Asymptotic behavior of spectral measures of Krein's and Kotani's strings

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Abstract We discuss the spectral theory of second-order differential operators that describe the vibration of strings, diffusion processes, and others. M. G. Krein established a one-to-one correspondence between the spectral measure and the string in the case of regular left boundaries, and this correspondence was extended by S. Kotani to a certain class of strings with singular left boundary. In this article we study the relationship between the asymptotic behavior of the spectral measure and that of the corresponding string. Although the results are basically for Kotani's strings, some are also applicable to Krein's.

1. Introduction

We first review quickly S. Kotani's generalization (see [5]) of M. G. Krein's spectral theory in order to explain the notation. By a *string* we mean a function

$$m : (-\infty, +\infty) \longrightarrow [0, +\infty]$$

which is nondecreasing, right-continuous, and normalized so that $m(-\infty) = 0$. We exclude the trivial case, where m vanishes identically. In this note a string m is referred to as a *Krein's string* if $m(-0) = 0$.

For a string m, we are interested in the spectral theory of the generalized Strum-Liouville operator

$$\mathcal{L} = -\frac{d}{dm(x)}\frac{d}{dx}, \quad -\infty < x < \ell,$$

where

$$\ell\bigl(=\ell(m)\bigr) = \sup\bigl\{x \bigm| m(x) < \infty\bigr\} \quad (\le +\infty).$$

This operator appears not only in the theory of vibration of strings but also in Feller's theory of diffusion processes, where $dm(x)$ is called the *speed measure*.

We say that a string m has left boundary of *limit circle type* if, for some c $(< \ell)$,

$$\int_{-\infty}^{c} x^2 \, dm(x) < \infty, \tag{1.1}$$

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or, equivalently, for some $c \in \mathbb{R}$,

$$\int_{-\infty}^{c} \Big(\int_{-\infty}^{x} m(u)\, du \Big)\, dx < \infty. \tag{1.2}$$

Although this terminology is compatible with Weyl's classification of boundaries, our framework is based on the idea of *inextensible measures* introduced in [7] and, hence, boundary conditions are implicitly included in the string itself when necessary. For example, the operator $-\frac{d^2}{dx^2}$ on $[0,1]$ with the boundary condition $u'(0)=0, u(1)=0$ corresponds to the string

$$m(x) = \begin{cases} 0 & (x<0), \\ x & (0 \le x < 1), \\ \infty & (x \ge 1). \end{cases}$$

Throughout the article we denote by $\mathcal{M}_{\mathrm{circ}}$ the totality of strings m satisfying the condition (1.1). Elements of $\mathcal{M}_{\mathrm{circ}}$ are referred to as *Kotani's strings.* Note that Kotani's strings include Krein's. For each $m \in \mathcal{M}_{\mathrm{circ}}$, we can define $\varphi_\lambda(x)$ $(x<\ell)$, for every $\lambda \in \mathbb{C}$, as the unique solution of

$$\mathcal{L}u = \lambda u, \quad u(-\infty) = 1,$$

or, precisely, of the following integral equation:

$$\varphi_\lambda(x) = 1 - \lambda \int_{-\infty}^{x} (x-y)\varphi_\lambda(y)\, dm(y) \quad (x<\ell).$$

Let $L^2_0((-\infty,\ell),dm)$ denote the space of all square integrable functions f such that $\mathrm{Supp}(f) \subset (-\infty,\ell)$, and for $f \in L^2_0((-\infty,\ell),dm)$, define the generalized Fourier transform by

$$\widehat{f}(\lambda) = \int_{-\infty}^{\ell} f(x)\varphi_\lambda(x)\, dm(x).$$

Then a nonnegative Radon measure $\sigma(d\lambda)$ on $[0,\infty)$ is called a *spectral measure* if

$$\|f\|_{L^2_0((-\infty,\ell),dm)} = \|\widehat{f}\|_{L^2([0,\infty),\sigma)}.$$

When a spectral measure is given, we have an eigenfunction expansion. For example, in probability theory the transition density (with respect to dm) of the diffusion process with generator $-\mathcal{L}$ is given by

$$p(t,x,y) = \int_{-0}^{\infty} e^{-t\lambda}\varphi_\lambda(x)\varphi_\lambda(y)\sigma(d\lambda) \quad (t>0),$$

and, hence, for Krein's string, it holds that

$$p(t,0,0) = \int_{-0}^{\infty} e^{-t\lambda}\sigma(d\lambda) \quad (t>0)$$

(see, e.g., [3]). Thus, the study of the transition function is sometimes reduced to that of the spectral measure. However, applications of our results to probability theory will be discussed in a future article.

Throughout the article we put $\sigma(x) = \int_{[0,x]} \sigma(d\xi)$ and $\sigma(-0) = 0$ so that the Lebesgue-Stieltjes measure $d\sigma(\xi)$ equals $\sigma(d\xi)$. We can compute $\sigma(d\lambda)$ by the following procedure (see [5]). Let

$$H(\lambda) = c + \int_{-\infty}^{c} \Big(\frac{1}{\varphi_\lambda(x)^2} - 1\Big)\, dx + \int_c^\ell \frac{dx}{\varphi_\lambda(x)^2}, \tag{1.3}$$

which exists for every $\lambda < 0$ and does not depend on the choice of c $(< \ell)$. Note that

$$H(-0) = \ell, \quad H(-\infty) = \inf\{x \mid m(x) > 0\}. \tag{1.4}$$

Then $H(\lambda)$ turns out to be a function with the following representation:

$$H(\lambda) = a + \int_{-0}^{\infty} \Big(\frac{1}{\xi - \lambda} - \frac{\xi}{\xi^2 + 1}\Big)\sigma(d\xi), \tag{1.5}$$

where a is a real number and $\sigma(d\xi)$ is a nonnegative Radon measure on $[0, \infty)$ satisfying

$$\int_{-0}^{\infty} \frac{\sigma(d\xi)}{\xi^2 + 1} < \infty.$$

Then $\sigma(d\xi)$ is the spectral measure we want. $H(\lambda)$ may of course be extended for $\lambda \in \mathbb{C} \backslash [0, \infty)$ and is called the *characteristic Herglotz function* of the string m. In order to keep the notation consistent with that of [3] and [6], we denote

$$h(s) = H(-s) \quad (s > 0)$$

throughout the article. Kotani [5] proved that the correspondence between m and H is not only one-to-one but also surjective: Let $\mathbb{H}$ be the totality of Herglotz functions H of the form (1.5). Then for every $H \in \mathbb{H}$, there exists a unique $m \in \mathcal{M}_{\mathrm{circ}}$ corresponding to H. As a special case, the constant function $H(\lambda) = a$ corresponds to the string $m(x) = \infty \cdot 1_{[a,\infty)}(x)$.

Further, note that if $\int_0^\infty \sigma(d\xi)/(1 + \xi) < \infty$, then H has a simpler representation

$$H(\lambda) = a_* + \int_{-0}^{\infty} \frac{\sigma(d\xi)}{\xi - \lambda}, \quad \text{where } a_* = a - \int_{-0}^{\infty} \frac{\xi}{\xi^2 + 1}\sigma(d\xi),$$

and such an H corresponds to the string m which satisfies $a_* = \inf\{x \mid m(x) > 0\}$, so that its translated string m_{a_*} defined by $m_{a_*}(x) = m(x + a_*)$ is a Krein's string. In particular, m is a Krein's string if $a_* \geq 0$. Keeping the above facts in mind, we set generally

$$a_* = a_*(m) := \inf\{x \in \mathbb{R} \mid m(x) > 0\} \quad (\geq -\infty) \tag{1.6}$$

for $m \in \mathcal{M}_{\mathrm{circ}}$ throughout the article. Note that $a_* = h(\infty-)$ by (1.4).

