Modulus of a rational map into a commutative algebraic group

Kazuya Kato and Henrik Russell

Abstract For a rational map $\phi: X \to G$ from a normal algebraic variety X to a commutative algebraic group G, we define the modulus of ϕ as an effective divisor on X. We study the properties of the modulus. This work generalizes the known theories for curves X to higher-dimensional varieties.

1. Introduction

Let X be a normal algebraic variety over a perfect field k, let G be a commutative algebraic group over k, and let $\varphi: X \to G$ be a rational map. In this article, we give a geometric definition of a modulus of φ as an effective divisor $\sum_v m(v)v$ on X. Here v ranges over all codimension 1 points of X at which φ is not defined as a morphism and $m(v)$ is a certain integer ≥ 1. In the curve case, this definition coincides with Serre's definition (see [Se]), which is based on the theory of local symbols. The case when k is of characteristic zero was explained in our previous article (see [KR, §5]). We discuss the positive characteristic case in this article. We study properties of this modulus.

An alternative way to define the modulus of φ is by using K-theoretic idèle class groups developed by Kato and Saito in [KS], as was done in [Ön] for surfaces. The coincidence of these two approaches follows from Proposition 7.5.

This article is related to the theory of generalized Albanese varieties developed by Russell in [Ru1] and [Ru2]. In particular, the following fact is proved in [Ru2] by using this article. If X is proper smooth and if Y is an effective divisor on X, φ factors through the generalized Albanese variety $\mathrm{Alb}(X,Y)$ of X with modulus Y if and only if (modulus of φ) $\leq Y$. In the case when k is of characteristic zero, this was proved in [KR, §5] as a consequence of the theory in [Ru1].

The definition of the modulus of φ is given in Section 3 assuming Theorem 3.3. The proof of this theorem is completed in Section 5. In Sections 6 and 7, we consider the relation of modulus with local symbols. In Section 8, we consider the relation of modulus with field extensions.

Kyoto Journal of Mathematics, Vol. 50, No. 3 (2010), 607–622
DOI 10.1215/0023608X-2010-006,
Received October 5, 2009. Revised March 5, 2010. Accepted March 9, 2009.
Mathematics Subject Classification: Primary 14L10; Secondary 11S15, 14E05.

2. Filtrations on additive groups and on Witt vector groups

Let K be a discrete valuation field, and let O_K be the valuation ring of K.

2.1

For $m \geq 0$, we define

$$\mathrm{fil}_m(K) = \{ f \in K \mid v_K(f) \geq -m \}.$$

Here v_K denotes the normalized valuation of K.

2.2

Let p be a prime number, and assume that K is of characteristic p. Let $W_n(K)$ be the set of Witt vectors of length n with entries in K. For $m \geq 0$, define

$$\begin{aligned} &\mathrm{fil}_m W_n(K) \\ &\quad = \{ (f_{n-1}, \ldots, f_0) \mid f_j \in K, p^j v_K(f_j) \geq -m \, (0 \leq j \leq n-1) \} \subset W_n(K). \end{aligned}$$

This filtration appeared in the article [Br] of Brylinski. In the case $n = 1$, this filtration coincides with the filtration on $K = W_1(K)$ in Section 2.1.

Let $F : W_n(K) \to W_n(K)$ be the map $(a_{n-1}, \ldots, a_0) \mapsto (a_{n-1}^p, \ldots, a_0^p)$. For $m \in \mathbb{N}$, let

$$\mathrm{fil}_m^F W_n(K) = \sum_{j \geq 0} F^j \big(\mathrm{fil}_m W_n(K) \big) \subset W_n(K).$$

We have $\mathrm{fil}_0 W_n(K) = \mathrm{fil}_0^F W_n(K) = W_n(O_K)$.

