Quantum continuous $\mathfrak{gl}_\infty$: Semiinfinite construction of representations

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To the memory of the late Professor Hiroshi Saito

Abstract We begin a study of the representation theory of quantum continuous $\mathfrak{gl}_\infty$, which we denote by $\mathcal{E}$. This algebra depends on two parameters and is a deformed version of the enveloping algebra of the Lie algebra of difference operators acting on the space of Laurent polynomials in one variable. Fundamental representations of $\mathcal{E}$ are labeled by a continuous parameter $u \in \mathbb{C}$. The representation theory of $\mathcal{E}$ has many properties familiar from the representation theory of $\mathfrak{gl}_\infty$: vector representations, Fock modules, and semiinfinite constructions of modules. Using tensor products of vector representations, we construct surjective homomorphisms from $\mathcal{E}$ to spherical double affine Hecke algebras $S\ddot{H}_N$ for all N. A key step in this construction is an identification of a natural basis of the tensor products of vector representations with Macdonald polynomials. We also show that one of the Fock representations is isomorphic to the module constructed earlier by means of the K-theory of Hilbert schemes.

1. Introduction

In this paper we begin to study the representation theory of an algebra $\mathcal{E}$, which we call the quantum continuous $\mathfrak{gl}_\infty$. This algebra is a deformation of the universal enveloping algebra of the Lie algebra of the q-difference operators in one variable. Its representation theory has a lot in common with that of the usual $\mathfrak{gl}_\infty$ with a central extension: vector representations, fundamental representations, and semiinfinite constructions of modules. Still there is an important new

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feature of $\mathcal{E}$: fundamental representations of this algebra are labeled by a continuous parameter u. This makes the representation theory of $\mathcal{E}$ very rich and interesting. We give some details below.

The algebra $\mathcal{E}$ is defined in terms of generators and relations. The generators are denoted by e_i, f_i $(i \in \mathbb{Z})$, and ψ_j^+, ψ_{-j}^- $(j \geq 0)$. The elements $\psi_0^\pm$ are central and invertible. The relations between generators depend symmetrically on three parameters q_1, q_2, q_3 which are assumed to satisfy $q_1 q_2 q_3 = 1$. These relations are given explicitly in terms of generating series (see Section 2). For example, let $e(z) = \sum_{i \in \mathbb{Z}} e_i z^{-i}$. Then the following relation holds in $\mathcal{E}$:

$$\begin{aligned} g(z,w)e(z)e(w) &= -g(w,z)e(w)e(z), \\ g(z,w) &= (z - q_1 w)(z - q_2 w)(z - q_3 w). \end{aligned} \tag{1.1}$$

This relation appears in different contexts (see [FO], [Kap], [FT], [SV2]). In terms of components, (1.1) is equivalent to the set of relations labeled by integers $m, n \in \mathbb{Z}$:

$$\begin{aligned} & e_{n+3}e_m - (q_1 + q_2 + q_3)e_{n+2}e_{m+1} + (q_1q_2 + q_1q_3 + q_2q_3)e_{n+1}e_{m+2} - e_n e_{m+3} \\ & \quad = -e_{m+3}e_n + (q_1 + q_2 + q_3)e_{m+2}e_{n+1} \\ & \qquad - (q_1q_2 + q_1q_3 + q_2q_3)e_{m+1}e_{n+2} + e_m e_{n+3}. \end{aligned}$$

As we have mentioned above, relation (1.1) has different origins. Let us explain the one important for us. Fix a parameter q, and consider the associative algebra $A = \mathbb{C}[Z, Z^{-1}, D, D^{-1}]$ with $DZ = qZD$. The algebra A acts on the space $\mathbb{C}[z, z^{-1}]$ by $Z(f)(z) = zf(z)$, $(Df)(z) = f(qz)$. Thus A can be identified with the algebra of q-difference operators and can be thought of as an algebra of special infinite matrices. The algebra A admits representations with a continuous parameter u on the space of delta functions $\bigoplus_{i \in \mathbb{Z}} \delta(q^i u/z)$ through the same action on a vector $f(z)$ in this space. Thus A may be called a continuous $\mathfrak{gl}_\infty$.

Consider the elements $\bar{e}_i = Z^i D \in A$. It is easy to check that these elements satisfy the relations

$$\begin{aligned} & \bar{e}_{n+3}\bar{e}_m - (1 + q + q^{-1})\bar{e}_{n+2}\bar{e}_{m+1} + (1 + q + q^{-1})\bar{e}_{n+1}\bar{e}_{m+2} - \bar{e}_n\bar{e}_{m+3} \\ & \quad = -\bar{e}_{m+3}\bar{e}_n + (1 + q + q^{-1})\bar{e}_{m+2}\bar{e}_{n+1} - (1 + q + q^{-1})\bar{e}_{m+1}\bar{e}_{n+2} + \bar{e}_m\bar{e}_{n+3}. \end{aligned}$$

Thus the relation (1.1) is a quantization of the relations above. In fact, all other relations of $\mathcal{E}$ (see Section 2) can be obtained in a similar way. Moreover, there exists a Poisson structure on A (considered as a Lie algebra) such that the usual quantization technique, applied to the universal enveloping algebra of A, gives $\mathcal{E}$. We do not discuss this construction in this paper and will return to it elsewhere.

We recall that in [DI] the authors constructed a class of quantum algebras generalizing quantum affine algebras. A particular example of their construction is the algebra $\mathcal{E}'$, which differs from $\mathcal{E}$ only by the absence of the cubic relations (see (2.6) below)

$$\big[e_0, [e_1, e_{-1}]\big] = 0, \qquad \big[f_0, [f_1, f_{-1}]\big] = 0.$$

We call these relations Serre relations for $\mathcal{E}$. The algebra $\mathcal{E}'$ was also considered in [FHH+] and [FT] and was called there the Ding-Iohara algebra. We note, however, that the Serre relations are important from the point of view of the representation theory of $\mathcal{E}$ and of the structure theory as well. We explain the reasons below.

We recall that in [SV2] and [FT] the equivariant localized K-theory of Hilbert schemes H_n of n points of $\mathbb{C}^2$ was studied. In particular, it was shown that the direct sum $\mathcal{F} = \bigoplus_{n \geq 0} K(H_n)$ is isomorphic to the space of symmetric polynomials in infinite numbers of variables and carries the structure of the $\mathcal{E}'$-module. We note, however, that this action factors through the surjection $\mathcal{E}' \to \mathcal{E}$, and therefore $\mathcal{F}$ has a natural structure of an $\mathcal{E}$-module. The space $\mathcal{F}$ has a natural basis labeled by fixed points of the action of the torus on H_n, and elements of this basis can be identified with the Macdonald polynomials. In addition the action of the generators e_i and f_i is given by Pieri-like formulas. In this paper we observe that the representation $\mathcal{F}$ can be constructed by means of a version of the semiinfinite wedge construction.

Recall that the main building block of the semiinfinite construction for $\mathfrak{gl}_\infty$ is its vector representation. We start by considering $\mathcal{E}$-modules $V(u)$ $(u \in \mathbb{C})$ which are spanned by the vectors $[u]_i$ $(i \in \mathbb{Z})$. They play the role of the vector representation. The usual $\mathfrak{gl}_\infty$ has only one vector representation, but $\mathcal{E}$ naturally has a continuous family of such representations. The algebra $\mathcal{E}$ is endowed with a structure of "comultiplication" (see [DI]). Strictly speaking, this "comultiplication" does not define a structure of an $\mathcal{E}$-module on an arbitrary tensor product $V \otimes W$ of $\mathcal{E}$-modules because some convergence conditions need to be satisfied (see Section 2 for details). We show that the tensor product $V(u_1) \otimes \cdots \otimes V(u_N)$ is well defined for general values of $u_1, \ldots, u_N$. We are mainly interested in the case when the parameters u_i form a geometric progression. We show that the tensor product

$$V(u) \otimes V(uq_2^{-1}) \otimes \cdots \otimes V(uq_2^{-N+1})$$

has a subrepresentation $W^N(u)$ spanned by the set of vectors $[u]_{i_1} \otimes [uq_2^{-1}]_{i_2} \otimes \cdots \otimes [uq_2^{-N+1}]_{i_N}$ with $i_1 > i_2 > \cdots > i_N$. The $\mathcal{E}$-modules $W^N(u)$ are analogues of the exterior powers of the vector representation for $\mathfrak{gl}_\infty$. We construct the structure of an $\mathcal{E}$-module on the limit $N \to \infty$ of $W^N(u)$, thus obtaining an analogue of the space of semiinfinite forms. We denote this representation by $\mathcal{F}(u)$ and call it the Fock representation.

The space $W^N(u)$ can be identified with the space of symmetric polynomials in N variables. We recall that the space $\mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]^{S_N}$ has a natural structure of faithful representation of the spherical double affine Hecke algebra $S\ddot{H}_N$. We show that the image of $\mathcal{E}$ coincides with spherical double affine Hecke algebras (DAHA) and thus obtain a surjective homomorphism $\mathcal{E} \to S\ddot{H}_N$ for any N. We recall that in [SV1] and [SV2] (see also [BS], [S]) the spherical DAHA of type GL_∞ was constructed as a projective limit of $S\ddot{H}_N$. It is natural to expect that our $\mathcal{E}$ is isomorphic to $\lim_{N \to \infty} S\ddot{H}_N$. (We plan to discuss this elsewhere.)

Because of the homomorphisms $\mathcal{E} \to S\ddot{H}_N$, any $S\ddot{H}_N$-module gives us a representation of $\mathcal{E}$. Consider now the resonance case $q_1^{1-r} q_3^{k+1} = 1$, $k > 0$, $r > 1$. In this case, the representation of $S\ddot{H}_N$ on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]^{S_N}$ has a subrepresentation $W^{k,r,N} \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]^{S_N}$ defined by

$$W^{k,r,N} = \big\{ f(x_1, \ldots, x_N) \bigm| f(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \text{ satisfies the wheel condition} \big\},$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ is said to satisfy the wheel condition if

$$x_i = x_1 q_3^{1-i} q_1^{s_1 + \cdots + s_{i-1}}, \quad i = 1, \ldots, k+1, s_1, \ldots, s_{k+1} \geq 0, s_1 + \cdots + s_{k+1} = r - 1.$$

In [FJM1] it is proved that $W^{k,r,N}$ has a basis labeled by the so-called (k,r)-admissible partitions, that is, partitions λ satisfying $\lambda_i - \lambda_{i+k} \geq r$ for all $i \geq 1$. Each element of the basis is a Macdonald (Laurent) polynomial. Thus we have an action of the algebra $\mathcal{E}$ on the space of polynomials satisfying the wheel condition. We construct a family of $\mathcal{E}$-modules $W^{k,r,N}(u)$, $u \in \mathbb{C}$, such that $W^{k,r,N}(1) \simeq W^{k,r,N}$. We also construct the inductive limit $N \to \infty$ of the modules $W^{k,r,N}$ and endow it with a structure of the $\mathcal{E}$-module. As a result, we construct a family of representations $W_{\mathbf{c}}^{k,r}(u)$ of $\mathcal{E}$ whose bases are labeled by infinite (k,r)-admissible partitions with a certain stability property at infinity. The parameter $\mathbf{c} = (c_1, \ldots, c_{k-1})$ $(1 \leq c_1 \leq \cdots \leq c_{k-1} \leq r)$ enters in the stability property.