The following example plays the most important role in the present article, and the notation is preserved throughout. For $0 < \alpha < 2$ $(\alpha \neq 1)$, define β by $\alpha^{-1} + \beta^{-1} = 1$, and define $m^{(\alpha)} \in \mathcal{M}_{\mathrm{circ}}$ as follows. Note that $m^{(\alpha)}$ is a Krein's string if and only if $0<\alpha<1$:

$$m^{(\alpha)}(x)=\begin{cases}(-\beta)^{\beta}x^{-\beta}1_{(0,\infty)}(x) & \text{if } 0<\alpha<1\ (\beta<0),\\ e^{x} & \text{if } \alpha=1,\\ \beta^{\beta}(-x)^{-\beta}1_{(-\infty,0)}(x)+\infty\cdot 1_{[0,\infty)}(x) & \text{if } 1<\alpha<2\ (\beta>2).\end{cases}$$

If $0<\alpha\le 1$, then $a_*=0,\ell=\infty$; if $1<\alpha<2$, then $a_*=-\infty,\ell=0$; and if $\alpha=1$, then $a_*=-\infty,\ell=\infty$. The characteristic Herglotz function is

$$H^{(\alpha)}(-s)=h^{(\alpha)}(s)=\begin{cases}\dfrac{\Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+\alpha)}\alpha^{2\alpha}s^{\alpha-1} & (0<\alpha<2,\alpha\neq 1),\\ -(\log s+2\gamma) & (\alpha=1),\end{cases}$$

where $\gamma=-\Gamma'(1)$ is Euler's constant. Inverting the Stieltjes transform, we obtain

$$\sigma^{(\alpha)}(\xi)=\frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^{\alpha}\quad(\xi\ge 0).$$

For details we refer to Kotani [5].

The aim of this article is to study the asymptotic behavior of the spectral function $\sigma(\xi)$; that is, we find necessary and sufficient conditions for $\sigma(\xi)\sim$ const. $\times\,\xi^{\alpha}$ as $\xi\to+0$ (or $\xi\to\infty$). (Throughout the article $f(x)\sim g(x)$ means that $\lim f(x)/g(x)=1$.)

In the case where $0<\alpha<1$, we already have the following result for Krein's strings.

THEOREM A ([3, THEOREM 3])

Let m be a Krein's string, and let $h(s),\sigma(\xi)$, and a_* be associated with m as above. Let $0<\alpha<1$, $C>0$, and $a\ge 0$. Define β by $\alpha^{-1}+\beta^{-1}=1$ so that $\beta=-\alpha/(1-\alpha)<0$. Then the following are equivalent:

(1.7) $$m(x)\sim C(-\beta)^{\beta}(x-a)^{-\beta}\quad \text{as } x\downarrow a,$$

(1.8) $$h(s)-a\sim C^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\alpha^{2\alpha}s^{\alpha-1}\quad \text{as } s\to\infty,$$

(1.9) $$a_*=a\qquad\text{and}\qquad \sigma(\xi)\sim C^{\alpha-1}\frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^{\alpha}\quad \text{as } \xi\to\infty.$$

Also, the following are equivalent:

(1.10) $$m(x)\sim C(-\beta)^{\beta}x^{-\beta}\quad \text{as } x\to\infty,$$

(1.11) $$\sigma(\xi)\sim C^{\alpha-1}\frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^{\alpha}\quad \text{as } \xi\downarrow 0,$$

(1.12) $$h(s)\sim C^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\alpha^{2\alpha}s^{\alpha-1}\quad \text{as } s\downarrow 0$$

(see also [3] for related bibliographies). The above result is stated only for Krein's strings, but we see later that this restriction is inessential when we are concerned with the case $x\to\infty$ (see Theorem 2.3).

The main aim of this article is to study similar problems for the case $1 < \alpha < 2$, and our main results are the following. Note that $h^{(\alpha)}(s) < 0$ $(s > 0)$ when $1 < \alpha < 2$.

THEOREM 1.1
Let $1 < \alpha < 2$ and $C > 0$. Define, as before, β by $\alpha^{-1} + \beta^{-1} = 1$, so that $\beta > 2$. Then the following are equivalent:

$$m(x) \sim C\beta^{\beta}|x|^{-\beta} \quad \text{as } x \to -\infty, \tag{1.13}$$

$$-h(s) \sim C^{\alpha-1}\frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)}\alpha^{2\alpha}s^{\alpha-1} \quad \text{as } s \to \infty, \tag{1.14}$$

$$\sigma(\xi) \sim C^{\alpha-1}\frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^{\alpha} \quad \text{as } \xi \to \infty. \tag{1.15}$$

THEOREM 1.2
Let α, β, and C be as in Theorem 1.1, and let $a \in \mathbb{R}$. Then the following are equivalent:

$$m(x) \sim C\beta^{\beta}(a-x)^{-\beta} \quad \text{as } x \uparrow a, \tag{1.16}$$

$$a - h(s) \sim C^{\alpha-1}\frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)}\alpha^{2\alpha}s^{\alpha-1} \quad \text{as } s \downarrow 0, \tag{1.17}$$

$$\ell(m) = a \quad \text{and} \quad \sigma(\xi) \sim C^{\alpha-1}\frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^{\alpha} \quad \text{as } \xi \downarrow 0. \tag{1.18}$$

Note that, in (1.18), the condition $\ell(m) = a$ is indispensable because σ does not determine m uniquely; it allows translations.

The above two theorems are proved in an extended form in Section 2. The case when we are interested in the orders but not in multiplicative constants is discussed in Section 3. The case $\alpha = 1$ is a little complicated and is discussed in Section 4. In this article we often use Tauberian theorems for Stieltjes transforms. The theory and results are found in the monograph [1], which is, however, not necessarily convenient for our present use. Therefore, in the appendix we sum up the necessary results with proofs for the convenience of the reader.

2. Main results

A positive, measurable function $L(x)$ defined on some interval (A, ∞) (or $(0, 1/A)$) is said to be *slowly varying* at ∞ (resp., $+0$) if

$$\lim_{c\to\infty[+0]} \frac{L(xc)}{L(c)} = 1 \quad (\forall x > 0).$$

Also, a positive, measurable function $\varphi(x)$ defined on some interval (A, ∞) (or $(0, 1/A)$) is said to be *regularly varying* at ∞ [resp., $+0$] with exponent $\rho (\in \mathbb{R})$ if

$$\lim_{c\to\infty[+0]} \frac{\varphi(xc)}{\varphi(c)} = x^{\rho} \quad (\forall x > 0).$$

A regularly varying function with exponent ρ can be expressed as $\varphi(x) = x^\rho L(x)$ with slowly varying $L(x)$. In many situations slowly varying functions behave as if they were constants. For example, the following facts are well known for regularly varying $\varphi(x) = x^\rho L(x)$. If $\rho > -1$, then

$$\int_0^x \varphi(u)\,du \sim \frac{x\varphi(x)}{\rho+1} \quad \text{as } x \to \infty \ [+0] \tag{2.1}$$

provided that the left-hand side makes sense. (In the case $x \to \infty$, the lower limit zero of the integral is of course inessential and may be replaced by other numbers.) Similarly, if $\rho < -1$, then

$$\int_x^\infty \varphi(u)\,du \sim -\frac{x\varphi(x)}{\rho+1} \quad \text{as } x \to \infty \ [+0] \tag{2.2}$$

provided that the left-hand side makes sense. (In the case $x \to +0$, the upper bound of the integral may be replaced by any finite positive number; see pp. 26–28 of [1] for details. Change the variable $u = 1/x$ to treat the case $x \to +0$.) Furthermore, if $\rho > 0$ and if φ is absolutely continuous with monotone derivative, then

$$\varphi'(x) \sim \frac{\rho\varphi(x)}{x} \quad \text{as } x \to \infty \ [+0] \tag{2.3}$$

(see [1, p. 36]). When $\rho \neq 0$, there exists a function f such that $f(\varphi(x)) \sim x$ and $\varphi(f(x)) \sim x$. Such an f, which varies regularly with exponent $1/\rho$, is called an *asymptotic inverse* of φ and is denoted by φ^{-1}.

Let $m \in \mathcal{M}_{\mathrm{circ}}$, and let $h, H, \sigma, \ell(=\ell(m)), a_*(=a_*(m))$ be as in Section 1. Our main results are the following three theorems. Theorems 1.1 and 1.2 in the introduction can be obtained as special cases when $L(x) = C$.

