - 4x_2 x_1, & & x_1 x_3 - x_3 x_1, & & x_1 x_4 - 4^{-1} x_4 x_1, \\ x_2 x_3 - 4x_3 x_2, & & x_2 x_4 - x_4 x_2, & & x_3 x_4 - 4x_4 x_3. \end{aligned}$$

Let $A' = \mathcal{A}(E', \sigma') = \mathbb{C}\langle x_1, x_2, x_3, x_4 \rangle$ be a 4-dimensional skew polynomial algebra with defining relations

$$\begin{aligned} x_1x_2 - 2ix_2x_1, & \quad x_1x_3 - x_3x_1, & \quad x_1x_4 - x_4x_1, \\ x_2x_3 - x_3x_2, & \quad x_2x_4 + x_4x_2, & \quad x_3x_4 - 2ix_4x_3, \end{aligned}$$

where $i = \sqrt{-1}$, that is,

$$\alpha'_{12} = \alpha'_{34} = 2i, \quad \alpha'_{13} = \alpha'_{14} = \alpha'_{23} = 1, \quad \alpha'_{24} = -1.$$

By the definition of $B' = \mathcal{A}(E, \nu^* \sigma^4)$ in the proof of Theorem 4.1,

$$\begin{aligned} \beta'_{12} &= \alpha'_{31} \alpha'_{12} \alpha'_{23} \alpha'_{41} \alpha'_{12} \alpha'_{24} = 4, & \beta'_{13} &= \alpha'_{21} \alpha'_{13} \alpha'_{32} \alpha'_{41} \alpha'_{13} \alpha'_{34} = 1, \\ \beta'_{14} &= \alpha'_{21} \alpha'_{14} \alpha'_{42} \alpha'_{31} \alpha'_{14} \alpha'_{43} = 4^{-1}, & \beta'_{23} &= \alpha'_{12} \alpha'_{23} \alpha'_{31} \alpha'_{42} \alpha'_{23} \alpha'_{34} = 4, \\ \beta'_{24} &= \alpha'_{12} \alpha'_{24} \alpha'_{41} \alpha'_{32} \alpha'_{24} \alpha'_{43} = 1, & \beta'_{34} &= \alpha'_{13} \alpha'_{34} \alpha'_{41} \alpha'_{23} \alpha'_{34} \alpha'_{42} = 4, \end{aligned}$$

so B' is $\mathbb{C}\langle x_1, x_2, x_3, x_4 \rangle$ with defining relations

$$\begin{aligned} x_1x_2 - 4x_2x_1, & \quad x_1x_3 - x_3x_1, & \quad x_1x_4 - 4^{-1}x_4x_1, \\ x_2x_3 - 4x_3x_2, & \quad x_2x_4 - x_4x_2, & \quad x_3x_4 - 4x_4x_3. \end{aligned}$$

Hence $B = B'$. However, for any $\theta \in S_4$,

$$\alpha'_{12} \alpha'_{23} \alpha'_{31} = 2i \neq 2^{\pm 1} = \alpha_{\theta(1)\theta(2)} \alpha_{\theta(2)\theta(3)} \alpha_{\theta(3)\theta(1)},$$

so $\text{GrMod } A \not\cong \text{GrMod } A'$.

THEOREM 4.3

If $A = \mathcal{A}(E, \sigma)$ is an n -dimensional skew polynomial algebra and $\nu \in \text{Aut}_k A$ is the generalized Nakayama automorphism, then $B = \mathcal{A}(E, \nu^* \sigma^n)$ is symmetric.

Proof

Since $\nu^* = (-1)^{n+1} \nu^! \in \text{Aut}_k V^*$ by [9, Theorem 9.2], it follows from (4.1) that the generalized Nakayama automorphism of B is given by

$$\nu_B(x_i)_{1 \leq i \leq n} = \left(\prod_{1 \leq s \leq n} \beta_{is} x_i \right)_{1 \leq i \leq n},$$

where $\beta_{ij} = \prod_{1 \leq t \leq n} (\alpha_{ti} \alpha_{ij} \alpha_{jt})$. It follows that

$$\begin{aligned} \nu_B(x_i) &= \prod_{1 \leq s \leq n} \beta_{is} x_i \\ &= (\alpha_{1i} \alpha_{i1} \alpha_{11} \cdot \alpha_{2i} \alpha_{i1} \alpha_{12} \cdots \alpha_{ni} \alpha_{i1} \alpha_{1n} \\ &\quad \cdot \alpha_{1i} \alpha_{i2} \alpha_{21} \cdot \alpha_{2i} \alpha_{i2} \alpha_{22} \cdots \alpha_{ni} \alpha_{i2} \alpha_{2n} \\ &\quad \vdots \quad \quad \quad \vdots \quad \quad \quad \ddots \quad \quad \quad \vdots \\ &\quad \cdot \alpha_{1i} \alpha_{in} \alpha_{n1} \cdot \alpha_{2i} \alpha_{in} \alpha_{n2} \cdots \alpha_{ni} \alpha_{in} \alpha_{nn}) x_i \end{aligned}$$

for any i . If we define $\bar{\alpha}_{pq}^i = \alpha_{qi}\alpha_{ip}\alpha_{pq}$, then $\bar{\alpha}_{ss}^i = \bar{\alpha}_{st}^i\bar{\alpha}_{ts}^i = 1$ for any $1 \leq s < t \leq n$, so $\nu_B(x_i) = x_i$ for any i . Thus we obtain the result. \square

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