Our paper is organized as follows. In Section 2 we give the definition of $\mathcal{E}$. In Section 3 the vector representations and their tensor products are constructed. In Section 4 we work out the semiinfinite construction for general parameters q_i. In Section 5 we establish a link between the tensor products of representations of $\mathcal{E}$ and representations of $S\ddot{H}_N$. In Section 6 we consider the semiinfinite construction in the resonance case $q_1^{1-r} q_3^{k+1} = 1$. In Section 7 we discuss further properties of the algebra $\mathcal{E}$.

2. Quantum continuous $\mathfrak{gl}_\infty$

In this section we introduce the algebra $\mathcal{E}$ which we call the quantum continuous $\mathfrak{gl}_\infty$.

2.1. Definition

Let q_1, q_2, q_3 be complex numbers satisfying $q_i \neq 1$ and $q_1 q_2 q_3 = 1$. Let

$$g(z,w) = (z - q_1 w)(z - q_2 w)(z - q_3 w).$$

Let $\mathcal{E}$ be an associative algebra over $\mathbb{C}$ generated by the elements e_i, f_i $(i \in \mathbb{Z})$, ψ_j^+, ψ_{-j}^- $(j > 0)$, and $(\psi_0^\pm)^{\pm 1}$ with defining relations depending on parameters q_1, q_2, q_3. (So strictly speaking, we have a family of algebras.) We use generating series

$$e(z) = \sum_{i \in \mathbb{Z}} e_i z^{-i}, \qquad f(z) = \sum_{i \in \mathbb{Z}} f_i z^{-i}, \qquad \psi^\pm(z) = \sum_{\pm i \geq 0} \psi_i^\pm z^{-i}.$$

The defining relations in $\mathcal{E}$ are

$$
\begin{aligned}
&(2.1) && g(z,w)e(z)e(w) = -g(w,z)e(w)e(z),\\
& && g(w,z)f(z)f(w) = -g(z,w)f(w)f(z),\\
&(2.2) && g(z,w)\psi^{\pm}(z)e(w) = -g(w,z)e(w)\psi^{\pm}(z),\\
& && g(w,z)\psi^{\pm}(z)f(w) = -g(z,w)f(w)\psi^{\pm}(z),\\
&(2.3) && [e(z),f(w)] = \frac{\delta(z/w)}{g(1,1)}\big(\psi^+(z)-\psi^-(z)\big),\\
&(2.4) && [\psi_i^{\pm},\psi_j^{\pm}] = 0, \qquad [\psi_i^{\pm},\psi_j^{\mp}] = 0,\\
&(2.5) && \psi_0^{\pm}(\psi_0^{\pm})^{-1} = (\psi_0^{\pm})^{-1}\psi_0^{\pm} = 1,\\
&(2.6) && \big[e_0,[e_1,e_{-1}]\big] = 0, \qquad \big[f_0,[f_1,f_{-1}]\big] = 0.
\end{aligned}
$$

Here $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ is the delta function.

REMARK 2.1

The form of relations (2.1), (2.2), and (2.3) is a convenient way of writing algebraic relations between generators. Namely, each relation is to be understood as generating functions for relations for Fourier coefficients of the right- and left-hand sides. For example, the relation (2.1) for $e(z)$ is equivalent to the following set of relations labeled by pairs $(n,m)\in\mathbb{Z}^2$:

$$
\begin{aligned}
e_{n+3}e_m &- (q_1+q_2+q_3)e_{n+2}e_{m+1} + (q_1q_2+q_2q_3+q_3q_1)e_{n+1}e_{m+2} - e_ne_{m+3}\\
&= -e_{m+3}e_n + (q_1+q_2+q_3)e_{m+2}e_{n+1}\\
&\quad - (q_1q_2+q_2q_3+q_3q_1)e_{m+1}e_{n+2} + e_me_{n+3},
\end{aligned}
$$

and (2.3) simply means that

$$
g(1,1)[e_i,f_j] = \begin{cases} \psi_{i+j}^+ & \text{if } i+j>0,\\ -\psi_{i+j}^- & \text{if } i+j<0,\\ \psi_0^+ - \psi_0^- & \text{if } i+j=0. \end{cases}
$$

REMARK 2.2

The algebra $\mathcal{E}$ can be considered an algebra over one of the fields of rational functions $\mathbb{C}(q_1,q_2)$, $\mathbb{C}(q_1,q_3)$, $\mathbb{C}(q_1,q_3)$. This is equivalent to saying that the parameters q_i are “general.” However, $\mathcal{E}$ is defined for arbitrary (except 1 and 0) values of parameters. Also, in Section 6 we consider the case when q_i satisfy an algebraic relation (different from $q_1q_2q_3=1$).

In what follows we call the algebra $\mathcal{E}$ the quantum continuous $\mathfrak{gl}_\infty$.

REMARK 2.3

In [DI] Ding and Iohara defined a class of algebras which are analogues of quantum affine algebras. Apart from the cubic Serre relations (2.6), the algebra $\mathcal{E}$

is a particular case of their construction. This algebra (without relations (2.6)) was also considered in [FT] and [FHH+].

LEMMA 2.4

The following are obvious:

- *the algebra $\mathcal{E}$ is invariant under permutations of parameters q_1, q_2, q_3,*
- *the elements $\psi_0^{\pm} \in \mathcal{E}$ are central,*
- *there is an antiinvolution of $\mathcal{E}$ sending e_i to f_{-i}, f_i to e_{-i}, and $\psi_i^{\pm}$ to $\psi_{-i}^{\mp}$,*
- *the algebra $\mathcal{E}$ is graded by the lattice $\mathbb{Z}^2$; the degrees of generators are given by*

$$\deg e_i = (1, i), \qquad \deg f_i = (-1, i), \qquad \deg \psi_i^{\pm} = (0, i).$$

We say that an $\mathcal{E}$-module is of level (l_+, l_-) if $\psi_0^{\pm}$ act on this representation by scalars $l_{\pm}$.

Let $\mathcal{E}'$ be the algebra defined in the same way as $\mathcal{E}$ without cubic relations (2.6) (see Remark 2.3). In [DI] the formal (see the explanations below) structure of the Hopf algebra on $\mathcal{E}'$ was constructed. In particular, the comultiplication is given by

$$\Delta e(z) = e(z) \otimes 1 + \psi^-(z) \otimes e(z), \tag{2.7}$$

$$\Delta f(z) = f(z) \otimes \psi^+(z) + 1 \otimes f(z), \tag{2.8}$$

$$\Delta \psi^{\pm}(z) = \psi^{\pm}(z) \otimes \psi^{\pm}(z). \tag{2.9}$$

We note that this "definition" does not define a comultiplication in the usual sense. The right-hand sides are not elements of $\mathcal{E} \otimes \mathcal{E}$ since they contain infinite sums. Still for certain classes of modules the formulas (2.7), (2.8), and (2.9) can be made precise. So in what follows, when talking about the tensor products $V_1 \otimes \cdots \otimes V_N$ of $\mathcal{E}$-modules, we construct the action of the generators e_i, f_i, and $\psi_i^{\pm}$ explicitly (based on the universal formulas (2.7), (2.8), (2.9)) and check that they satisfy the relations of quantum continuous $\mathfrak{gl}_\infty$.

We close this section with the following statement.

LEMMA 2.5

In $\mathcal{E}'$ the cubic element $[e_0, [e_1, e_{-1}]]$ belongs to the kernel of $\mathrm{ad} f(z)$. *Similarly, $[f_0, [f_1, f_{-1}]]$ belongs to the kernel of* $\mathrm{ad} e(z)$.

The proof will be given elsewhere. Using this lemma, it is not difficult to prove the Serre relations in each representation we discuss in this paper.

3. The modules $V(u)$

In this section we define vector representations $V(u)$ of $\mathcal{E}$. We also construct tensor products of vector representations and certain submodules inside tensor products.

3.1. Vector representations

For a parameter $u \in \mathbb{C}$ we consider the space $V(u)$ spanned by basis vectors $[u]_i$ $(i \in \mathbb{Z})$. In the following lemma we define representations of the quantum continuous $\mathfrak{gl}_\infty$ depending on parameter u. We call $V(u)$ a vector representation.

PROPOSITION 3.1

The assignment

$$(1-q_1)e(z)[u]_i = \delta(q_1^i u/z)[u]_{i+1},$$

$$-(1-q_1^{-1})f(z)[u]_i = \delta(q_1^{i-1}u/z)[u]_{i-1},$$

$$\psi^+(z)[u]_i = \frac{(1-q_1^i q_3 u/z)(1-q_1^i q_2 u/z)}{(1-q_1^i u/z)(1-q_1^{i-1}u/z)}[u]_i,$$

$$\psi^-(z)[u]_i = \frac{(1-q_1^{-i}q_3^{-1}z/u)(1-q_1^{-i}q_2^{-1}z/u)}{(1-q_1^{-i}z/u)(1-q_1^{-i+1}z/u)}[u]_i$$

defines a structure of a level $(1,1)$ $\mathcal{E}$*-module on* $V(u)$*.*

REMARK 3.2

An important feature of the representations $V(u)$ is that $\psi^\pm(z)$ act on $[u]_i$ via multiplication by the expansions at $z=\infty$ and $z=0$ of the function

$$\frac{(1-q_1^{-i}q_3^{-1}z/u)(1-q_1^{-i}q_2^{-1}z/u)}{(1-q_1^{-i}z/u)(1-q_1^{-i+1}z/u)}.$$

For the proof of Proposition 3.1 we need a simple lemma. We use the following notation: for a rational function $\gamma(z)$ we denote by $\gamma^\pm(z)$ the expansions of $\gamma(z)$ at $z=\infty$ and $z=0$; that is, $\gamma^\pm(z)$ are Taylor series in $z^{\mp 1}$.

LEMMA 3.3

Let $\gamma(z)$ *be a rational function regular at* $z=0,\infty$ *and with simple poles. Then we have the formal series identity*

$$\gamma^+(z)-\gamma^-(z) = \sum_t \gamma^{(t)}\delta(z/z^{(t)}),$$

where the sum runs over all poles $z^{(t)}$ *of* $\gamma(z)$ *and* $\gamma^{(t)} = \operatorname{res}_{z=z^{(t)}} \gamma(z)\frac{dz}{z}$*.*

We now prove Proposition 3.1.

Proof

Since $e_m[u]_j = (1-q_1)^{-1}q_1^{jm}u^m[u]_{j+1}$, the relations (2.6) are obviously satisfied. We show now that (2.1) and (2.3) hold; all other relations are proved similarly. In what follows we often use the formula

$$\gamma(z)\delta(z/w) = \gamma(w)\delta(z/w). \tag{3.1}$$

So let us show that

$$g(z,w)e(z)e(w) = -g(w,z)e(w)e(z).$$

In fact, we prove that both sides vanish on $V(u)$. By definition,

$$\begin{aligned}(1-q_1)^2 g(z,w)e(z)e(w)[u]_i &= g(z,w)\delta(q_1^{i+1}u/z)\delta(q_1^i u/w)[u]_{i+2}\\ &= g(q_1^{i+1}u, q_1^i u)\delta(q_1^{i+1}u/z)\delta(q_1^i u/w)[u]_{i+2}\\ &= 0.\end{aligned}$$

Similarly, $g(w,z)e(w)e(z) = 0$.