THEOREM 2.1

Let $1 < \alpha < 2$, and define β, as before, by $\alpha^{-1} + \beta^{-1} = 1$, so that $\beta > 2$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a measurable function varying regularly at ∞ with exponent $\alpha - 1$, then the following are equivalent:

$$m(x) \sim \beta^\beta \frac{1}{|x|\varphi^{-1}(|x|)} \quad \text{as } x \to -\infty, \tag{2.4}$$

$$-h(s) \sim \frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)} \alpha^{2\alpha}\varphi(s) \quad \text{as } s \to \infty, \tag{2.5}$$

$$\sigma(\xi) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} \xi\varphi(\xi) \quad \text{as } \xi \to \infty. \tag{2.6}$$

Notice here (and in the next theorem) that $x\varphi^{-1}(x)$ and $\xi\varphi(\xi)$ are regularly varying at ∞ with exponent $1 + 1/(\alpha-1) = \beta$ and α, respectively.

THEOREM 2.2

Let α and β be as in Theorem 2.1, and let $a \in R$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a function varying regularly at $+0$ with exponent $\alpha - 1$, then the following are equivalent:

$$m(x) \sim \beta^{\beta} \frac{1}{(a-x)\varphi^{-1}(a-x)} \quad \textit{as } x \uparrow a, \tag{2.7}$$

$$a - h(s) \sim \frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)} \alpha^{2\alpha} \varphi(s) \quad \textit{as } s \downarrow 0, \tag{2.8}$$

$$\ell(m) = a \qquad \textit{and} \qquad \sigma(\xi) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} \xi \varphi(\xi) \quad \textit{as } \xi \downarrow 0. \tag{2.9}$$

An extension of Theorem A to Kotani's strings is the following. Although it is essentially due to [3], we write it here to stress that it also holds not only for Krein's strings but also for Kotani's (and another purpose is to correct a misprint in [3]).

THEOREM 2.3
Let $0 < \alpha < 1$ and $\alpha^{-1} + \beta^{-1} = 1$, so that $\beta < 0$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a measurable function varying regularly at $+0$ with exponent $\alpha - 1$; then the following are equivalent:

$$m(x) \sim (-\beta)^{\beta} \frac{1}{x\varphi^{-1}(x)} \quad \textit{as } x \to \infty, \tag{2.10}$$

$$h(s) \sim \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \alpha^{2\alpha} \varphi(s) \quad \textit{as } s \downarrow 0, \tag{2.11}$$

$$\sigma(\xi) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} \xi \varphi(\xi) \quad \textit{as } \xi \downarrow 0. \tag{2.12}$$

THEOREM 2.4
Let α and β be as in Theorem 2.3, and let $a \in \mathbb{R}$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a measurable function varying regularly at ∞ with exponent $\alpha - 1$, then the following are equivalent:

$$m(x) \sim (-\beta)^{\beta} \frac{1}{(x-a)\varphi^{-1}(x-a)} \quad \textit{as } x \downarrow a, \tag{2.13}$$

$$h(s) - a \sim \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \alpha^{2\alpha} \varphi(s) \quad \textit{as } s \to \infty, \tag{2.14}$$

$$a_* = a \qquad \textit{and} \qquad \sigma(\xi) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} \xi \varphi(\xi) \quad \textit{as } \xi \to \infty. \tag{2.15}$$

REMARK 1
Since in these two theorems $\varphi(x)$ varies regularly with *negative* exponent, $\varphi(x)$ is essentially decreasing. Therefore, the asymptotic behavior of $\varphi(x)$ as $x \to +0$ or $x \to \infty$ corresponds to that of $\varphi^{-1}(x)$ as $x \to \infty$ or $x \to +0$, respectively.

Our idea of the proofs of these theorems is based on [3]: we use the scaling property to reduce the problem to the continuity of the correspondence.

THEOREM B (KOTANI [5, THEOREM 8])

Let $m_\infty, m_1, m_2, \ldots \in \mathcal{M}_{\mathrm{circ}}$, and let H_n be their characteristic Herglotz functions. Then $H_n(\lambda) \to H_\infty(\lambda)$ as $n \to \infty$ for all $\lambda < 0$ if and only if the following two conditions hold:

$$\text{(M1)} \quad m_n(x) \Longrightarrow m_\infty(x), \quad x \in \mathbb{R},$$

$$\text{(M2)} \quad \lim_{c\to-\infty} \sup_{n\ge1} \int_{-\infty}^{c} \Big(\int_{-\infty}^{y} m_n(u)\,du \Big)\,dy = 0.$$

Here $\Longrightarrow$ denotes the convergence at all continuity points of the limit function. Notice that (M2) is equivalent to

$$\text{(M2a)} \quad \lim_{c\to-\infty} \limsup_{n\to\infty} \int_{-\infty}^{c} x^2\,dm_n(x) = 0$$

(see (2.16); for the case when (M2) fails, see [4]). We also note the following fact.

PROPOSITION 2.1

For $m_n \in \mathcal{M}_{\mathrm{circ}}$ $(n = \infty, 1, 2, \ldots)$, define $N_n : \mathbb{R} \to [0, \infty]$ as follows:

$$N_n(x) = \int_{-\infty}^{x} \Big(\int_{-\infty}^{y} m_n(u)\,du \Big)\,dy = \frac{1}{2} \int_{-\infty}^{x} (x-u)^2\,dm_n(u). \tag{2.16}$$

Then the conditions (M1) and (M2) hold if and only if

$$\text{(N)} \qquad N_n(x) \Longrightarrow N_\infty(x), \quad x \in \mathbb{R}\ (n \to \infty).$$

Proof

Suppose that (N) holds. Then (M2) may be written as

$$\lim_{c\to-\infty} \sup_{n\ge1} N_n(c) = 0,$$

which can easily be seen because $N_\infty(-\infty) = N_n(-\infty) = 0$. So let us prove (M1). Let

$$M_n(x) = N_n'(x) \Big(= \int_{-\infty}^{x} m_n(u)\,du \Big).$$

By the monotonicity in x of $M_n(x)$ $(n \ge 1)$, the convergence of N_n implies that of M_n. Indeed, since, for every $\epsilon > 0$,

$$M_n(x) \le \frac{1}{\epsilon} \big(N_n(x+\epsilon) - N_n(x) \big),$$

we see that

$$\limsup_{n\to\infty} M_n(x) \le \frac{1}{\epsilon} \big(N_\infty(x+\epsilon) - N_\infty(x) \big) \le M_\infty(x+\epsilon).$$

Letting $\epsilon \to 0$, we have

$$\limsup_{n\to\infty} M_n(x) \le M_\infty(x+0).$$

Similarly, we have

$$\liminf_{n\to\infty} M_n(x) \ge M_\infty(x-0),$$

and, hence, $M_n(x) \Longrightarrow M_\infty(x)$. Repeating the same argument, we conclude that (M1) holds.

Conversely, let us see that (M1) and (M2) imply (N). Let us fix a continuity point $x \in \mathbb{R}$ of m_∞. Then (M1) implies that, for every continuity point $c(< x)$ of m_∞,

$$\lim_{n\to\infty} \int_c^x (x-u)^2\, dm_n(u) = \int_c^x (x-u)^2\, dm_\infty(u).$$

Therefore, in order to see (N), it suffices to show that

$$\lim_{c\to-\infty} \limsup_{n\to\infty} \int_{-\infty}^c (x-u)^2\, dm_n(u) = 0.$$

However, this follows from (M2a). □

We next note the following scaling property, which can be proved by simple changes of variables.

LEMMA 2.1 ([3, LEMMA 2])

Let $a > 0, c > 0, b \in \mathbb{R}$, and let $m \in \mathcal{M}_{\mathrm{circ}}$, for which we define N as in Proposition 2.1. Then, if $m \in \mathcal{M}_{\mathrm{circ}}$ corresponds to $H \in \mathbb{H}$, then $m_{a,b,c}(x) = acm(ax+b)$ and $N_{a,b,c}(x) = (c/a)N(ax+b)$ correspond to $H_{a,b,c}(\lambda) = (1/a)(H(c\lambda) - b)$.

Especially, $c\varphi(c)m(\varphi(c)x)$ and $(c/(\varphi(c)))N(\varphi(c)x)$ correspond to $(1/(\varphi(c))) \times H(cs)$ for any function $\varphi(c) > 0$. This fact is used repeatedly.

We are now ready to prove Theorems 2.1–2.4. Since the proofs of these four theorems are essentially the same, we prove Theorem 2.2 only. In fact, the proofs of the other three are even simpler.