If we regard $W_n(K)$ as a subgroup of $W_{n+1}(K)$ via $V : W_n(K) \to W_{n+1}(K)$; $(a_{n-1}, \ldots, a_0) \mapsto (0, a_{n-1}, \ldots, a_0)$, we have

$$\mathrm{fil}_m W_{n+1}(K) \cap W_n(K) = \mathrm{fil}_m W_n(K), \qquad \mathrm{fil}_m^F W_{n+1}(K) \cap W_n(K) = \mathrm{fil}_m^F W_n(K).$$

3. Modulus

3.1

Let X be a normal algebraic variety over a perfect field k. We regard X as a scheme. Let G be a commutative smooth connected algebraic group over k, and let $\varphi : X \to G$ be a rational map. We define the modulus

$$\mathrm{mod}(\varphi) = \sum_v \mathrm{mod}_v(\varphi) v$$

of φ as an effective divisor on X, where v ranges over all points of X of codimension one and $\mathrm{mod}_v(\varphi) \in \mathbb{N}$ is as follows.

The case when k is of characteristic zero is already explained in [KR]. (In [KR], we assumed that X is proper smooth over k, but this condition is not used in the definition.)

3.2

First, assume that k is algebraically closed.

Let $0 \to L \to G \to A \to 0$ be the canonical exact sequence of commutative algebraic groups, where A is an abelian variety and L is an affine smooth connected algebraic group. Write $L = L_m \times L_u$, where L_m is multiplicative and L_u is unipotent. Then since k is algebraically closed, $L_m \cong (\mathbb{G}_m)^t$ for some $t \geq 0$. If k is of characteristic zero, $L_u \cong (\mathbb{G}_a)^s$ for some $s \geq 0$. Fix such an isomorphism. If k is of characteristic $p > 0$, L_u is embedded into a finite direct sum $\bigoplus_{i=1}^s W_{n_i}$ of Witt vector groups for some $s \geq 0$ and for some $n_i \geq 1$. Fix such an embedding.

Let K be the function field of X. Since $H^1_{\mathrm{fppf}}(\mathrm{Spec}(\mathcal{O}_{X,x}), \mathbb{G}_m) = 0$ and $H^1_{\mathrm{fppf}}(\mathrm{Spec}(\mathcal{O}_{X,x}), L_u) = 0$ for any point x of X, we have exact sequences

$$0 \to L(K) \to G(K) \to A(K) \to 0, \qquad 0 \to L(\mathcal{O}_{X,x}) \to G(\mathcal{O}_{X,x}) \to A(\mathcal{O}_{X,x}) \to 0.$$

If v is a point of X of codimension one, since A is proper and $\mathcal{O}_{X,v}$ is a discrete valuation ring, we have $A(K) = A(\mathcal{O}_{X,v})$. Hence the canonical map $L(K)/L(\mathcal{O}_{X,v}) \to G(K)/G(\mathcal{O}_{X,v})$ is bijective. Take an element $l \in L(K)$ whose image in $G(K)/G(\mathcal{O}_{X,v})$ coincides with the class of $\varphi \in G(K)$. In the case when k is of characteristic zero, let $(l_i)_{1 \leq i \leq s}$ be the image of l in $(\mathbb{G}_a(K))^s = K^s$ under $L \to L_u \cong (\mathbb{G}_a)^s$. In the case when k is of characteristic $p > 0$, let $(l_i)_{1 \leq i \leq s}$ be the image of l in $\bigoplus_{i=1}^s W_{n_i}(K)$ under $L \to L_u \overset{\subset}{\to} \bigoplus_{i=1}^s W_{n_i}$.

If $\varphi \in G(\mathcal{O}_{X,v})$, then we define $\mathrm{mod}_v(\varphi) = 0$. If $\varphi \notin G(\mathcal{O}_{X,v})$ and if the characteristic of k is zero (resp., $p > 0$), then we define

$$\mathrm{mod}_v(\varphi) = 1 + \max\{r(l_i) \mid 1 \leq i \leq n\}, \quad \text{where for } f \in K \text{ (resp., } W_{n_i}(K)),$$
$$r(f) = \min\{r \in \mathbb{N} \mid f \in \mathrm{fil}_r(K)\} \quad (\text{resp.}, r(f) = \min\{r \in \mathbb{N} \mid f \in \mathrm{fil}^F_r W_{n_i}(K)\}).$$

In the case when k is of characteristic zero, it is easy to see that $\mathrm{mod}_v(\varphi)$ is independent of the choice of the isomorphism $L_u \cong (\mathbb{G}_a)^s$. In the case when k is of characteristic $p > 0$, however, it is not so easy to prove

THEOREM 3.3

Let the notation be as above, and assume that k is of characteristic $p > 0$. Then $\mathrm{mod}_v(\varphi)$ is independent of the choice of the embedding $L_u \to \bigoplus_{i=1}^s W_{n_i}$.