Now we show that

$$[e(z), f(w)][u]_i = \frac{\delta(z/w)}{g(1,1)}(\psi^+(z) - \psi^-(z))[u]_i. \tag{3.2}$$

The left-hand side reads as

$$\begin{aligned}&\frac{q_1}{(1-q_1)^2}\big(\delta(q_1^{i-1}u/w)\delta(q_1^{i-1}u/z) - \delta(q_1^i u/w)\delta(q_1^i u/z)\big)[u]_i\\ &= \frac{q_1}{(1-q_1)^2}\delta(z/w)\big(\delta(q_1^{i-1}u/z) - \delta(q_1^i u/z)\big).\end{aligned} \tag{3.3}$$

The right-hand side of (3.2) equals

$$\begin{aligned}&\frac{\delta(z/w)}{g(1,1)}\Big(\frac{1-q_1^i q_3 u/z}{1-q_1^i u/z} \times \frac{1-q_1^i q_2 u/z}{1-q_1^{i-1}u/z}\\ &\qquad - \frac{1-q_1^{-i}q_3^{-1}z/u}{1-q_1^{-i}z/u} \times \frac{1-q_1^{-i}q_2^{-1}z/u}{1-q_1^{-i+1}z/u}\Big)[u]_i.\end{aligned} \tag{3.4}$$

Since the expression in the round brackets is of the form $\gamma^+(z) - \gamma^-(z)$ for a rational function $\gamma(z)$, we can apply Lemma 3.3, which proves (3.2). □

We define the rational functions

$$\gamma_{i,u}(z) = \frac{(1-q_3 q_1^i u/z)(1-q_2 q_1^i u/z)}{(1-q_1^{-1}q_1^i u/z)(1-q_1^i u/z)}.$$

Then we have $\psi^{\pm}(z)[u]_i = \gamma_{i,u}^{\pm}(z)[u]_i$.

3.2. Tensor products

Consider the tensor product of vector representations $V(u_1)\otimes\cdots\otimes V(u_N)$. We define the following generating series of operators on this space:

$$\begin{aligned}&(1-q_1)e(z)([u_1]_{i_1}\otimes\cdots\otimes[u_N]_{i_N})\\ &= \sum_{s=1}^{N}\Big(\prod_{l=1}^{s-1}\gamma_{i_l,u_l}(q_1^{i_s}u_s)\Big)\delta(q_1^{i_s}u_s/z)[u_1]_{i_1}\\ &\quad \otimes\cdots\otimes[u_{s-1}]_{i_{s-1}}\otimes[u_s]_{i_s+1}\otimes[u_{s+1}]_{i_{s+1}}\otimes\cdots\otimes[u_N]_{i_N},\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&-(1-q_1^{-1})f(z)([u_1]_{i_1}\otimes\cdots\otimes[u_N]_{i_N})\\
&=\sum_{s=1}^{N}\delta(q_1^{i_s-1}u_s/z)\Big(\prod_{l=s+1}^{N}\gamma_{i_l,u_l}(q_1^{i_s-1}u_s)\Big)[u_1]_{i_1}\\
&\quad\otimes\cdots\otimes[u_{s-1}]_{i_{s-1}}\otimes[u_s]_{i_s-1}\otimes[u_{s+1}]_{i_{s+1}}\otimes\cdots\otimes[u_N]_{i_N},
\end{aligned}
\tag{3.6}
$$

$$
\psi^{\pm}(z)([u_1]_{i_1}\otimes\cdots\otimes[u_N]_{i_N})=\psi^{\pm}(z)[u_1]_{i_1}\otimes\cdots\otimes\psi^{\pm}(z)[u_N]_{i_N}. \tag{3.7}
$$

The formulas above are read from the universal formulas (2.7), (2.8), and (2.9). In fact, formula (2.7) gives (formally) the action of $e(z)$ on the tensor product $V(u_1)\otimes\cdots\otimes V(u_N)$:

$$
e(z)=\sum_{s=1}^{N}\underbrace{\psi^-(z)\otimes\cdots\psi^-(z)}_{s-1}\otimes e(z)\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}. \tag{3.8}
$$

By definition, $e(z)[u]_i=(1-q_1)^{-1}\delta(q_1^iu/z)[u]_{i+1}$ and $\psi^{\pm}(z)$ acts on $[u]_i$ via multiplication by certain series. The product of delta functions with these series is in general not defined. The series in question are expansions of given rational functions. It is therefore natural to regularize (3.8) by substituting the support $z=q_1^iu$ of the delta function into rational functions. We thus obtain formula (3.5). Similar arguments lead to (3.6). Note, however, that the expression $\gamma_{i_l,u_l}(q_1^{i_s}u_s)$ is not defined if the argument is a pole of $\gamma_{i_l,u_l}(z)$. Therefore formulas (3.5) and (3.6) do not always produce well-defined operators.

For an element $\mathbf{a}=(a_1,\ldots,a_N)\in\mathbb{Z}^N$, let $\mathbf{u_a}\in\bigotimes_{s=1}^N V(u_s)$ denote the vector $\bigotimes_{s=1}^N[u_s]_{a_s}$. The vectors $\mathbf{u_a}$ constitute a basis of the tensor product $\bigotimes_{s=1}^N V(u_s)$. For a linear operator A on this tensor product we denote by $\langle\mathbf{u_{a'}}|e(z)|\mathbf{u_a}\rangle$ the matrix element of A in terms of this basis.

LEMMA 3.4
Let $A\subset\mathbb{Z}^N$ be a subset such that

- *for all $\mathbf{a}\in\mathbb{Z}^N$, $\mathbf{a}'\in A$ the matrix coefficients $\langle\mathbf{u_{a'}}|e(z)|\mathbf{u_a}\rangle$ and $\langle\mathbf{u_{a'}}|f(z)|\times\mathbf{u_a}\rangle$ are well defined;*
- *for all $\mathbf{a}\in A$, $\mathbf{b}\notin A$ the matrix coefficients $\langle\mathbf{u_b}|e(z)|\mathbf{u_a}\rangle$ and $\langle\mathbf{u_b}|f(z)|\mathbf{u_a}\rangle$ vanish.*

Then formulas (3.5), (3.6), and (3.7) define a structure of the $\mathcal{E}$-module on $\mathrm{span}\{\mathbf{u_a}\}_{\mathbf{a}\in A}$.

Proof
It suffices to show that the defining relations of $\mathcal{E}$ are satisfied. We check (2.3) and (2.6). The rest can be checked similarly.

We start with relation (2.3). Let us compute the matrix coefficient

$$
\langle\mathbf{u_{a'}}|[e(z),f(w)]|\mathbf{u_a}\rangle. \tag{3.9}
$$

We first show that it vanishes unless $\mathbf{a}'=\mathbf{a}$. Introduce the notation $\mathbf{a}\pm\mathbf{1}_s=(\ldots,a_s\pm1,\ldots)$. Then clearly (3.9) vanishes unless $\mathbf{a}'=\mathbf{a}+\mathbf{1}_t-\mathbf{1}_s$ for some

$s,t=1,\dots,N$. If $s>t$, then (3.9) vanishes because formulas (3.5) and (3.6) give identical expressions for $\langle \mathbf{u}_{\mathbf{a}'}|e(z)f(w)|\mathbf{u}_{\mathbf{a}}\rangle$ and for $\langle \mathbf{u}_{\mathbf{a}'}|f(w)e(z)|\mathbf{u}_{\mathbf{a}}\rangle$. Assume now $s<t$. Then from formulas (3.5) and (3.6) we obtain that (3.9) is equal to (up to some constant multiple)

$$\begin{aligned}&\bigl(\gamma_{a_t,u_t}(q_1^{a_s-1}u_s)\gamma_{a_s-1,u_s}(q_1^{a_t}u_t)-\gamma_{a_s,u_s}(q_1^{a_t}u_t)\gamma_{a_t+1,u_t}(q_1^{a_s-1}u_s)\bigr)\\ &\quad\times\prod_{s\neq l<t}\gamma_{a_l,u_l}(q_1^{a_t}u_t)\prod_{t\neq l>s}\gamma_{a_l,u_l}(q_1^{a_s-1}u_s)\delta(q_1^{a_s-1}u_s/w)\delta(q_1^{a_t}u_t/z),\end{aligned}$$

which vanishes thanks to a simple relation

$$\gamma_{a_t,u_t}(q_1^{a_s-1}u_s)\gamma_{a_s-1,u_s}(q_1^{a_t}u_t)=\gamma_{a_s,u_s}(q_1^{a_t}u_t)\gamma_{a_t+1,u_t}(q_1^{a_s-1}u_s).$$

So we only have the terms with $s=t$ and, thus $\mathbf{a}\neq\mathbf{a}'$ implies that (3.9) is zero.

Now assume $\mathbf{a}=\mathbf{a}'$. Then (3.9) is equal to

$$\begin{aligned}&\sum_{s=1}^{N}\delta(q_1^{a_s-1}u_s/w)\delta(q_1^{a_s-1}u_s/z)\prod_{l\neq s}\gamma_{a_l,u_l}(q_1^{a_s-1}u_s)\\ &\quad-\sum_{s=1}^{N}\delta(q_1^{a_s}u_s/w)\delta(q_1^{a_s}u_s/z)\prod_{l\neq s}\gamma_{a_l,u_l}(q_1^{a_s}u_s)\\ &=\delta(z/w)\Bigl(\sum_{s=1}^{N}\delta(q_1^{a_s-1}u_s/z)\prod_{l\neq s}\gamma_{a_l,u_l}(q_1^{a_s-1}u_s)\\ &\quad-\sum_{s=1}^{N}\delta(q_1^{a_s}u_s/z)\prod_{l\neq s}\gamma_{a_l,u_l}(q_1^{a_s}u_s)\Bigr).\end{aligned}\tag{3.10}$$

Assume for a moment that

$$q_1^i u_l\neq q_1^j u_m\quad\text{unless } i=j, l=m.\tag{3.11}$$

Then all poles of the function $\prod_{s=1}^{N}\gamma_{a_s,u_s}(z)$ are simple, and Lemma 3.3 proves

$$[e(z),f(w)]\mathbf{u}_{\mathbf{a}}=\frac{\delta(z/w)}{g(1,1)}\bigl(\psi^{+}(z)-\psi^{-}(z)\bigr)\mathbf{u}_{\mathbf{a}}.\tag{3.12}$$

We note also that if relation (3.12) holds for parameters satisfying (3.11), then it holds for all values of parameters.

We now prove relation (2.6). Let $E=[e_0,[e_1,e_{-1}]]$. Let $\mathbf{a}\pm\mathbf{1}_j=(a_1,\dots,a_j\pm 1,\dots,a_N)$. From Lemma 2.5 it follows that for all $1\le i\le j\le N$,

$$\begin{aligned}&\sum_{n=1}^{N}\langle \mathbf{u}_{\mathbf{a}+\mathbf{1}_i+\mathbf{1}_j}|f(z)|\mathbf{u}_{\mathbf{a}+\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_n}\rangle\langle \mathbf{u}_{\mathbf{a}+\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_n}|E|\mathbf{u}_{\mathbf{a}}\rangle\\ &\quad=\sum_{n=1}^{N}\langle \mathbf{u}_{\mathbf{a}+\mathbf{1}_i+\mathbf{1}_j}|E|\mathbf{u}_{\mathbf{a}-\mathbf{1}_n}\rangle\langle \mathbf{u}_{\mathbf{a}-\mathbf{1}_n}|f(z)|\mathbf{u}_{\mathbf{a}}\rangle.\end{aligned}$$

For generic $u_1,\dots,u_N$, it is easy to see that $\langle \mathbf{u}_{\mathbf{a}+\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_n}|E|\mathbf{u}_{\mathbf{a}}\rangle=0$ comparing the coefficients of the delta functions. As far as the actions of e_m are well defined, the Serre relations $E=0$ is valid in the limiting case, too. $\square$

The following lemma is dual to Lemma 3.4.