Proof of Theorem 2.2

Recall (1.4). Since $h(s) - b$ corresponds to $m(x+b)$ by Lemma 2.1, we may and do assume that $\ell = h(+0) = a = 0$ without loss of generality.

We first see the equivalence of (2.7) and (2.8). Let

$$N(x) = \int_{-\infty}^x \Big(\int_{-\infty}^y m(u)\, du\Big)\, dy = \frac{1}{2}\int_{-\infty}^x (x-u)^2\, dm(u) \quad (x \in \mathbb{R}). \tag{2.17}$$

Then, since $1/(x\varphi^{-1}(x))$ varies regularly with exponent $-(1 + 1/(\alpha-1)) = -\beta$, (2.7) is equivalent to

$$N(x) \sim \frac{\beta^\beta}{(\beta-2)(\beta-1)} \frac{x^2}{\varphi^{-1}(-x)} \quad (x \uparrow 0) \tag{2.18}$$

by the property of regularly varying functions (see (2.2)). Thus, in order to prove the equivalence of (2.7) and (2.8), it suffices to see that of (2.18) and (2.8) (with $a = 0$). By the regularly varying property, (2.18) is equivalent to

$$\frac{\varphi^{-1}(\lambda)}{\lambda} N(\lambda x) \Longrightarrow N^{(\alpha)}(x) := \frac{\beta^\beta}{(\beta-2)(\beta-1)}(-x)^{2-\beta} \quad (\lambda \downarrow 0). \tag{2.19}$$

Indeed, (2.19) follows from (2.18) if we note that $\lambda/\varphi^{-1}(\lambda)$ varies regularly with exponent $2-\beta$. Notice that $((\varphi^{-1}(\lambda))/\lambda)N(\lambda x)$ corresponds to $(1/(\varphi(\lambda)))h(\lambda s)$ by Lemma 2.1. Then we apply Kotani's continuity theorem (see Theorem B and Proposition 2.1) to see that (2.19) implies

$$\frac{1}{\varphi(c)}h(cs) \to h^{(\alpha)}(s), \quad \forall s>0,\ c\downarrow 0. \tag{2.20}$$

Putting $s=1$, we have (2.8). The converse can be shown similarly.

We next see the equivalence of (2.9) and (2.8). Note that

$$-h'(s) = \int_{-0}^{\infty} \frac{d\sigma(\xi)}{(\xi+s)^2}. \tag{2.21}$$

Therefore, by Karamata's Tauberian theorem (see Theorem C in the appendix), (2.9) is equivalent to

$$-h'(s) \sim \frac{\Gamma(2-\alpha)}{\Gamma(1+\alpha)}\alpha^{2\alpha}\varphi(s)/s \quad (s\downarrow 0). \tag{2.22}$$

But since $a-h(s) = -\int_0^s h'(u)\,du$, (2.22) is equivalent to (2.8) by the property of regularly varying functions with monotone derivatives that we stated at the beginning of this section (see (2.1) and (2.3)). □

3. The case of oscillations

Throughout the article $f(x) \asymp g(x)$ means that $f(x) = O(g(x))$ and $g(x) = O(f(x))$; that is,

$$0 < \liminf f(x)/g(x) \le \limsup f(x)/g(x) < \infty.$$

Let $m \in \mathcal{M}_{\mathrm{circ}}$, and let $N, h, H, \sigma, \ell(=\ell(m)), a_*(=a_*(m))$ be as in the previous sections.

THEOREM 3.1

Let $1<\alpha<2$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a function varying regularly at ∞ with exponent $\alpha-1$, then the following are equivalent:

$$m(x) \asymp \frac{1}{|x|\varphi^{-1}(|x|)} \quad \text{as } x\to-\infty, \tag{3.1}$$

$$-h(s) \asymp \varphi(s) \quad \text{as } s\to\infty, \tag{3.2}$$

$$\sigma(\xi) \asymp \xi\varphi(\xi) \quad \text{as } \xi\to\infty. \tag{3.3}$$

THEOREM 3.2

Let $1<\alpha<2$ and $a\in R$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a measurable function varying regularly at $+0$ with exponent $\alpha-1$, then the following are equivalent:

$$m(x) \asymp \frac{1}{(a-x)\varphi^{-1}(a-x)} \quad \text{as } x\uparrow a, \tag{3.4}$$

$$a-h(s) \asymp \varphi(s) \quad \text{as } s\downarrow 0, \tag{3.5}$$

$$\ell(m) = a \qquad \textit{and} \qquad \sigma(\xi) \asymp \xi\varphi(\xi) \quad \textit{as } \xi \downarrow 0. \tag{3.6}$$

THEOREM 3.3

Let $0 < \alpha < 1$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a measurable function varying regularly at $+0$ with exponent $\alpha - 1$, then the following are equivalent:

$$m(x) \asymp \frac{1}{x\varphi^{-1}(x)} \quad \textit{as } x \to \infty, \tag{3.7}$$

$$h(s) \asymp \varphi(s) \quad \textit{as } s \downarrow 0, \tag{3.8}$$

$$\sigma(\xi) \asymp \xi\varphi(\xi) \quad \textit{as } \xi \downarrow 0. \tag{3.9}$$

We write below the counterpart of Theorem 3.3 for reference. However, since it may be reduced to the case of Krein's strings, it is also an easy corollary of I. S. Kac's inequality, and we do not claim that it is new (cf. [3]).

THEOREM 3.4

Let $0 < \alpha < 1$, and let $a \in \mathbb{R}$. If $\varphi(x) = x^{\alpha-1}L(x) > 0$ is a measurable function varying regularly at ∞ with exponent $\alpha - 1$, then the following are equivalent:

$$m(x) \asymp \frac{1}{(x-a)\varphi^{-1}(x-a)} \quad \textit{as } x \downarrow a, \tag{3.10}$$

$$h(s) - a \asymp \varphi(s) \quad \textit{as } s \to \infty, \tag{3.11}$$

$$a_*(m) = a \qquad \textit{and} \qquad \sigma(\xi) \asymp \xi\varphi(\xi) \quad \textit{as } \xi \to \infty. \tag{3.12}$$

For the proof of the Abelian implication, we prepare the following.

PROPOSITION 3.1

Let $m_1, m_2 \in \mathcal{M}_{\mathrm{circ}}$, and let $H_1, H_2 \in \mathbb{H}$ be their characteristic Herglotz functions. If

$$m_1(x) \le m_2(x) \quad (\forall x \in \mathbb{R}),$$

then

$$H_1(-s) \ge H_2(-s) \quad (\forall s > 0).$$

Proof

By Kotani's continuity theorem (Theorem B), it suffices to show the case where m_1 and m_2 are step functions. However, for such strings the assertion is already well known because, for such strings, we can compute the characteristic Herglotz functions explicitly using continued fractions (cf. [6, Examples 1.1, 1.3]). $\square$

Since the proofs of Theorems 3.1–3.3 are essentially the same, we prove the first one only.

Proof of Theorem 3.1
We first show that (3.1) implies (3.2). By (3.1), there exist $C_1, C_2 > 0$ and large $A > 0$ such that

$$\frac{C_1}{|x|\varphi^{-1}(|x|)} < m(x) < \frac{C_2}{|x|\varphi^{-1}(|x|)} \quad (\forall x < -A).$$

Now define $m_1, m_2 \in \mathcal{M}_{\mathrm{circ}}$ as follows:

$$m_1(x) = \min\Big\{m(x), \frac{C_1}{|x|\varphi^{-1}(|x|)}\Big\}$$

and

$$m_2(x) = \max\Big\{m(x), \frac{C_2}{|x|\varphi^{-1}(|x|)}\Big\}.$$

Then

$$m_1(x) \le m(x) \le m_2(x) \quad (\forall x \in \mathbb{R})$$

and

$$m_j(x) \sim \frac{C_j}{|x|\varphi^{-1}(|x|)} \quad (x \to -\infty,\ j = 1, 2).$$

Now let h_1 and h_2 correspond to m_1 and m_2, respectively. Then, by Proposition 3.1, it holds that

$$h_1(s) \ge h(s) \ge h_2(s) \quad (\forall s > 0).$$

Since we can apply Theorem 2.1 to m_1, m_2 to get the asymptotic behavior of h_1, h_2, we have (3.2).