This theorem is proved in Section 5.

3.4

Now we do not assume k is algebraically closed. Then by Galois descent for $\mathrm{Gal}(\bar{k}/k)$, we see that there is a unique effective divisor $\mathrm{mod}(\varphi)$ on X whose pullback to $X \otimes_k \bar{k}$ is the modulus of the rational map $X \otimes_k \bar{k} \to G \otimes_k \bar{k}$.

4. Quotients of the filtrations

Let p be a prime number, and let K be a discrete valuation field of characteristic p with residue field κ.

We study $\mathrm{fil}^F_m W_n(K)/\mathrm{fil}^F_{[m/p]} W_n(K)$ and its quotient $\mathrm{fil}^F_m W_n(K)/\mathrm{fil}^F_{m-1} W_n(K)$, for $m \geq 1$. Here for $x \in \mathbb{R}$, $[x]$ denotes $\max\{a \in \mathbb{Z} \mid a \leq x\}$ as usual.

PROPOSITION 4.1

(1) *The following sequence is exact.*

$$0 \to \bigoplus_{j\geq 0} \mathrm{fil}_{[m/p]} W_n(K) \xrightarrow{h} \bigoplus_{j\geq 0} \mathrm{fil}_m W_n(K) \to \mathrm{fil}_m^F W_n(K) \to 0,$$

where the third arrow is $(x_j)_j \mapsto \sum_j F^j(x_j)$, *and* h *is the map* $(x_j)_j \mapsto (y_j)_j$ *with* $y_0 = F(x_0)$, $y_j = F(x_j) - x_{j-1}$ *for* $j \geq 1$.

(2) *We have an isomorphism*

$$\bigoplus_{i\geq 0} \frac{\mathrm{fil}_m W_n(K)}{\mathrm{fil}_{[m/p]} W_n(K) + F(\mathrm{fil}_{[m/p]} W_n(K))} \xrightarrow{\simeq} \frac{\mathrm{fil}_m^F W_n(K)}{\mathrm{fil}_{[m/p]}^F W_n(K)};$$

$$(x_i)_i \mapsto \sum_i F^i(x_i).$$

Proof

(1) We prove that for each $i \geq 0$, the sequence

$$0 \to \bigoplus_{j=0}^{i-1} \mathrm{fil}_{[m/p]} W_n(K) \xrightarrow{h_i} \bigoplus_{j=0}^{i} \mathrm{fil}_m W_n(K) \to \sum_{j=0}^{i} F^j \mathrm{fil}_m W_n(K) \to 0$$

is exact, where h_i is the restriction of h. We prove this by induction on i. The case $i = 0$ is trivial. Assume that $i \geq 1$. The nontrivial point is the exactness at the central term. Let $x = (x_j)_j$ be an element of $\bigoplus_{j=0}^{i} \mathrm{fil}_m W_n(K)$ such that $\sum_j F^j(x_j) = 0$. We prove that x belongs to the image of h_i. We have $F^i(x_i) = -\sum_{j=0}^{i-1} F^j(x_j) \in \mathrm{fil}_{mp^{i-1}} W_n(K)$. Hence $x_i \in \mathrm{fil}_{[m/p]} W_n(K)$. Let $y = (y_j)_j$ be the element of $\bigoplus_{j=0}^{i-1} \mathrm{fil}_{[m/p]} W_n(K)$ defined by $y_{i-1} = x_i$ and $y_j = 0$ for $0 \leq j < i-1$, and let $x' = x + h_i(y)$. Then $x' \in \bigoplus_{j=0}^{i-1} \mathrm{fil}_m W_n(K)$. By induction on i, $(x'_j)_j$ is in the image of h_i.

(2) This follows from (1) easily.