LEMMA 3.5

Let $A \subset \mathbb{Z}^N$ be a subset such that

- *for all $\mathbf{a} \in A$, $\mathbf{a}' \in \mathbb{Z}^N$, the matrix coefficients $\langle \mathbf{u}_{\mathbf{a}'} | e(z) | \mathbf{u}_{\mathbf{a}} \rangle$ and $\langle \mathbf{u}_{\mathbf{a}'} | f(z) | \mathbf{u}_{\mathbf{a}} \rangle$ are well defined;*
- *for all $\mathbf{a} \notin A$, $\mathbf{b} \in A$, the matrix coefficients $\langle \mathbf{u}_{\mathbf{b}} | e(z) | \mathbf{u}_{\mathbf{a}} \rangle$ and $\langle \mathbf{u}_{\mathbf{b}} | f(z) | \mathbf{u}_{\mathbf{a}} \rangle$ vanish.*

Then formulas (3.5), (3.6), and (3.7) define a structure of the $\mathcal{E}$-module on $\mathrm{span}\{\mathbf{u}_{\mathbf{a}}\}_{\mathbf{a} \in A}$.

Proof

This is similar to the proof of Lemma 3.4. □

In the following lemma we check that for generic values of parameters $u_1, \dots, u_N$, the tensor product $V(u_1) \otimes \cdots \otimes V(u_N)$ is well defined.

LEMMA 3.6

Let $u_1, \dots, u_N \in \mathbb{C}$ be some numbers with the property

$$\frac{u_i}{u_j} \neq q_1^k \quad \text{for all } 1 \le i < j \le N, k \in \mathbb{Z}. \tag{3.13}$$

Then the comultiplication rules (2.7), (2.8), and (2.9) define the structure of the $\mathcal{E}$-module on the tensor product $V(u_1) \otimes \cdots \otimes V(u_N)$.

Proof

The action of $\psi^{\pm}(z)$ is obviously well defined. They have only simple poles because of the condition (3.13). We check that the action of $e(z)$ is also well defined. (The case of $f(z)$ is similar.)

By definition, we have

$$\begin{aligned}
&(1 - q_1) e(z)([u_1]_{i_1} \otimes \cdots \otimes [u_N]_{i_N}) \\
&\quad = \sum_{s=1}^{N} \Bigl(\prod_{l=1}^{s-1} \gamma_{i_l, u_l}(q_1^{i_s} u_s) \Bigr) \delta(q_1^{i_s} u_s / z) [u_1]_{i_1} \\
&\qquad \otimes \cdots \otimes [u_{s-1}]_{i_{s-1}} \otimes [u_s]_{i_s+1} \otimes [u_{s+1}]_{i_{s+1}} \otimes \cdots \otimes [u_N]_{i_N}.
\end{aligned} \tag{3.14}$$

After substitution, the denominators take the form $1 - q_1^k u_t / u_s$, $k \in \mathbb{Z}$, which do not vanish because of condition (3.13). □

3.3. Submodules of tensor products

We now consider the tensor product of modules $V(u)$, where the evaluation parameters form a geometric progression with ratio q_2.

Let $V^N(u)$ be the $\mathcal{E}$-module defined by

$$V^N(u) = V(u) \otimes V(uq_2^{-1}) \otimes \cdots \otimes V(uq_2^{-N+1}).$$

Set

$$\mathcal{P}^N = \{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N\}.$$

Let $W^N(u) \hookrightarrow V^N(u)$ be the subspace spanned by the vectors

$$|\lambda\rangle_u = [u]_{\lambda_1} \otimes [uq_2^{-1}]_{-1+\lambda_2} \otimes \cdots \otimes [uq_2^{-N+1}]_{-N+1+\lambda_N}, \tag{3.15}$$

where $\lambda \in \mathcal{P}^N$. In what follows, if the value of u is clear from the context, we abbreviate $|\lambda\rangle_u = |\lambda\rangle$.

LEMMA 3.7

$W^N(u)$ is a level $(1,1)$ submodule of $V^N(u)$.

Proof

We prove that $W^N(u)$ is invariant with respect to e_i. The case of f_i is similar.

Recall the comultiplication rule

$$\Delta e(z) = e(z) \otimes 1 + \psi^-(z) \otimes e(z).$$

Since $e(z)[u]_j$ is proportional to $[u]_{j+1}$, it suffices to check that

$$\psi^-(z)[u]_j \otimes e(z)[uq_2^{-1}]_{j-1} = 0.$$

By definition,

$$\psi^-(z)[u]_j \otimes e(z)[uq_2^{-1}]_{j-1} = \frac{(1-q_1^{-j}q_3^{-1}z/u)(1-q_1^{-j}q_2^{-1}z/u)}{(1-q_1^{-j}z/u)(1-q_1^{-j+1}z/u)}\delta(q_1^{j-1}q_2^{-1}u/z),$$

which vanishes because of (3.1). The lemma is proved. $\square$

In the following proposition we write down the action of generators of $\mathcal{E}$ on $|\lambda\rangle$ explicitly. We introduce the notation

$$\lambda \pm \mathbf{1}_j = (\lambda_1, \ldots, \lambda_j \pm 1, \ldots, \lambda_N).$$

PROPOSITION 3.8

The action of $e(z)$ is given by the formula

$$\begin{aligned}(1-q_1)e(z)|\lambda\rangle = \sum_{i=1}^{N}\prod_{j=1}^{i-1} &\frac{(1-q_1^{\lambda_i-\lambda_j}q_3^{i-j-1})(1-q_1^{\lambda_i-\lambda_j+1}q_3^{i-j+1})}{(1-q_1^{\lambda_i-\lambda_j}q_3^{i-j})(1-q_1^{\lambda_i-\lambda_j+1}q_3^{i-j})}\\ &\times \delta(q_1^{\lambda_i}q_3^{i-1}u/z)|\lambda+\mathbf{1}_i\rangle.\end{aligned} \tag{3.16}$$

The action of $f(z)$ is given by

$$\begin{aligned}-(1-q_1^{-1})f(z)|\lambda\rangle = \sum_{i=1}^{N}\prod_{j=i+1}^{N} &\frac{(1-q_1^{\lambda_j-\lambda_i+1}q_3^{j-i+1})(1-q_1^{\lambda_j-\lambda_i}q_3^{j-i-1})}{(1-q_1^{\lambda_j-\lambda_i+1}q_3^{j-i})(1-q_1^{\lambda_j-\lambda_i}q_3^{j-i})}\\ &\times \delta(q_1^{\lambda_i-1}q_3^{i-1}u/z)|\lambda-\mathbf{1}_i\rangle.\end{aligned} \tag{3.17}$$

For ψ operators, one has

$$\psi^+(z)|\lambda\rangle = \prod_{i=1}^{N} \frac{(1-q_1^{\lambda_i}q_3^{i}u/z)(1-q_1^{\lambda_i-1}q_3^{i-2}u/z)}{(1-q_1^{\lambda_i}q_3^{i-1}u/z)(1-q_1^{\lambda_i-1}q_3^{i-1}u/z)}|\lambda\rangle, \tag{3.18}$$

$$\psi^-(z)|\lambda\rangle = \prod_{i=1}^{N} \frac{(1-q_1^{-\lambda_i}q_3^{-i}z/u)(1-q_1^{-\lambda_i+1}q_3^{-i+2}z/u)}{(1-q_1^{-\lambda_i}q_3^{-i+1}z/u)(1-q_1^{-\lambda_i+1}q_3^{-i+1}z/u)}|\lambda\rangle. \tag{3.19}$$

Proof

The proof follows from the comultiplication rules and the definition of the modules $V(u)$. □

4. Semiinfinite construction

In this section we construct the Fock modules $\mathcal{F}(u)$ by using the inductive limit of certain subspaces in the finite tensor products of vector representations. To construct the inductive limit consistently, we need to modify the operators $f(z)$ and $\psi^{\pm}(z)$, and this modification results in the nontrivial level $(1,q_2)$ in the inductive limit.

4.1. Modified operators

Recall that for any $N \geq 1$ the basis of the space $W^N(u)$ is labeled by the sequences $\lambda = (\lambda_1,\dots,\lambda_N) \in \mathcal{P}^N$. The corresponding vectors $|\lambda\rangle \in W^N(u)$ are given by formula (3.15). Set

$$\mathcal{P}^{N,+} = \{\lambda \in \mathcal{P}^N \mid \lambda_N \geq 0\},$$

and define $W^{N,+}(u)$ to be the subspace of $W^N(u)$ spanned by the vectors $|\lambda\rangle$ for $\lambda \in \mathcal{P}^{N,+}$. Our goal in this section is to construct a semiinfinite tensor product $\mathcal{F}(u)$ as an inductive limit of $W^{N,+}(u)$ and to endow it with a structure of an $\mathcal{E}$-module. For this purpose we need to vary N and take the limit $N \to \infty$. Our strategy is as follows. Let $\tau_N : \mathcal{P}^{N,+} \to \mathcal{P}^{N+1,+}$ be the mapping given by

$$\tau_N(\lambda) = (\lambda_1,\dots,\lambda_N,0),$$

and induce the embedding $\tau_N : W^{N,+}(u) \hookrightarrow W^{N+1,+}(u)$. Define the inductive limit

$$\mathcal{F}(u) = \lim_{N\to\infty} W^{N,+}(u). \tag{4.1}$$

This space is spanned by the vectors $|\lambda\rangle (\lambda \in \mathcal{P}^+)$, where the sets of infinite partitions $\mathcal{P}, \mathcal{P}^+$ are defined by

$$\mathcal{P} = \big\{\lambda = (\lambda_1,\lambda_2,\dots) \,\big|\, \lambda_i \geq \lambda_{i+1}, \lambda_i \in \mathbb{Z}\big\},$$

$$\mathcal{P}^+ = \{\lambda \in \mathcal{P} \mid \lambda_i = 0 \text{ for sufficiently large } i\}.$$

In what follows we refer to $\mathcal{F}(u)$ as a Fock module. We define an action of $\mathcal{E}$ on $\mathcal{F}(u)$ in the following way. Consider the operators acting on the space $W^N(u)$:

$$e^{[N]}(z) = e(z), \qquad f^{[N]}(z) = \frac{1 - q_2 q_3^N u/z}{1 - q_3^N u/z} f(z), \tag{4.2}$$

$$\begin{aligned} \psi^{+[N]}(z) &= \frac{1 - q_2 q_3^N u/z}{1 - q_3^N u/z} \psi^+(z), \\ \psi^{-[N]}(z) &= q_2 \frac{1 - q_2^{-1} q_3^{-N} z/u}{1 - q_3^{-N} z/u} \psi^-(z). \end{aligned} \tag{4.3}$$

REMARK 4.1

The action of $f(z)$ splits into a sum of delta functions. By definition, the change caused by the multiplication of the rational function

$$\beta_N(z) = \frac{1 - q_2 q_3^N u/z}{1 - q_3^N u/z}$$

is that each delta function is multiplied by the value of $\beta_N(z)$ at its support. The changes in $\psi^+(z)$ and $\psi^-(z)$ are the multiplication by $\beta_N(z)$ as a series in z^{-1} and in z, respectively. Since $\beta_N(z)$ has no pole at $z = \infty$ or $z = 0$, the regularity of the series is not violated in either case.