We next show that, conversely, (3.2) implies (3.1). Suppose that

$$0 < C_1 = \liminf_{\lambda\to\infty} \frac{-h(\lambda)}{\varphi(\lambda)} \le \limsup_{\lambda\to\infty} \frac{-h(\lambda)}{\varphi(\lambda)} = C_2 < \infty. \tag{3.13}$$

If we change the variable λ in (3.13) by λs and then use the assumption that φ varies regularly with exponent $\alpha - 1$, we have, for every $s > 0$,

$$0 < C_1 s^{\alpha-1} = \liminf_{\lambda\to\infty} \frac{-h(\lambda s)}{\varphi(\lambda)} \le \limsup_{\lambda\to\infty} \frac{-h(\lambda s)}{\varphi(\lambda)} = C_2 s^{\alpha-1} < \infty. \tag{3.14}$$

Now let $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$ be an arbitrary sequence. Since (3.14) implies that $h(\lambda_n)/\varphi(\lambda_n)$ and $h(2\lambda_n)/\varphi(\lambda_n)$ are bounded, by Lemma A.2 in the appendix, we can choose a subsequence $\{\lambda_{n_j}\}_j$ such that

$$\lim_{j\to\infty} \frac{h(\lambda_{n_j} s)}{\varphi(\lambda_{n_j})} = -vs + H_\infty(-s) \quad (s > 0) \tag{3.15}$$

for some $v \ge 0$ and an $H_\infty \in \mathbb{H}$. By (3.14), we see that

$$0 < C_1 s^{\alpha-1} \le vs - H_\infty(-s) \le C_2 s^{\alpha-1} < \infty. \tag{3.16}$$

Divide by s, and let $s \to \infty$. Then since $\alpha < 2$, we see that, in fact, $v = 0$. Here we used $H_\infty(-s)/s \to 0$ ($s \to \infty$), which fact can be seen by the representation (1.5).

Since $h(\lambda s)/\varphi(\lambda)$ corresponds to $\lambda\varphi(\lambda)m(\varphi(\lambda)x)$ (see Lemma 2.1), (3.15) with $v=0$ implies

$$\lambda_{n_j}\varphi(\lambda_{n_j})m\big(\varphi(\lambda_{n_j})x\big) \Longrightarrow m_\infty(x), \tag{3.17}$$

where $m_\infty \in \mathcal{M}_{\mathrm{circ}}$ is the string corresponding to H_∞. Putting $x=-1$, we have

$$\begin{aligned} m_\infty(-1-0) &\le \liminf_{j\to\infty} \lambda_{n_j}\varphi(\lambda_{n_j})m\big(-\varphi(\lambda_{n_j})\big) \\ &\le \limsup_{j\to\infty} \lambda_{n_j}\varphi(\lambda_{n_j})m\big(-\varphi(\lambda_{n_j})\big) \le m_\infty(-1). \end{aligned}$$

Let us confirm that $m_\infty(-1-0)>0$ and $m_\infty(-1)<\infty$. In fact, it holds that $0<m_\infty(x)<\infty$ on $(-\infty,0)$. To see this fact it is sufficient to show that $\inf\{\mathrm{Supp}(dm_\infty)\}=-\infty$ and $\ell(m_\infty)=0$ or, equivalently, $H(-\infty)=-\infty$ and $H(-0)=0$ (cf. (1.4)). But these facts can easily be seen from (3.16) with $v=0$ and the condition $\alpha>1$.

Thus, we have seen that, for any given $0<\lambda_1<\lambda_2<\cdots\to\infty$, we can choose a subsequence such that

$$0<\liminf_{j\to\infty} \lambda_{n_j}\varphi(\lambda_{n_j})m\big(-\varphi(\lambda_{n_j})\big) \le \limsup_{j\to\infty} \lambda_{n_j}\varphi(\lambda_{n_j})m\big(-\varphi(\lambda_{n_j})\big)<\infty.$$

This is possible only when

$$0<\liminf_{\lambda\to\infty} \lambda\varphi(\lambda)m\big(-\varphi(\lambda)\big) \le \limsup_{\lambda\to\infty} \lambda\varphi(\lambda)m\big(-\varphi(\lambda)\big)<\infty.$$

Changing the variable $x=-\varphi(\lambda)$, we have (3.1).

We next see the equivalence of (3.3) and (3.2). If (3.3) holds, then by Theorem D in the appendix, it holds that

$$-h'(s)=\int_{-0}^{\infty}\frac{d\sigma(\xi)}{(s+\xi)^2} \asymp \frac{\varphi(s)}{s} \quad (s\to\infty). \tag{3.18}$$

Therefore, integrating both sides, we can deduce (3.2) (for the integration of the right-hand side, see the beginning of Section 2). Conversely, if (3.2) holds, then there exist $C_1, C_2>0$ such that

$$C_1\varphi(s) \le -h(s) \le C_2\varphi(s) \tag{3.19}$$

for all large $s>0$. Let

$$c=\frac{1}{3}\left(\frac{C_1}{C_2}\right)^{1/(\alpha-1)} \;(<1).$$

Then, by (3.19),

$$\limsup_{s\to\infty}\frac{-h(cs)}{-h(s)} \le \frac{C_2}{C_1}\limsup_{s\to\infty}\frac{\varphi(cs)}{\varphi(s)}=\frac{C_2}{C_1}c^{\alpha-1}=\frac{1}{3}.$$

Thus, for all large $s>0$, we may assume that

$$0<\frac{-h(cs)}{-h(s)} \le \frac{1}{2}.$$

Therefore, it holds that, as $s \to \infty$,

$$\frac{-h(s)}{s} \asymp \frac{-h(s)+h(cs)}{(1-c)s} = \int_{-0}^{\infty} \frac{d\sigma(\xi)}{(cs+\xi)(s+\xi)} \asymp \int_{-0}^{\infty} \frac{d\sigma(\xi)}{(s+\xi)^2}.$$

Thus, (3.2) also implies

$$\int_{-0}^{\infty} \frac{d\sigma(\xi)}{(s+\xi)^2} \asymp \frac{\varphi(s)}{s},$$

from which we can deduce (3.3) by Theorem D in the appendix. □

4. The case $\alpha = 1$

In this section we discuss the case $\alpha = 1$. It is rather hard to give a simple condition as in the other cases.

Let $L(x) > 0$ be a slowly varying function (at $+\infty$ or at $+0$). Since $f(x) = xL(x)$ is a regularly varying function with exponent 1, so is its asymptotic inverse $f^{-1}(x)$. Therefore, there exists a slowly varying function L^* such that $f^{-1}(x) \sim xL^*(x)$. L^*, which is unique up to asymptotic equivalence, is called the *de Bruijn conjugate* of L. Thus, L^* is a function such that $cL(c)L^*(cL(c)) \sim c$. This relation can be rewritten as $c^*L^*(c^*) \sim c = c^*/L(c)$ if we put $c^* = cL(c)$. Therefore, in general,

$$L^*(c^*) \sim 1/L(c). \tag{4.1}$$

If L varies slowly enough so that

$$L\big(xL(x)\big) \sim L(x), \tag{4.2}$$

then $L(c^*) \sim L(c)$ and, therefore, (4.1) implies that $L^*(c^*) \sim 1/L(c) \sim 1/L(c^*)$ and hence $L^*(x) \sim 1/L(x)$ (see [1, p. 78]). For example, if $L(x) = C$, then (4.2) is trivially satisfied, and hence, $L^*(x) = 1/C$. Similarly, if $L(x) \sim \log x$ as $x \to \infty$, then $L^*(x) \sim 1/\log x$, and if $L(x) \sim \log(1/x)$ as $x \to +0$, then $L^*(x) \sim 1/\log(1/x)$.

Our main result for this section is the following. The reader should note that we do not assume that either $L(c) \to \infty$ or $L(c) \to 0$.