□

4.2

For a commutative ring R, let $\Omega_R^1 = \Omega_{R/\mathbb{Z}}^1$ be the differential module of R. Then for any commutative ring R over $\mathbb{F}_p$, there is a homomorphism

$$\delta : W_n(R) \to \Omega_R^1;\ (a_{n-1}, \ldots, a_0) \mapsto \sum_i a_i^{p^i - 1}\, da_i.$$

4.3

Let $\Omega_{O_K}^1(\log)$ be the differential module of O_K with log poles defined by

$$\Omega_{O_K}^1(\log) = \big(\Omega_{O_K}^1 \oplus (O_K \otimes_{\mathbb{Z}} K^\times)\big)/N,$$

where N is the O_K-submodule of $\Omega_{O_K}^1 \oplus (O_K \otimes_{\mathbb{Z}} K^\times)$ generated by $(da, -a \otimes a)$ for $a \in O_K - \{0\}$. We have canonical homomorphisms $\Omega_{O_K}^1 \to \Omega_{O_K}^1(\log)$ and $K^\times \to \Omega_{O_K}(\log)$; $a \mapsto \mathrm{class}(0, 1 \otimes a)$. We denote the latter map by $d\log$. If the condition

(i) the completion of K is separable over K

is satisfied, then for a lifting $(b_i)_i$ of a p-base of κ to O_K and for a prime element t of K, $\Omega^1_{O_K}$ (resp., $\Omega^1_{O_K}(\log)$) is a free O_K-module with base $(db_i)_i$ and dt (resp., $(db_i)_i$ and $d\log(t)$).

(Condition (i) is equivalent to the condition that $(b_i)_i$ and t form a p-base of K. Recall that for a field F of characteristic p, a family $(b_i)_{i\in I}$ of elements of F is called a p-base of F if F is generated over F^p by b_i $(i \in I)$ as a field and for any subset J of I such that $J \neq I$, F is not generated over F^p by b_j $(j \in J)$. Recall also that if $(b_i)_i$ is a p-base of F, $(db_i)_i$ is a base of the F-module Ω^1_F.)

Without assumption (i), for any integer $j \geq 0$, $\Omega^1_{O_K} \otimes_{O_K} O_K/m_K^j$ (resp., $\Omega^1_{O_K}(\log) \otimes_{O_K} O_K/m_K^j$) is a free O_K/m_K^j-module with base $(db_i)_i$ and dt (resp., $(db_i)_i$ and $d\log(t)$). This is because this group is invariant under the completion of K, and the condition (i) is satisfied of course if K is complete.

PROPOSITION 4.4

For $m \geq 1$, the homomorphism δ (see Section 4.2) for K induces an injective homomorphism

$$\delta_m : \frac{\mathrm{fil}_m W_n(K)}{\mathrm{fil}_{[m/p]} W_n(K) + F(\mathrm{fil}_{[m/p]} W_n(K))} \to \Omega^1_{O_K}(\log) \otimes_{O_K} \frac{m_K^{-m}}{m_K^{-[m/p]}}.$$

Proof

The problem is the injectivity. By induction on m, it is reduced to the injectivity of

$$A := \frac{\mathrm{fil}_m W_n(K)}{\mathrm{fil}_{m-1} W_n(K) + F(\mathrm{fil}_{[m/p]} W_n(K))} \to \Omega^1_{O_K}(\log) \otimes_{O_K} \frac{m_K^{-m}}{m_K^{1-m}}.$$

We assume that $K = \kappa((t))$ without a loss of generality. Note that

$$\Omega^1_{O_K}(\log) \otimes_{O_K} m_K^{-m}/m_K^{1-m} \cong \Omega^1_\kappa \oplus \kappa,$$

$$adb \otimes t^{-m} \leftrightarrow (adb, 0) \quad (a, b \in \kappa), \qquad ad\log(t) \otimes t^{-m} \leftrightarrow (0, a) \quad (a \in \kappa).$$

We define an increasing filtration $(A_i)_{-1\leq i\leq n-1}$ on A as follows. For $-1 \leq i \leq n-1$, let A_i be the image of $\mathrm{fil}_m W_{i+1}(K)$ in A under $V^{n-1-i} : W_{i+1}(K) \to W_n(K)$. Then as is easily seen, $A_i = A$ if $i \geq \mathrm{ord}_p(m)$, $A_{-1} = 0$, and for $0 \leq i \leq r := \min(\mathrm{ord}_p(m), n-1)$, we have an isomorphism

$$\kappa\,(\text{resp., } \kappa/\kappa^p) \xrightarrow{\cong} A_i/A_{i-1} \quad \text{in the case } i = \mathrm{ord}_p(m) \ (\text{resp., } i < \mathrm{ord}_p(m)),$$

$$a \mapsto (f_{n-1}, \ldots, f_0) \quad \text{with } f_j = at^{-mp^{-i}} \text{ if } j = i, f_j = 0 \text{ otherwise.}$$