Another point is why we choose $\beta_N(z)$ to multiply. This factor has the following meaning. Consider the eigenvalue of $\psi^+(z)$ on the tensor component $[q_2^{-N+1} u]_{\lambda_N - N+1}$ when $\lambda_N = 0$. It has four factors, say, of the form $(\alpha_N^{(1)}(z) \times \alpha_N^{(2)}(z))/(\alpha_N^{(3)}(z)\alpha_N^{(4)}(z))$. In fact, we have

$$\beta_N(z) = \frac{\alpha_N^{(4)}(z)}{\alpha_N^{(2)}(z)} = \frac{\alpha_{N+1}^{(1)}(z)}{\alpha_{N+1}^{(3)}(z)}.$$

Namely, we have removed part of the factors from the tail (we say the Nth tensor component is in the tail if $\lambda_N = 0$), those which disappear when we extend the tail from N to $N + 1$.

It turns out that the operators $x^{[N]}(z)$, $x = e, f, \psi^\pm$ are stable and define an $\mathcal{E}$-module structure on $\mathcal{F}(u)$. Let us give precise definitions.

First we prepare the following.

LEMMA 4.2

Suppose that for $\lambda \in \mathcal{P}^{N,+}$ the equality $\lambda_N = 0$ is valid. Then, for $x = e, f, \psi^+, \psi^-$, we have $x^{[N]}(z)|\lambda\rangle \in W^{N,+}(u)$ and

$$\tau_N\bigl(x^{[N]}(z)|\lambda\rangle\bigr) = x^{[N+1]}(z)\tau_N(|\lambda\rangle).$$

Proof

For $x = e$ our lemma is trivial because in the right-hand side the $(N + 1)$st term in the comultiplication of $e(z)$ acts trivially. Let $x = \psi^+$. We need to prove that the eigenvalue of the operator

$$\beta_N(z)\psi^+(z) \quad \text{on the vector } |\lambda_1,\dots,\lambda_N\rangle$$

coincides with that of the operator

$$\beta_{N+1}(z)\psi^+(z) \quad \text{on the vector } |\lambda_1,\dots,\lambda_N,0\rangle.$$

Since

$$\psi^+(z)[uq_2^{-N}]_{-N} = \frac{(1-q_1^{-N}q_3q_2^{-N}u/z)(1-q_1^{-N}q_2^{-N+1}u/z)}{(1-q_1^{-N}q_2^{-N}u/z)(1-q_1^{-N-1}q_2^{-N}u/z)}[uq_2^{-N}]_{-N},$$

the statement follows from the equality

$$\beta_N(z) = \beta_{N+1}(z)\frac{(1-q_3^{N+1}u/z)(1-q_2q_3^Nu/z)}{(1-q_3^Nu/z)(1-q_3^{N+1}q_2u/z)}.$$

The case $x=\psi^-$ is similar. For $x=f$ the comparison of the ith terms in the left- and right-hand sides is similar for $1\le i\le N-1$. It is easy to see that the rest of the terms, that is, $i=N$ in the left-hand side and $i=N,N+1$ in the right-hand side, are zero. So the equality for $x=f$ follows. □

4.2. Fock modules

We now endow each space $\mathcal{F}(u)$ with a structure of an $\mathcal{E}$-module. For any $\lambda=(\lambda_1,\lambda_2,\dots)\in\mathcal{P}^+$ we set

$$x(z)|\lambda\rangle = \lim_{N\to\infty} x^{[N]}(z)|\lambda_1,\dots,\lambda_N\rangle, \tag{4.4}$$

where $x=e,f,\psi^+,\psi^-$ and the right-hand side is considered as an element of $\mathcal{F}(u)$ via (4.1).

THEOREM 4.3

Formula (4.4) endows $\mathcal{F}(u)$ with the structure of a level $(1,q_2)$ $\mathcal{E}$-module.

Proof

We have to check that all relations of $\mathcal{E}$ are satisfied. The only nontrivial check is the commutation relation

$$[e(z),f(w)] = \frac{\delta(z/w)}{g(1,1)}(\psi^+(z)-\psi^-(z)).$$

We prove this relation using the structure of the $\mathcal{E}$-module on $W^N(u)$. For given $|\lambda\rangle\in\mathcal{F}(u)$, choose N large enough so that $\lambda_{N-1}=\lambda_N=0$. Set $\lambda^{[N]}=(\lambda_1,\dots,\lambda_N)$. Because of Lemma 4.2, it suffices to prove the equality

$$[e^{[N]}(z),f^{[N]}(w)]|\lambda^{[N]}\rangle = \frac{\delta(z/w)}{g(1,1)}\big(\psi^{+[N]}(z)-\psi^{-[N]}(z)\big)|\lambda^{[N]}\rangle. \tag{4.5}$$

We start with the equality in $W^N(u)$:

$$[e(z),f(w)]|\lambda^{[N]}\rangle = \frac{\delta(z/w)}{g(1,1)}\big(\psi^+(z)-\psi^-(z)\big)|\lambda^{[N]}\rangle.$$

This is an equality for the coefficients of $\delta(z/w)|\lambda^{[N]}\rangle$. The coefficients are Laurent series in z. In the left-hand side the coefficient is a sum of delta functions:

$$\mathrm{LHS} = \sum_{i=1}^{N} c_i \delta(q_1^{\lambda_i} q_3^{i-1} u/z) + \sum_{i=1}^{N} c_i' \delta(q_1^{\lambda_i - 1} q_3^{i-1} u/z).$$

The right-hand side is expressed in terms of a rational function $a_N(z)$: the operator $\psi^+(z)$ has an eigenvalue on $|\lambda^{[N]}\rangle$ which is equal to the series expansion of $a_N(z)$ in z^{-1}, while the eigenvalue of $\psi^-(z)$ is the series expansion of the same rational function $a_N(z)$ in z^{-1}. The rational function $a_N(z)$ is regular at both $z=0$ and $z=\infty$. The equality implies that if $z \in \mathbb{C}\backslash\{0\}$, $a_N(z)\,dz/z$ has only simple poles at $z = q_1^{\lambda_i} q_3^{i-1} u$ with the residue c_i, and at $z = q_1^{\lambda_i - 1} q_3^{i-1} u$ with the residue c_i'. Moreover, since $\lambda_N = 0$, it has a zero at $z = q_3^N u$.

Our aim is to prove (4.5). To obtain the right-hand side of this equality, we expand $a_N(z)\beta_N(z)$ in z^{-1} for $\psi^+(z)$, and in z for $\psi^-(z)$, and then take the difference. We get a Laurent series as the difference of these two expansions. Note that the only poles of $a_N(z)\beta_N(z)$ in $\mathbb{C} \sqcup \{\infty\}$ are still at $z = q_1^{\lambda_i} q_3^{i-1} u$ and $z = q_1^{\lambda_i - 1} q_3^{i-1} u$ after the multiplication of $\beta_N(z)$. Therefore the nth Fourier coefficient of this series is calculated by taking the sum of the residues of $a_N(z)\beta_N(z) z^{-n-1}\,dz$ at these poles. On the other hand, the same procedure applied to the partial fraction of $a_N(z)\beta_N(z)$ gives rise to the change of the coefficients of the delta functions by multiplication of the values of $\beta_N(z)$ at their supports. Therefore we have the equality of two Laurent series.

We give a remark on the proof of the Serre relations. For $E = [e_0, [e_1, e_{-1}]]$, the proof follows from the case of the finite tensor product. For $F = [f_0, [f_1, f_{-1}]]$, it is the same because the modification of the actions is such that $\langle \lambda - \mathbf{1}_i - \mathbf{1}_j - \mathbf{1}_n \mid F|\lambda\rangle$ changes by a constant multiple. $\square$

COROLLARY 4.4

The nonzero matrix coefficients of the action of the generators on $\mathcal{F}(u)$ are given by the following formulas. For $e(z)$,

$$(1-q_1)\langle \lambda + \mathbf{1}_i | e(z) | \lambda\rangle = \prod_{j=1}^{i-1} \frac{(1 - q_1^{\lambda_i - \lambda_j} q_3^{i-j-1})(1 - q_1^{\lambda_i - \lambda_j + 1} q_3^{i-j+1})}{(1 - q_1^{\lambda_i - \lambda_j} q_3^{i-j})(1 - q_1^{\lambda_i - \lambda_j + 1} q_3^{i-j})} \times \delta(q_1^{\lambda_i} q_3^{i-1} u/z).$$

For $f(z)$,

$$-(1-q_1^{-1})\langle \lambda - \mathbf{1}_i | f(z) | \lambda\rangle = \frac{1 - q_1^{\lambda_{i+1} - \lambda_i}}{1 - q_1^{\lambda_{i+1} - \lambda_i + 1} q_3} \prod_{j=i+1}^{\infty} \frac{(1 - q_1^{\lambda_j - \lambda_i + 1} q_3^{j-i+1})((1 - q_1^{\lambda_{j+1} - \lambda_i} q_3^{j-i})}{(1 - q_1^{\lambda_{j+1} - \lambda_i + 1} q_3^{j-i+1})(1 - q_1^{\lambda_j - \lambda_i} q_3^{j-i})} \times \delta(q_1^{\lambda_i - 1} q_3^{i-1} u/z).$$

For $\psi^{\pm}(z)$,

$$\psi^+(z)|\lambda\rangle = \frac{1-q_1^{\lambda_1-1}q_3^{-1}u/z}{1-q_1^{\lambda_1}u/z}\prod_{i=1}^{\infty}\frac{(1-q_1^{\lambda_i}q_3^i u/z)(1-q_1^{\lambda_{i+1}-1}q_3^{i-1}u/z)}{(1-q_1^{\lambda_{i+1}}q_3^i u/z)(1-q_1^{\lambda_i-1}q_3^{i-1}u/z)}|\lambda\rangle,$$

$$\psi^-(z)|\lambda\rangle = q_2\frac{1-q_1^{-\lambda_1+1}q_3 z/u}{1-q_1^{-\lambda_1}z/u}\prod_{i=1}^{\infty}\frac{(1-q_1^{-\lambda_i}q_3^{-i}z/u)(1-q_1^{-\lambda_{i+1}+1}q_3^{-i+1}z/u)}{(1-q_1^{-\lambda_{i+1}}q_3^{-i}z/u)(1-q_1^{-\lambda_i+1}q_3^{-i+1}z/u)}|\lambda\rangle.$$

We define the vacuum vector $|\lambda^0\rangle$ with $\lambda_i^0 = 0$. Let $\psi_{\emptyset}^{\pm}(z)$ be the expansions of a rational function $(1-q_2 z)/(1-z)$ as a series in $z^{\pm 1}$. Explicitly,

$$\psi_{\emptyset}^{+}(z) = \frac{1-q_2 z}{1-z}, \qquad \psi_{\emptyset}^{-}(z) = q_2\frac{1-q_2^{-1}z^{-1}}{1-z^{-1}}.$$

Then $\psi_{\emptyset}^{\pm}(u/z)$ are the eigenvalues of $\psi^{\pm}(z)$ on $|\lambda^0\rangle$, that is,

$$\psi^{\pm}(z)|\lambda^0\rangle = \psi_{\emptyset}^{\pm}(u/z)|\lambda^0\rangle.$$