THEOREM 4.1

Let $m \in \mathcal{M}_{\mathrm{circ}}$, and define N as in (2.17). Let $L(c) > 0$ $(c > 0)$ be a function varying slowly at ∞ (or $+0$), and let $L^*(c) > 0$ be its de Bruijn conjugate. Then the following conditions are equivalent as $c \to \infty$ (resp., $+0$):

$$cL(c)N\Big(\frac{x}{L(c)} + f(c)\Big) \longrightarrow e^x \quad (\forall x \in \mathbb{R}) \text{ for some } f(c), \tag{4.3}$$

$$\frac{1}{L^*(c)}\Big\{N^{-1}\Big(\frac{x}{c}\Big) - N^{-1}\Big(\frac{1}{c}\Big)\Big\} \longrightarrow \log x \quad (\forall x > 0), \tag{4.4}$$

$$L(c)\big(h(cs) - h(c)\big) \longrightarrow -\log s \quad (\forall s > 0), \tag{4.5}$$

$$\sigma(c) \sim \frac{c}{L(c)}. \tag{4.6}$$

Proof
We first prove the equivalence of (4.3) and (4.4). Considering the inverse functions, we see that (4.3) is equivalent to

$$L(c)\Big\{N^{-1}\Big(\frac{x}{cL(c)}\Big) - f(c)\Big\} \longrightarrow \log x \quad (\forall x \in \mathbb{R}). \tag{4.7}$$

If we change the variable $c^* = cL(c)$ as before and recall (4.1), then (4.7) may be written as

$$\frac{1}{L^*(c^*)}\Big\{N^{-1}\Big(\frac{x}{c^*}\Big) - g(c^*)\Big\} \longrightarrow \log x \quad (\forall x \in \mathbb{R})$$

for suitable g. But changing the notation of the parameter from c^* to c, we see that this implies (4.4). The converse is essentially the same.

Next, let us see the equivalence of (4.3) and (4.5). Note that (4.5) is equivalent to the existence of f such that

$$L(c)\big(h(cs) - f(c)\big) \to -(\log s + 2\gamma)\big(=h^{(1)}(s)\big). \tag{4.8}$$

Indeed, if (4.5) holds, then put $f(c) = h(c) + 2\gamma/L(c)$. The converse is trivial. Applying Lemma 2.1 with $a = 1/L(c)$ and $b = f(c)$, we see that the left-hand side of (4.8) corresponds to $cL(c)N(x/(L(c)) + f(c))$. Therefore, the equivalence of (4.8) and (4.3) follows from the continuity theorem (see Theorem B and Proposition 2.1).

It remains to show the equivalence of (4.5) and (4.6). By Theorem C in the appendix with $\rho = 2$ (and hence $C_{\alpha,\rho} = 1$), (4.6) is equivalent to

$$-h'(c)\Big(=\int_{-0}^{\infty} \frac{d\sigma(\xi)}{(c+\xi)^2}\Big) \sim \frac{1}{cL(c)}. \tag{4.9}$$

(The current $1/L$ corresponds to L of Theorem C.) Therefore, the assertion may be reduced to the following lemma. □

LEMMA 4.1
Let $L(c) > 0$ $(c > 0)$ be a function varying slowly at ∞ [or $+0$], and let $f(x)$ $(x \in \mathbb{R})$ be a function which is absolutely continuous with nonincreasing derivative f'. Then, the following conditions are equivalent as $x \to \infty$ [resp., $+0$]:

$$L(c)\big(f(cx) - f(c)\big) \longrightarrow \log x \quad (\forall x > 0), \tag{4.10}$$

$$cL(c)f'(c) \longrightarrow 1. \tag{4.11}$$

Proof
Since L varies slowly, (4.11) is equivalent to

$$cL(c)f'(cx) \longrightarrow \frac{1}{x} \quad (\forall x > 0). \tag{4.12}$$

As is well known, the convergence is locally uniform for $u > 0$. Therefore, this implies (4.10) as follows:

$$L(c)\big(f(cx) - f(c)\big) = \int_1^x cL(c)f'(cu)\,du \longrightarrow \int_1^x \frac{du}{u} = \log x. \tag{4.13}$$

Conversely, suppose that (4.10) (and hence (4.13)) holds. If we differentiate both sides of (4.13), then we have (4.11). To justify this argument, note that f' is nonincreasing, and then see the proof of Proposition 2.1. □

As we mentioned at the beginning of this section, it holds that $L^*(x) \sim 1/L(x)$ when (4.2) is satisfied. Therefore, we have the following.

COROLLARY 4.1

If, further, $L(xL(x)) \sim L(x)$, then the following conditions are equivalent as $c \to \infty$ (or $+0$):

$$L(c)\Big\{N^{-1}\Big(\frac{x}{c}\Big) - N^{-1}\Big(\frac{1}{c}\Big)\Big\} \longrightarrow \log x \quad (\forall x > 0), \tag{4.14}$$

$$L(c)\big(h(cs) - h(c)\big) \longrightarrow -\log s \quad (\forall s > 0), \tag{4.15}$$

$$\sigma(c) \sim \frac{c}{L(c)}. \tag{4.16}$$

EXAMPLE 4.1

Let $m(x) \sim Ax^\gamma e^{Bx}$ $(x \to \infty)$, where $A, B > 0$ and $\gamma \in \mathbb{R}$. Then $N(x) \sim (A/B^2) \times x^\gamma e^{Bx}$. In other words,

$$\log N(x) = Bx + \gamma \log x + \log(A/B^2) + o(1) \quad (x \to +\infty).$$

Changing the variable $x = N^{-1}(u)$, we see that

$$\log u = BN^{-1}(u) + \gamma \log N^{-1}(u) + \log(A/B^2) + o(1) \quad (u \to +\infty),$$

and, hence, as $c \to +0$,

$$B\Big\{N^{-1}\Big(\frac{x}{c}\Big) - N^{-1}\Big(\frac{1}{c}\Big)\Big\} = \log x - \gamma \log \frac{N^{-1}(x/c)}{N^{-1}(1/c)} + o(1).$$

Since N^{-1} varies slowly, the second term of the right-hand side is $o(1)$. Thus, (4.14) holds with $L(x) = B$. Therefore,

$$\sigma(c) \sim \frac{c}{B} \quad (c \to +0).$$

EXAMPLE 4.2

Let $m(x) \sim A|x|^\gamma e^{Bx}$ $(x \to -\infty)$, where $A, B > 0$ and $\gamma \in \mathbb{R}$. (This does not happen in Krein's strings but is possible for Kotani's.) Then, as in Example 4.1, we have

$$\sigma(c) \sim \frac{c}{B} \quad (c \to \infty).$$

EXAMPLE 4.3

Let $m(x) \sim Ax^\gamma e^{\sqrt{Bx}}$ $(x \to \infty)$, where $A, B > 0$ and $\gamma \in \mathbb{R}$. Then $N(x) \sim 4A \times x^{\gamma+1} e^{\sqrt{Bx}}$ $(x \to \infty)$, and therefore,

$$\log N(x) = \sqrt{Bx} + (\gamma + 1) \log x + C + o(1) \quad (x \to +\infty).$$

Changing the variable, we see that

$$\sqrt{BN^{-1}(u)} = \log u - (\gamma+1)\log N^{-1}(u) - C + o(1) \quad (u \to +\infty). \tag{4.17}$$

Since $N^{-1}(u)$ varies slowly, it is easy to see that $\log N^{-1}(u) = o(\log u)$. (A slowly varying function is dominated by u^ϵ for all $\epsilon > 0$.) Therefore, (4.17) implies

$$\sqrt{BN^{-1}(u)} \sim \log u \quad (u \to +\infty) \tag{4.18}$$

as well as

$$\sqrt{BN^{-1}\Big(\frac{x}{c}\Big)} - \sqrt{BN^{-1}\Big(\frac{1}{c}\Big)} \tag{4.19}$$
$$= \log x - (\gamma+1)\log\Big\{N^{-1}\Big(\frac{x}{c}\Big)\Big/N^{-1}\Big(\frac{1}{c}\Big)\Big\} + o(1) \to \log x.$$

Here, we used the fact that N^{-1} varies slowly. Considering $a-b$ as $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$, we have the following by (4.18) and (4.19):

$$\frac{B}{2\log(1/c)}\Big\{N^{-1}\Big(\frac{x}{c}\Big) - N^{-1}\Big(\frac{1}{c}\Big)\Big\} \longrightarrow \log x \quad (c \to +0).$$

Therefore,

$$\sigma(c) \sim \frac{2}{B} c\log(1/c) \quad (c \to +0).$$

REMARK 2

Functions N satisfying (4.4) appear in de Haan theory (see [1, Section 3.10]).

A. Appendix

Tauberian theorems for Stieltjes transforms are well known. They are due to Hardy and Littlewood, Karamata, de Haan, and others. The theory and results are found in the monograph [1], which is, however, not necessarily sufficient enough for our present use. Therefore, we restate some of the results in our own way with direct proofs. Let $\sigma : \mathbb{R} \to [0,\infty)$ $(n = 1,2,\ldots)$ be a nondecreasing right-continuous function, and set $\sigma(-0) = 0$ for the sake of convenience. Then its *generalized Stieltjes transform* of order ρ (>0) is defined by

$$S_\rho(\sigma; s) = \int_{-0}^{\infty} \frac{d\sigma(\xi)}{(s+\xi)^\rho} \quad (s > 0) \tag{A.1}$$

provided that the right-hand side exists. Of course, $S_\rho(\sigma; s)$ uniquely determines $\sigma(\cdot)$. A typical example is, for $0 \le \alpha < \rho$,

$$S_\rho(\xi^\alpha; s) = \int_0^\infty \frac{d\xi^\alpha}{(s+\xi)^\rho} = C_{\alpha,\rho} s^{\alpha-\rho} \quad (s > 0), \tag{A.2}$$

where

$$C_{\alpha,\rho} = \int_0^\infty \frac{d\xi^\alpha}{(1+\xi)^\rho} = \frac{\Gamma(\alpha+1)\Gamma(\rho-\alpha)}{\Gamma(\rho)}.$$

For the relationship between the asymptotic behavior of $\sigma(\cdot)$ and that of $S_\rho(\sigma;\cdot)$, we have the following.

THEOREM C (CF. [1, P. 40])

Let $0 \le \alpha < \rho$*, and let* L *be a slowly varying function at* $\infty[+0]$*. Then*

$$\sigma(\xi) \sim \xi^{\alpha} L(\xi) \quad (\xi \to \infty[+0]) \tag{A.3}$$

if and only if

$$S_\rho(\sigma; s) \sim S_\rho(\xi^{\alpha}; s) L(s) \quad (s \to \infty[+0]), \tag{A.4}$$

that is,

$$S_\rho(\sigma; s) \sim C_{\alpha,\rho} s^{\alpha-\rho} L(s) \quad (s \to \infty[+0]). \tag{A.5}$$

THEOREM D (CF. [8, APPENDIX])

Let $0 \le \alpha < \rho$*, and let* L *be a slowly varying function at* $\infty[+0]$*. Then*

$$\sigma(\xi) \asymp \xi^{\alpha} L(\xi) \quad (\xi \to \infty[+0]) \tag{A.6}$$

if and only if

$$S_\rho(\sigma; s) \asymp s^{\alpha-\rho} L(s) \quad (s \to \infty[+0]). \tag{A.7}$$

In order to prove Theorems C and D, we prepare a few lemmas. Let

$$\mathbb{H}_\rho = \Big\{ f(\lambda) = v + \int_{-0}^{\infty} \frac{d\sigma(\xi)}{(\lambda+\xi)^\rho}; v \ge 0, \int_{-0}^{\infty} \frac{d\sigma(\xi)}{(1+\xi)^\rho} < \infty \Big\}.$$

LEMMA A.1

Let $\rho > 0$*, and let* $f_1, f_2, \ldots \in \mathbb{H}_\rho$*. That is,*

$$f_n(\lambda) = v_n + S_\rho(\sigma_n; \lambda) = v_n + \int_{-0}^{\infty} \frac{d\sigma_n(\xi)}{(\lambda+\xi)^\rho} \quad (v_n \ge 0).$$

If $\{f_n(c)\}_n$ *is bounded for some* $c > 0$*, then we can choose a subsequence* $1 \le n_1 < n_2 < \cdots$ *and an* $f \in \mathbb{H}_\rho$ *such that, for all* $\lambda \in \mathbb{C}\backslash[0,\infty)$*,*

$$f(\lambda) = \lim_{j\to\infty} f_{n_j}(\lambda).$$

Furthermore, it also holds that

$$\sigma_{n_j}(\xi) \Longrightarrow \sigma(\xi) \quad (n \to \infty),$$

where σ *corresponds to* f*; that is,*

$$f(\lambda) = v + S_\rho(\sigma; \lambda) = v + \int_{-0}^{\infty} \frac{d\sigma(\xi)}{(\lambda+\xi)^\rho}.$$

Proof

Let us consider the *finite* Borel measures on $[0,\infty]$ defined by

$$\Xi_n(d\xi) = 1_{[0,\infty)}(\xi) \frac{d\sigma_n(\xi)}{(c+\xi)^\rho} + v_n \delta_\infty(d\xi) \quad (n = 1, 2, \ldots),$$

where $\delta_\infty(d\xi)$ denotes the unit mass at ∞. Then f_n can be written as

$$f_n(\lambda) = \int_{[0,\infty]} \frac{(c+\xi)^\rho}{(\lambda+\xi)^\rho} \Xi_n(d\xi).$$

Here notice that the integrand is bounded, continuous in $\xi \in [0,\infty]$. With this representation, our assertion of the lemma is now reduced to the relative compactness of the family of finite measures $\{\Xi_n(d\xi)\}_n$. However, since $\Xi_n([0,\infty])$ $(=f_n(c))$ is bounded by assumption, the rest of the proof is routine. □

LEMMA A.2

Let $H_1, H_2, \ldots \in \mathbb{H}$; that is,

$$H_n(\lambda) = a_n + \int_{-0}^{\infty} \Big(\frac{1}{\xi-\lambda} - \frac{\xi}{\xi^2+1}\Big)\, d\sigma_n(\xi). \tag{A.8}$$

If both $\{H_n(-1)\}_n$ and $\{H_n(-2)\}_n$ are bounded, then we can choose a subsequence $1 \le n_1 < n_2 < \cdots$, an $H \in \mathbb{H}_\rho$, and a $v \ge 0$ such that, for all $\lambda \in \mathbb{C}\backslash[0,\infty)$,

$$\lim_{j\to\infty} H_{n_j}(\lambda) = -v\lambda + H(\lambda).$$

Proof

Let

$$f_n(s) = \frac{H_n(-s) - H_n(-1)}{s+1} = \int_{-0}^{\infty} \frac{1}{\xi+s}\,\frac{d\sigma_n(\xi)}{\xi+1}.$$

Then f_n is the usual Stieltjes transform of $\tau_n(d\xi) = d\sigma_n(\xi)/(\xi+1)$. Since $\{H_n(-1)\}_n$ and $\{-H_n(-2)\}_n$ are bounded, so is $\{f_n(1)\}_n$. Therefore, we can apply Lemma A.1 with $\rho = 1$. We can choose a subsequence such that

$$\frac{H_{n_j}(\lambda) - H_{n_j}(-1)}{-\lambda+1} \to v + \int_{-0}^{\infty} \frac{1}{\xi-\lambda}\,\frac{d\sigma(\xi)}{\xi+1} \tag{A.9}$$

for some σ, for which we define

$$H_\infty(\lambda) = \int_{-0}^{\infty} \Big(\frac{1}{\xi-\lambda} - \frac{\xi}{\xi^2+1}\Big)\, d\sigma(\xi) \quad (\in \mathbb{H}),$$

so that

$$\frac{H_{n_j}(\lambda) - H_{n_j}(-1)}{-\lambda+1} \to v + \frac{H_\infty(\lambda) - H_\infty(-1)}{-\lambda+1}. \tag{A.10}$$

Choosing a subsequence again if necessary, we can assume that $\{H_{n_j}(-1)\}_j$ also converges to some $a \in \mathbb{R}$. Then (A.10) implies

$$H_{n_j}(\lambda) \to -v\lambda + H(\lambda),$$

where $H(\lambda) = (v + a - H_\infty(-1)) + H_\infty(\lambda)$. □

LEMMA A.3

Let $\rho > 0$, and let $\sigma_n(\cdot)$ $(n = 1,2,\ldots)$, $\sigma_\infty(\cdot)$ be functions such that $S_\rho(\sigma_n; s)$ and $S_\rho(\sigma_\infty; s)$ exist. Then

$$S_\rho(\sigma_n; s) \to S_\rho(\sigma_\infty; s) \quad (\forall s > 0) \tag{A.11}$$

if and only if

$$\sigma_n(\xi) \Longrightarrow \sigma_\infty(\xi) \qquad \textit{and} \qquad S_\rho(\sigma_n; 1) \to S_\rho(\sigma_\infty; 1). \tag{A.12}$$