For a partition λ we denote by $\langle\lambda|\psi^{\pm}(z)_i|\lambda\rangle$ the eigenvalue of the series $\psi^{\pm}(z)$ on the vector $[uq_2^{-i+1}]_{\lambda_i-i+1} \in V(uq_2^{-i+1})$, that is,

$$\psi^{\pm}(z)[uq_2^{-i+1}]_{\lambda_i-i+1} = \langle\lambda|\psi^{\pm}(z)_i|\lambda\rangle[uq_2^{-i+1}]_{\lambda_i-i+1}.$$

The index i in $\psi^{\pm}(z)_i$ indicates the component $V(uq_2^{-i+1})$, and the shift $\lambda_i \mapsto \lambda_i - i + 1$ as well. Similarly, we introduce the matrix coefficients $\langle\lambda+\mathbf{1}_i|e(z)_i|\lambda\rangle$ and $\langle\lambda|f(z)_i|\lambda+\mathbf{1}_i\rangle$. Then the formulas from Corollary 4.4 can be rewritten in the following way:

$$\langle\lambda|\psi^{\pm}(z)|\lambda\rangle = \psi_{\emptyset}^{\pm}(u/z)\prod_{i\geq 1}\frac{\langle\lambda|\psi^{\pm}(z)_i|\lambda\rangle}{\langle\lambda^0|\psi^{\pm}(z)_i|\lambda^0\rangle},$$

$$\langle\lambda+\mathbf{1}_i|e(z)|\lambda\rangle = \langle\lambda+\mathbf{1}_i|e(z)_i|\lambda\rangle\prod_{j=1}^{i-1}\langle\lambda|\psi^-(z)_j|\lambda\rangle,$$

$$\langle\lambda|f(z)|\lambda+\mathbf{1}_i\rangle = \langle\lambda|f(z)_i|\lambda+\mathbf{1}_i\rangle \times \psi_{\emptyset}^{+}(q_3^i u/z)\prod_{j=i+1}^{\infty}\frac{\langle\lambda|\psi^+(z)_j|\lambda\rangle}{\langle\lambda^0|\psi^+(z)_j|\lambda^0\rangle}.$$

COROLLARY 4.5

The module $\mathcal{F}(1)$ *is isomorphic to the module constructed in [FT, Theorem 3.5].*

Proof

We recall that in [FT] the operators e_i, f_j, and $\psi_i^{\pm}$ were constructed on the space $\mathcal{F}$ with basis $[\lambda]$ labeled by infinite partitions λ. This space is defined as the direct sum of localized equivariant K groups of Hilbert schemes of points of $\mathbb{C}^2$. The matrix coefficients of these operators are given in [FT, Proposition 3.7]. We prove that $\mathcal{F}(1) \simeq \mathcal{F}$. To this end we identify $t_1 = q_1$, $t_2 = q_3$ and do a change of basis as follows. Consider constants c_λ defined by $c_{\lambda^0}=1$ and

$$\frac{c_{\lambda+\mathbf{1}_i}}{c_\lambda}=(1-q_1q_3)\prod_{j=i}^{\infty}\frac{1-q_1^{\lambda_{j+1}-\lambda_i}q_3^{j-i+1}}{1-q_1^{\lambda_j-\lambda_i}q_3^{j-i+1}}\prod_{j=1}^{i-1}\frac{1-q_1^{\lambda_j-\lambda_i-1}q_3^{j-i-1}}{1-q_1^{\lambda_j-\lambda_i-1}q_3^{j-i}}. \tag{4.6}$$

It is straightforward to check that c_λ are well defined, that is, that the right-hand sides $d_{\lambda,i}$ of (4.6) satisfy

$$d_{\lambda+\mathbf{1}_i,k}d_{\lambda,i}=d_{\lambda+\mathbf{1}_k,i}d_{\lambda,k}.$$

Another straightforward check shows that the linear map $\mathcal{F}\to\mathcal{F}(1)$, $[\lambda]\mapsto c_\lambda|\lambda\rangle$ is the isomorphism of modules of $\mathcal{E}$. □

REMARK 4.6

Strictly speaking, in [FT] the authors proved that the operators $e(z)$, $f(z)$, and $\psi^{\pm}(z)$ acting on $\mathcal{F}$ satisfy the relations of the Ding-Iohara algebra $\mathcal{E}'$. However, since Serre relations are satisfied on $\mathcal{F}(1)$, the representation $\mathcal{F}$ factors through the surjection $\mathcal{E}'\to\mathcal{E}$.

5. Macdonald polynomials and spherical DAHA

In this section we establish a link between the spherical DAHA and quantum continuous $\mathfrak{gl}_\infty$. Throughout the section we consider the algebra $\mathcal{E}$ over the field of rational functions $\mathbb{C}(q_1,q_3)$.

5.1. Macdonald polynomials

Our basic reference in this section is Macdonald's book [M]. However, we use the Laurent polynomials version of Macdonald polynomials.

The Macdonald operators D_N^r are mutually commuting q-difference operators acting on the ring of symmetric Laurent polynomials $\mathbb{C}(q,t)[x_1^{\pm1},\dots,x_N^{\pm1}]^{\mathfrak{S}_N}$, where $\mathfrak{S}_N$ denotes the symmetric group of N letters. These operators are given by the formula

$$D_N^r=\sum_{|I|=r}A_I(x;t)T_I,$$

where $I\subset\{1,\dots,N\}$ runs over subsets of cardinality r,

$$A_I(x;t)=t^{r(r-1)/2}\prod_{i\in I,j\notin I}\frac{tx_i-x_j}{x_i-x_j},$$

$$T_I=\prod_{i\in I}T_{q,x_i},$$

and $(T_{q,x_i}f)(x_1,\dots,x_N)=f(x_1,\dots,qx_i,\dots,x_N)$. Let $D_N(X;q,t)=\sum_{r=0}^N D_N^r\times X^r$ be their generating function.

The Macdonald polynomials P_λ form a basis in the space of symmetric polynomials. They are uniquely characterized by the following defining properties:

$$D_N(X;q,t)P_\lambda = \prod_{i=1}^{N}(1+Xq^{\lambda_i}t^{N-i})\cdot P_\lambda,$$

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu}m_\mu \quad u_{\lambda\mu}\in\mathbb{C}(q,t),$$

where m_λ denotes the monomial symmetric function and we write $\mu<\lambda$ if $\mu\neq\lambda$ and $\mu_1+\cdots+\mu_i\le\lambda_1+\cdots+\lambda_i$ for $i=1,\dots,N$.

We define $P_\lambda(x;q,t)$ for partitions $\lambda\in\mathcal{P}^N$ with possibly negative entries by the formula

$$P_\lambda(x;q,t)=\prod_{i=1}^{N}x_i^{\lambda_N}\cdot P_{\lambda_1-\lambda_N,\dots,\lambda_{N-1}-\lambda_N,0}(x;q,t).$$

In what comes next, the following Macdonald operators are of special importance for us:

$$D_N^1(q,t)=\textstyle\sum_{i=1}^N \displaystyle\prod_{j(\neq i)}\frac{tx_i-x_j}{x_i-x_j}T_{q,x_i},$$

$$D_N^{-1}(q,t)=D_N^1(q^{-1},t^{-1}).$$

Set $W^N=W^N(1)$ (i.e., $u=1$).

PROPOSITION 5.1

Choose $q_1=q, q_2=q^{-1}t, q_3=t^{-1}$. Under the isomorphism of vector spaces

$$W^N \xrightarrow{\sim} \mathbb{C}[x_1^{\pm1},\dots,x_N^{\pm1}]^{\mathfrak{S}_N},\qquad |\lambda\rangle\mapsto P_\lambda(x),$$

we have the identification

$$(1-q_1)e_0 = \textit{multiplication by } \textstyle\sum_{i=1}^N x_i,$$

$$-(1-q_1^{-1})f_0 = \textit{multiplication by } \textstyle\sum_{i=1}^N x_i^{-1},$$

$$\psi_1^+ = q_3^{N-1}(1-q_2)(1-q_3)D_N^1(q,t),$$

$$\psi_{-1}^- = q_3^{-N+1}(1-q_2^{-1})(1-q_3^{-1})D_N^{-1}(q,t).$$

Proof

We first look at the operator ψ_1^+ (the ψ_{-1}^--case is similar). Formula (3.18) gives

$$\psi_1^+|\lambda\rangle = q_3^{N-1}(1-q_2)(1-q_3)\sum_{i=1}^N q_1^{\lambda_i}q_3^{i-N}|\lambda\rangle,$$

which agrees with the formula for the eigenvalues of the operator D_N^1 (see [M]).

To prove that $(1-q_1)e_0$ acts as $\sum x_i$, we compare the matrix coefficients of $(1-q_1)e_0$ in the basis $|\lambda\rangle$ and of $\sum x_i$ in the basis P_λ. The latter is given by the

Pieri formulas. From (3.16) we obtain

$$
\begin{aligned}
(5.1)\qquad &(1-q_1)e_0\cdot|\lambda\rangle\\
&=\sum_{i=1}^{N}\prod_{j=1}^{i-1}\frac{(1-q_1^{\lambda_i-\lambda_j}q_3^{i-j-1})(1-q_1^{\lambda_i-\lambda_j+1}q_3^{i-j+1})}{(1-q_1^{\lambda_i-\lambda_j}q_3^{i-j})(1-q_1^{\lambda_i-\lambda_j+1}q_3^{i-j})}\cdot|\lambda+\mathbf{1}_i\rangle.
\end{aligned}
$$

The matrix coefficients as above coincide with the Pieri rule formulas (see [M]). Similarly one proves that $-(1-q_1^{-1})f_0$ acts as multiplication by $\sum_{i=1}^{N}x_i^{-1}$. $\square$

5.2. Spherical DAHA

We first recall the definition of the DAHA of type GL_N (see [C]). This is a $\mathbb{C}(q,v)$-algebra generated by elements $T_i^{\pm1}$, $X_j^{\pm1}$, and $Y_j^{\pm1}$ for $1\le i\le N-1, 1\le j\le N$, subject to the following relations:

$$
\begin{gathered}
(T_i+v^{-1})(T_i-v)=0,\quad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\\
T_iT_k=T_kT_i\quad\text{if } |i-k|>1,\\
X_jX_k=X_kX_j,\qquad Y_jY_k=Y_kY_j,\\
T_iX_iT_i=X_{i+1},\qquad T_i^{-1}Y_iT_i^{-1}=Y_{i+1},\\
T_iX_k=X_kT_i,\qquad T_iY_k=Y_kT_i\quad\text{if } k\ne i,i+1,\\
Y_1X_1\cdots X_N=qX_1\cdots X_NY_1,\\
X_1^{-1}Y_2=Y_2X_1^{-1}T_1^{-2}.
\end{gathered}
$$

Denote this algebra by $\ddot{H}_N$. Let $S\in\ddot{H}_N$ be the idempotent given by

$$
S=\frac{1}{[N]!}\sum_{w\in\mathfrak{S}_N}v^{l(w)}T_w,\quad T_w=T_{i_1}\cdots T_{i_r},
$$

for a reduced decomposition $w=s_{i_1}\cdots s_{i_r}$. Here s_i denotes the transposition $(i,i+1)$, $l(w)$ is the length of w, and

$$
[N]!=\prod_{i=1}^{N}[i],\quad [i]=\frac{v^{2i}-1}{v^2-1}.
$$

The algebra $S\ddot{H}_NS$ is called the spherical DAHA and is denoted by $S\ddot{H}_N$. In [SV1] Schiffmann and Vasserot defined the elements $P_{0,m}^N, P_{m,0}^N\in S\ddot{H}_N$ $(m\in\mathbb{Z})$ by the formulas

$$
P_{0,l}^N=S\sum_{i=1}^{N}Y_i^lS,\qquad P_{0,-l}^N=q^lS\sum_{i=1}^{N}Y_i^{-l}S,
$$

$$
P_{l,0}^N=q^lS\sum_{i=1}^{N}X_i^lS,\qquad P_{-l,0}^N=S\sum_{i=1}^{N}X_i^{-l}S
$$

with $l>0$. They proved that these elements generate $S\ddot{H}_N$. We need the following modification of their result.

LEMMA 5.2
Four elements $P^N_{0,1}, P^N_{0,-1}, P^N_{1,0}$, and $P^N_{-1,0}$ generate $S\ddot{H}_N$.

Proof
Consider the degeneration $v=1$ of $\ddot{H}_N$. (In fact, one has to be careful with such a degeneration since $\ddot{H}_N$ is defined over the field of rational functions $\mathbb{C}(q,v)$, which may have a pole at $v=1$. To make everything precise, one has to pass to the analogue of $\ddot{H}_N$, defined over $\mathbb{C}[q^{\pm 1}, v^{\pm 1}]$. We omit the details here and refer the reader to [SV1, Section 2].) We prove that for $v=1$, the corresponding spherical DAHA is generated by our four elements. This would imply the lemma. If $v=1$, the idempotent S commutes with $\sum_{i=1}^N X_i^{\pm 1}$ and $\sum_{i=1}^N Y_i^{\pm 1}$, and $X_iY_j = q^{-\delta_{i,j}} Y_j X_i$. Let

$$P_{\pm 1} = S\sum_{i=1}^N X_i^{\pm 1} S, \qquad Q_{\pm 1} = S\sum_{i=1}^N Y_i^{\pm 1} S.$$

Then we have

$$\begin{aligned}(adP_{\pm 1})^k Q_1 &= S\sum_{i=1}^N (adX_i^{\pm 1})^k Y_i S \\ &= (1-q^{\pm 1})^k S\sum_{i=1}^N X_i^{\pm k} Y_i S,\end{aligned}$$

and for an arbitrary $m \in \mathbb{Z}\setminus\{0\}$,

$$\begin{aligned}(adQ_{\pm 1})^m \Big(S\sum_{i=1}^N X_i^k Y_i S\Big) &= S\sum_{i=1}^N (adY_i^{\pm 1})^m \Big(X_i^k Y_i S\Big) \\ &= (q^{\pm k}-1)^m S\sum_{i=1}^N X_i^k Y_i^{1\pm m} S.\end{aligned}$$

We thus obtain that all $P^N_{m,0}$ and $P^N_{0,m}$ can be obtained as linear combinations of products of $P^N_{\pm 1,0}$ and $P^N_{0,\pm 1}$. □

Similar results holds for the quantum continuous $\mathfrak{gl}_\infty$.

LEMMA 5.3
For $c^\pm \in \mathbb{C}^\times$, the algebra $\mathcal{E}/\langle \psi_0^\pm - c^\pm\rangle$ is generated by four elements $e_0, \psi_1^+, f_0, \psi_{-1}^-$.

Proof
The proof follows directly from relations in $\mathcal{E}$. For example, the $z^{\mp 1}w^{-m}$-terms of the relations (2.2) give

$$[\psi_1^+, e_m] = c^+\Big(\sum_{j=1}^3 q_j - \sum_{j=1}^3 q_j^{-1}\Big) e_{m+1},$$

$$[\psi^-_{-1}, e_m] = c^- \Big(\sum_{j=1}^{3} q_j - \sum_{j=1}^{3} q_j^{-1}\Big) e_{m-1}.$$

Therefore all $e_m, m \in \mathbb{Z}$, can be obtained via e_0, ψ^+_1, and ψ^-_{-1}. Similarly one gets $f_m, m \in \mathbb{Z}, \psi^+_n, \psi^-_{-n}, n \geq 0$. □

THEOREM 5.4
For any N there exists a surjective homomorphism of algebras $\mathcal{E} \to S\ddot{H}_N$, where the parameters q, v of $S\ddot{H}_N$ are related to q_1, q_3 by $q = q_1$, $v^2 = q_3^{-1}$.

Proof
We recall that the algebra $S\ddot{H}_N$ can be faithfully represented on the space

$$\mathbb{C}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]^{S_N}$$

with $t = v^2$ in such a way that $P^N_{\pm 1,0}$ acts as a multiplication by $\sum_{i=1}^N x_i^{\pm 1}$ and $P^N_{0,\pm 1}$ acts as Macdonald difference operators D^1_N and D^{-1}_N. Therefore Proposition 5.1 and Lemmas 5.2 and 5.3 show that the assignment

$$(1-q_1)e_0 \mapsto S\sum_{i=1}^N X_i S, \qquad -(1-q_1^{-1})f_0 \mapsto S\sum_{i=1}^N X_i^{-1} S,$$

$$q_3^{N-1}(1-q_2)(1-q_3)\psi^+_1 \mapsto S\sum_{i=1}^N Y_i S,$$

$$q_3^{1-N}(1-q_2^{-1})(1-q_3^{-1})\psi^-_{-1} \mapsto S\sum_{i=1}^N Y_1^{-1} S$$

extends to the surjective homomorphism of algebras. □

6. Resonance case

Let $k \geq 1, r \geq 2$ be positive integers. In this section we impose the following condition on the parameters q_1, q_2, q_3:

$$q_1^{1-r} q_3^{k+1} = 1. \tag{6.1}$$

As usual we assume that $q_1 q_2 q_3 = 1$. We refer to the condition (6.1) as the resonance condition. We also assume that $q_1^n q_3^m = 1$ if and only if there exists an integer α such that $n = (1-r)\alpha$, $m = (k+1)\alpha$.

In this section we establish a link between the representations of $\mathcal{E}$ and ideals in polynomial algebra spanned by the Macdonald polynomials (see [FJM2], [FJM1]).

6.1. Finite tensor products

If the resonance condition holds, the action of $\mathcal{E}$ on $W^N(u)$ becomes ill defined (since the denominators of the formulas determining $e(z)w$ and $f(z)w$ vanish for some vectors $w \in W^N(u)$). We still find a subspace inside $W^N(u)$ on which the action is well defined.

Set

$$S^{k,r,N} = \{\lambda \in \mathcal{P}^N \mid \lambda_i - \lambda_{i+k} \geq r (1 \leq i \leq N-k)\}. \tag{6.2}$$

We call partitions satisfying the condition (6.2) (k,r)-admissible partitions. Let $W^{k,r,N}(u) \hookrightarrow W^N(u)$ be the subspace spanned by the vectors $|\lambda\rangle$ for $\lambda \in S^{k,r,N}$.

Induce the actions of the operators $e(z), f(z), \psi^{\pm}(z)$ on $W^{k,r,N}(u)$ from those on $W^N(u)$ for the generic values of the parameters.

REMARK 6.1

In fact, one has to be careful defining the matrix coefficients $\langle\lambda + \mathbf{1}_i|e(z)|\lambda\rangle$ and $\langle\lambda - \mathbf{1}_i|f(z)|\lambda\rangle$. Namely, both contain factors of the form $1 - q_1^s q_3^l$. If the condition (6.1) holds and $s = \alpha(1-r)$, $l = \alpha(k+1)$, such a factor vanishes. The prescription is first to cancel all factors of the form $1 - q_1^{1-r} q_3^{k+1}$ (if they appear simultaneously in the numerator and in the denominator) and then impose the resonance condition.

LEMMA 6.2

The comultiplication rule makes the subspace $W^{k,r,N}(u) \hookrightarrow W^N(u)$ into a level $(1,1)$ $\mathcal{E}$-module.

Proof

We need to check that matrix coefficients $\langle\lambda + \mathbf{1}_i|e(z)|\lambda\rangle$ ($\langle\lambda - \mathbf{1}_i|f(z)|\lambda\rangle$) are well defined provided $\lambda + \mathbf{1}_i \in S^{k,r,N}$ ($\lambda - \mathbf{1}_i \in S^{k,r,N}$). We check this for $e(z)$. (The case of $f(z)$ is similar.) Formula (3.16) gives

$$\begin{aligned}(1-q_1)&\langle\lambda + \mathbf{1}_i|e(z)|\lambda\rangle \\ &= \prod_{j=1}^{i-1} \frac{(1 - q_1^{\lambda_i - \lambda_j} q_3^{i-j-1})(1 - q_1^{\lambda_i - \lambda_j + 1} q_3^{i-j+1})}{(1 - q_1^{\lambda_i - \lambda_j} q_3^{i-j})(1 - q_1^{\lambda_i - \lambda_j + 1} q_3^{i-j})} \delta(q_1^{\lambda_i} q_3^{i-1} u/z).\end{aligned} \tag{6.3}$$

The denominator vanishes if for some m satisfying $1 \leq m \leq i-1$,

$$q_1^{\lambda_i - \lambda_m} q_3^{i-m} = 1 \qquad \text{or} \qquad q_1^{\lambda_i - \lambda_m + 1} q_3^{i-m} = 1.$$

In the first case there exists some positive integer α such that

$$\lambda_i - \lambda_m = \alpha(1-r), \quad i - m = \alpha(k+1).$$

This is impossible since $\lambda + \mathbf{1}_i$ is (k,r)-admissible. In fact,

$$\lambda_i + 1 \leq \lambda_{i-\alpha k} - \alpha r = \lambda_{m+\alpha} - \alpha r \leq \lambda_m - \alpha r,$$

and hence $\lambda_i - \lambda_m \leq -1 - \alpha r < \alpha(1-r)$. Now assume $q_1^{\lambda_i - \lambda_m + 1} q_3^{i-m} = 1$. Then there exists a positive integer α such that

$$\lambda_i - \lambda_m + 1 = \alpha(1-r), \quad i - m = \alpha(k+1).$$

As above, we have $\lambda_i + 1 - \lambda_m \leq -\alpha r < \alpha(1-r)$, which is a contradiction.

We show now that the action of $\mathcal{E}$ preserves the linear span of vectors corresponding to the (k,r)-admissible partitions. In fact, let $\lambda_i - \lambda_{i+k} = r$. Then we need to show that

$$\langle \lambda + \mathbf{1}_{i+k} | e(z) | \lambda \rangle = 0 \tag{6.4}$$

(since $\lambda + \mathbf{1}_{i+k}$ is not (k,r)-admissible). But this zero comes from the factor $1 - q_1^{\lambda_i - \lambda_j + 1} q_3^{i-j+1}$ in the formula (3.16), where i and j are replaced with $i+k$ and j, respectively. The check for $f(z)$ is similar. $\square$

REMARK 6.3

We recall that in [FJM1] the vector space spanned by the Macdonald polynomials P_λ with (k,r)-admissible partitions λ was considered (see also [Kas]). It was proved that this space is $S\ddot{H}_N$-stable. The lemma above gives yet another proof of this statement by using the representation theory of $\mathcal{E}$.

6.2. Semiinfinite limit

In this subsection we define a semiinfinite representation in the resonance case. The construction is similar to the construction for $\mathcal{F}(u)$, although certain modifications are needed.