Proof

Suppose that (A.12) holds. Then, by Lemma A.1 and by the latter half of (A.12), choosing a subsequence if necessary, we have

$$S_\rho(\sigma_n; s) \to v + S_\rho(\sigma; s) \quad (\forall s > 0) \tag{A.13}$$

for some $v \geq 0$ and σ. Since Lemma A.1 insists that $\sigma_n(\xi) \Longrightarrow \sigma(\xi)$ as well, we have in fact $\sigma = \sigma_\infty$ and, hence, $S_\rho(\sigma_n; s) \to v + S_\rho(\sigma_\infty; s)$, $s > 0$. Thus, it remains only to show that $v = 0$. However, this follows immediately from the latter half of assumption (A.12). The converse can be proved in a similar way. □

Proof of Theorem C

We first prove for the case $c \to +0$. For every $c > 0$, define

$$\sigma_c(\xi) = \frac{\sigma(c\xi)}{c^\alpha L(c)}.$$

Then it is easy to see that (A.3) is equivalent to

$$\sigma_c(\xi) \to \xi^\alpha \quad (\forall \xi > 0). \tag{A.14}$$

Also, (A.4) can be rewritten as

$$S_\rho\big(\sigma_c(\xi); 1\big) \to S_\rho(\xi^\alpha; 1). \tag{A.15}$$

Change the variables (ξ, c) by $(\xi/s, s\xi)$ in both sides, noting that L varies slowly; then we see that this is also equivalent to

$$S_\rho\big(\sigma_c(\xi); s\big) \to S_\rho(\xi^\alpha; s) \quad (\forall s > 0). \tag{A.16}$$

Thus, the assertion of Theorem C may be reduced to the equivalence of (A.14) and (A.16). This kind of problem is already discussed in Lemma A.3. We directly have that (A.16) implies (A.14). Conversely, let us see that (A.16) follows from (A.14). However, as we have seen already, it suffices to show (A.15), which can also be rewritten as

$$\lim_{c \to +0} \int_0^\infty \frac{\sigma_c(\xi)}{(1+\xi)^{\rho+1}}\, d\xi = \int_0^\infty \frac{\xi^\alpha}{(1+\xi)^{\rho+1}}\, d\xi. \tag{A.17}$$

To prove (A.17) under the condition (A.14), choose $0 < \epsilon < 1$ such that $0 < \alpha + \epsilon < \rho$. Since σ varies regularly with exponent α, by Potter's theorem (see [1, p. 25]), there exists $C > 0$ and $A > 0$ such that

$$\frac{\sigma(x)}{\sigma(y)} \leq C\Big\{\Big(\frac{x}{y}\Big)^{\alpha+\epsilon} + \Big(\frac{x}{y}\Big)^{\alpha-\epsilon}\Big\}, \quad x, y \in (0, A].$$

Therefore,

$$\sigma_c(\xi) = \frac{\sigma(c\xi)}{\sigma(c)} \frac{\sigma(c)}{c^\alpha L(c)} \leq C_1(\xi^{\alpha+\epsilon} + \xi^{\alpha-\epsilon}), \quad c, c\xi \leq A, \tag{A.18}$$

and, hence, by the dominated convergence theorem, we obtain

$$\lim_{c \to +0} \int_0^{A/c} \frac{\sigma_c(\xi)}{(1+\xi)^{\rho+1}}\, d\xi = \int_0^\infty \frac{\xi^\alpha}{(1+\xi)^{\rho+1}}\, d\xi. \tag{A.19}$$

Therefore, it remains to show that

$$\lim_{c\to+0}\int_{A/c}^{\infty}\frac{\sigma_c(\xi)}{(1+\xi)^{\rho+1}}\,d\xi=0. \tag{A.20}$$

However,

$$\int_{A/c}^{\infty}\frac{\sigma_c(\xi)}{\xi^{\rho+1}}\,d\xi=\int_{A/c}^{\infty}\frac{\sigma(c\xi)}{\xi^{\rho+1}}\frac{d\xi}{c^{\alpha}L(c)}=\frac{c^{\rho-\alpha}}{L(c)}\int_{A}^{\infty}\frac{\sigma(\xi)}{\xi^{\rho+1}}\,d\xi\to 0\quad(c\to+0)$$

because $\rho>\alpha$. We next treat the case $c\to\infty$. The idea is the same except that (A.19) and (A.20) should be replaced by

$$\lim_{c\to\infty}\int_{A/c}^{\infty}\frac{\sigma_c(\xi)}{(1+\xi)^{\rho+1}}\,d\xi=\int_{0}^{\infty}\frac{\xi^{\alpha}}{(1+\xi)^{\rho+1}}\,d\xi \tag{A.21}$$

and

$$\lim_{c\to\infty}\int_{0}^{A/c}\frac{\sigma_c(\xi)}{(1+\xi)^{\rho+1}}\,d\xi=0, \tag{A.22}$$

respectively. However, the proof of (A.21) is the same as (A.19), while (A.22) is almost trivial (for another proof, see [1, p. 40]). □

Proof of Theorem D

To begin, note that

$$\int_{-0}^{\infty}\frac{d\sigma(\xi)}{(s+\xi)^{\rho}}=\rho\int_{-0}^{\infty}\frac{\sigma(\xi)}{(s+\xi)^{\rho+1}}\,d\xi.$$

Therefore, if $C_1\tau(\xi)\le\sigma(\xi)\le C_2\tau(\xi)$, then

$$C_1S_{\rho}(\tau;s)\le S_{\rho}(\sigma;s)\le C_2S_{\rho}(\tau;s).$$

Thus, the Abelian implication is reduced to Theorem C by choosing a suitable regular varying function $\tau(\xi)$. The proof of the Tauberian implication is essentially the same as in the proof of Theorem C. Let $\sigma_c(\xi)=\sigma(c\xi)/(c^{\rho}L(c))$, as before. Then instead of (A.15), we have

$$C_1\le\liminf_{c\to0[\infty]}S_{\rho}(\sigma_c;1)\le\limsup_{c\to0[\infty]}S_{\rho}(\sigma_c;1)\le C_2 \tag{A.23}$$

for some $C_1,C_2>0$. Since L varies slowly, this implies, instead of (A.16), for every $s>0$,

$$C_1s^{\alpha-\rho}\le\liminf_{c\to0[\infty]}S_{\rho}\big(\sigma_c(\xi);s\big)\le\limsup_{c\to0[\infty]}S_{\rho}\big(\sigma_c(\xi);s\big)\le C_2s^{\alpha-\rho}. \tag{A.24}$$

Therefore, for any given $c_1>c_2>\cdots\to0$ (or $0<c_1<c_2<\cdots\to\infty$) we can choose a subsequence $\{c_{n_j}\}$ and a function $\sigma_{\infty}(\cdot)$ such that

$$S_{\rho}(\sigma_{c_{n_j}};s)\to v+S_{\rho}(\sigma_{\infty};s),\quad \sigma_{c_{n_j}}(\xi)\Longrightarrow\sigma_{\infty}(\xi),$$

for some $v\ge0$. (In fact, $v=0$ by (A.24).) The latter implies

$$\lim_{j\to\infty}\frac{\sigma(c_{n_j})}{c_{n_j}^{\alpha}L(c_{n_j})}=\sigma_{\infty}(1).$$

It remains to show that $\sigma_\infty(1-) > 0$ ($\sigma_\infty(1) < \infty$ holds in general). If $\sigma_\infty(1-) = 0$, then $S_\rho(\sigma_\infty(\cdot); \infty) < \infty$, which contradicts the first inequality in (A.24). Since $\{c_n\}$ was arbitrary, we conclude that

$$0 < \liminf_{c\to 0[\infty]} \frac{\sigma(c)}{c^\alpha L(c)} \le \limsup_{c\to 0[\infty]} \frac{\sigma(c)}{c^\alpha L(c)} < \infty.$$

□

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