Fix a sequence of integers $\mathbf{c} = (c_1, \dots, c_{k-1})$ satisfying $0 = c_0 \le c_1 \le \cdots \le c_{k-1} \le r$, and define the tail

$$\lambda^0_{\nu k+i+1} = -\nu r - c_i \, (\nu \ge 0, 0 \le i \le k-1).$$

We define the sets of partitions $S^{k,r}_{\mathbf{c}}, S^{k,r,N,+}_{\mathbf{c}}$ and the mapping $\tau_{k,r,N} : S^{k,r,N,+}_{\mathbf{c}} \to S^{k,r,N+k,+}_{\mathbf{c}}$ as follows:

$$S^{k,r}_{\mathbf{c}} = \big\{ \lambda \in \mathcal{P} \bigm| \lambda_j - \lambda_{j+k} \ge r \ (j \ge 1), \ \lambda_j = \lambda^0_j \text{ for sufficiently large } i \big\},$$
$$S^{k,r,N,+}_{\mathbf{c}} = \big\{ \lambda \in S^{k,r,N} \bigm| \lambda_j \ge \lambda^0_j (1 \le j \le N) \big\},$$
$$\tau_{k,r,N}(\lambda) = (\lambda_1, \dots, \lambda_N, \lambda^0_{N+1}, \dots, \lambda^0_{N+k}).$$

Let $W^{k,r}_{\mathbf{c}}(u)$ be the space spanned by the vectors $|\lambda\rangle$ for $\lambda \in S^{k,r}_{\mathbf{c}}$. To endow it with the structure of the $\mathcal{E}$-module, we introduce the subspaces $W^{k,r,N,+}_{\mathbf{c}}(u) \hookrightarrow W^{k,r,N}(u)$ spanned by the vectors $|\lambda\rangle$ for $\lambda \in S^{k,r,N,+}_{\mathbf{c}}$ and induce the embeddings

$$\tau_{k,r,N} : W^{k,r,N,+}_{\mathbf{c}}(u) \to W^{k,r,N+k,+}_{\mathbf{c}}(u) \tag{6.5}$$

by the formula $\tau_{k,r,N}|\lambda\rangle = |\tau_{k,r,N}(\lambda)\rangle$. These embeddings give the identification

$$W^{k,r}_{\mathbf{c}}(u) \simeq \lim_{N \to \infty} W^{k,r,N+}_{\mathbf{c}}(u). \tag{6.6}$$

We define an action of $\mathcal{E}$ on $W^{k,r}_{\mathbf{c}}(u)$ in the following way. Let $\beta_{k,N}(z)$ be the rational function defined by

$$\beta_{k,N}(z) = \prod_{j=0}^{i} \frac{1 - q_1^{-c_j-\nu-1} q_3^{-\nu+j} u/z}{1 - q_1^{-c_j-\nu-1} q_3^{-\nu+j-1} u/z} \prod_{j=i+1}^{k-1} \frac{1 - q_1^{-c_j-\nu} q_3^{-\nu+j+1} u/z}{1 - q_1^{-c_j-\nu} q_3^{-\nu+j} u/z},$$

where $N = \nu k + i + 1$ $(0 \le i \le k-1, \nu \ge 0)$. Consider the operators acting on the space $W_{\mathbf{c}}^{k,r,N,+}(u)$:

$$e^{[k,N]}(z) = e(z), \qquad f^{[k,N]}(z) = \beta_{k,N}(z) f(z), \tag{6.7}$$

$$\psi^{+[k,N]}(z) = \beta_{k,N}(z)\psi^{+}(z), \qquad \psi^{-[k,N]}(z) = \beta^{+}_{k,N}(z)\psi^{-}(z), \tag{6.8}$$

where $\beta^{+}_{k,N}(z)$ is the expansion of $\beta_{k,N}(z)$ as series in z. These operators turn out to be stable and define the $\mathcal{E}$-module structure on $W_{\mathbf{c}}^{k,r}(u)$.

In the following, that is, Lemma 6.4 and Theorem 6.5, the arguments are very similar to those for Lemma 4.2 and Theorem 4.3. We omit the proofs for them.

LEMMA 6.4

Suppose that for $\lambda \in S_{\mathbf{c}}^{k,r,N,+}$, the equality $(\lambda_{N-k+1}, \dots, \lambda_N) = (\lambda^0_{N-k+1}, \dots, \lambda^0_N)$ is valid. Then, for $x = e, f, \psi^+, \psi^-$, we have $x^{[k,N]}(z)|\lambda\rangle \in W_{\mathbf{c}}^{k,r,N,+}(u)$ and

$$\tau_{k,r,N}\bigl(x^{[k,N]}(z)|\lambda\rangle\bigr) = x^{[k,N+k]}(z)\tau_{k,r,N}(|\lambda\rangle).$$

For $\lambda \in S_{\mathbf{c}}^{k,r}$ we set

$$x(z)|\lambda\rangle = \lim_{N\to\infty} x^{[k,N]}(z)|\lambda_1, \dots, \lambda_N\rangle, \tag{6.9}$$

where $x = e, f, \psi^+, \psi^-$ and the right-hand side is considered as an element of $W_{\mathbf{c}}^{k,r}(u)$ via (6.6).

THEOREM 6.5

Formula (6.9) endows $W_{\mathbf{c}}^{k,r}(u)$ with the structure of a level $(1, q_3^k)$ $\mathcal{E}$-module.

We now write down explicit formulas for the nonzero matrix coefficients of operators $e(z)$, $f(z)$, and $\psi^{\pm}(z)$ acting on $W_{\mathbf{c}}^{k,r}(u)$. For $e(z)$,

$$\begin{aligned}
&(1-q_1)\langle \lambda + \mathbf{1}_i | e(z) | \lambda\rangle \\
&\quad = \prod_{j=1}^{i-1} \frac{(1 - q_1^{\lambda_i - \lambda_j} q_3^{i-j-1})(1 - q_1^{\lambda_i - \lambda_j + 1} q_3^{i-j+1})}{(1 - q_1^{\lambda_i - \lambda_j} q_3^{i-j})(1 - q_1^{\lambda_i - \lambda_j + 1} q_3^{i-j})} \delta(q_1^{\lambda_i} q_3^{i-1} u/z).
\end{aligned}$$

For $f(z)$,

$$\begin{aligned}
&-(1-q_1^{-1})\langle \lambda - \mathbf{1}_i | f(z) | \lambda\rangle \\
&\quad = \prod_{j=i+1}^{i+k} \frac{1 - q_1^{\lambda_j - \lambda_i + 1} q_3^{j-i+1}}{1 - q_1^{\lambda_j - \lambda_i + 1} q_3^{j-i}} \\
&\qquad \times \prod_{j=i+1}^{\infty} \frac{(1 - q_1^{\lambda_{j+k} - \lambda_i + 1} q_3^{j+k-i+1})(1 - q_1^{\lambda_j - \lambda_i} q_3^{j-i-1})}{(1 - q_1^{\lambda_j - \lambda_i} q_3^{j-i})(1 - q_1^{\lambda_{j+k} - \lambda_i + 1} q_3^{j+k-i})} \delta(q_1^{\lambda_i - 1} q_3^{i-1} u/z).
\end{aligned}$$

For $\psi^{\pm}(z)$,

$$\psi^{+}(z)|\lambda\rangle=\prod_{i=1}^{k}\frac{1-q_1^{\lambda_i}q_3^{i}u/z}{1-q_1^{\lambda_i}q_3^{i-1}u/z}\prod_{i=1}^{\infty}\frac{(1-q_1^{\lambda_i-1}q_3^{i-2}u/z)(1-q_1^{\lambda_{i+k}}q_3^{i+k}u/z)}{(1-q_1^{\lambda_{i+k}}q_3^{i+k-1}u/z)(1-q_1^{\lambda_i-1}q_3^{i-1}u/z)}|\lambda\rangle,$$

$$\psi^{-}(z)|\lambda\rangle=q_3^{k}\prod_{i=1}^{k}\frac{1-q_1^{-\lambda_i}q_3^{-i}z/u}{1-q_1^{-\lambda_i}q_3^{-i+1}z/u}$$
$$\times\prod_{i=1}^{\infty}\frac{(1-q_1^{-\lambda_i+1}q_3^{-i+2}z/u)(1-q_1^{-\lambda_{i+k}}q_3^{-i-k}z/u)}{(1-q_1^{-\lambda_{i+k}}q_3^{-i-k+1}z/u)(1-q_1^{-\lambda_i+1}q_3^{-i+1}z/u)}|\lambda\rangle.$$

Define series $\varphi^{\pm}_{\emptyset}(u/z)$ in $z^{\mp1}$ by

$$\varphi^{+}_{\emptyset}(u/z)=\frac{1-q_3u/z}{1-u/z},\qquad \varphi^{-}_{\emptyset}(u/z)=q_3\frac{1-q_3^{-1}z/u}{1-z/u}.$$

Then the formulas above can be rewritten in the following way:

$$\langle\lambda|\psi^{\pm}(z)|\lambda\rangle=\prod_{i=0}^{k-1}\varphi^{\pm}_{\emptyset}(q_3^{i}q_1^{-c_i}u/z)\prod_{i\geq1}\frac{\langle\lambda|\psi^{\pm}(z)_i|\lambda\rangle}{\langle\lambda^0|\psi^{\pm}(z)_i|\lambda^0\rangle},$$

$$\langle\lambda+\mathbf{1}_i|e(z)|\lambda\rangle=\langle\lambda+\mathbf{1}_i|e(z)_i|\lambda\rangle\prod_{j=1}^{i-1}\langle\lambda|\psi^{-}(z)_j|\lambda\rangle,$$

$$\langle\lambda|f(z)|\lambda+\mathbf{1}_i\rangle=\langle\lambda|f(z)_i|\lambda+\mathbf{1}_i\rangle\prod_{j=i+1}^{\infty}\frac{\langle\lambda|\psi^{+}(z)_j|\lambda\rangle}{\langle\lambda^0|\psi^{+}(z)_j|\lambda^0\rangle}\prod_{j=0}^{k-1}\varphi^{+}_{\emptyset}(q_3^{i+j}q_1^{\lambda^0_{i+j+1}}u/z).$$

7. Quantum continuous $\mathfrak{gl}_\infty$: further directions

As we have shown above, there exists a surjective homomorphism from the quantum continuous $\mathfrak{gl}_\infty$ to the spherical DAHA of type GL_N. It is therefore natural to expect that $\mathcal{E}$ is isomorphic to the stable limit $S\ddot{H}_\infty=\lim_{N\to\infty}S\ddot{H}_N$ constructed in [SV1] and [SV2]. The related statement is that the representation obtained as the direct sum of the modules $W^N(u)$ is faithful.

There is also a link between $\mathcal{E}$ and the so-called shuffle algebra $\mathbf{S}$ (see [FO], [FT]). The latter is an algebra generated by variables $\tilde{e}_i$, $i\in\mathbb{Z}$. Let $\mathcal{E}^+$ be the subalgebras of $\mathcal{E}$ generated by e_i, $i\in\mathbb{Z}$. We conjecture that there exists an isomorphism $\pi:\mathcal{E}^+\to\mathbf{S}$, $\pi^+(e_i)=\tilde{e}_i$.

In the paper we have studied tensor products of the vector representations of $\mathcal{E}$. We have also considered limits of these tensor products, thus constructing Fock modules $\mathcal{F}(u)$. It is a natural question to study the tensor products of the modules $\mathcal{F}(u)$. It turns out that the structure of these tensor products is very rich. In particular, there exist submodules inside $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_N)$ whose characters coincide with those of the minimal representations of the $\mathcal{W}_N$-algebras up to a trivial factor.

We plan to return to all these questions elsewhere.

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