If $a_i \in \kappa$ $(0 \leq i \leq r)$ and $f_i = a_i t^{-mp^{-i}}$ for $0 \leq i \leq r$ and $f_i = 0$ for $r < i < n$, then the image of $(f_{n-1}, \ldots, f_0) \in \mathrm{fil}_m W_n(K)$ in $\Omega^1_{O_K}(\log) \otimes_{O_K} m_K^{-m}/m_K^{1-m} \cong \Omega^1_\kappa \oplus \kappa$ is

$$\Big(\sum_{i=0}^{r} a_i^{p^i-1} da_i, -\frac{m}{p^r} \cdot a_r^{p^r}\Big) \in \Omega^1_\kappa \oplus \kappa.$$

For $i \geq 0$, let B_i be the subgroup of Ω^1_κ generated by elements of the form $a^{p^j-1}da$ with $a \in \kappa$ and $0 \leq j \leq i$. For example, $B_0 = d\kappa$. Let $B_{-1} = 0$. The theory of Cartier isomorphisms shows

$$\kappa/\kappa^p \xrightarrow{\cong} B_i/B_{i-1}, \qquad a \mapsto a^{p^i-1}\, da \tag{4.1}$$

for $i \geq 0$. For $0 \leq i \leq r$, the image of the composition $A_i \to \Omega^1_\kappa \oplus \kappa \to \Omega^1_\kappa$ is contained in B_i, and the composition $\kappa/\kappa^p \xrightarrow{\cong} A_i/A_{i-1} \to B_i/B_{i-1}$ is nothing but the isomorphism (4.1). If $\mathrm{ord}_p(m) \leq n-1$ and $i = \mathrm{ord}_p(m)$, the composition $A_i \to \Omega^1_\kappa \oplus \kappa \to \kappa$ kills A_{i-1}, and the composition $\kappa \xrightarrow{\cong} A_i/A_{i-1} \to \kappa$ coincides with injective map $a \mapsto -m/p^r \cdot a^{p^r}$. This completes the proof of injectivity in the proposition. □

4.5

Let $O_K[F]$ be the noncommutative polynomial ring defined by

$$O_K[F] = \Big\{ \sum_{j \geq 0} F^j a_j ; a_j \in O_K \Big\}, \quad Fa = a^p F \,(a \in O_K).$$

For $m \in \mathbb{N}$, let

$$D_m = O_K[F] \otimes_{O_K} \Omega^1_{O_K}(\log) \otimes_{O_K} m_K^{-m}/m_K^{-[m/p]},$$

$$\bar{D}_m = \kappa[F] \otimes_\kappa \big(\Omega^1_{O_K}(\log) \otimes_{O_K} m_K^{-m}/m_K^{1-m}\big).$$

4.6

For $m \in \mathbb{N}$, by Propositions 4.1(2) and 4.4, we have an injective homomorphism

$$\theta_m : \mathrm{fil}^F_m W_n(K)/\mathrm{fil}^F_{[m/p]} W_n(K) \to D_m(K) :$$

$$\sum_{j \geq 0} F^j(x_j) \mapsto \sum_j F^j \otimes \delta_m(x_j)$$

for $x \in \mathrm{fil}_m W_n(K)$.

For $m \geq 1$, θ_m induces an injective homomorphism

$$\bar{\theta}_m : \mathrm{fil}^F_m W_n(K)/\mathrm{fil}^F_{m-1} W_n(K) \to \bar{D}_m.$$

4.7

For $m \geq 0$, we define a subgroup ${}^\flat\mathrm{fil}^F_m W_n(K)$ of $\mathrm{fil}^F_m W_n(K)$ as follows.

Let ${}^\flat\bar{D}_m$ be the image of $\kappa[F] \otimes_\kappa (\Omega^1_{O_K} \otimes_{O_K} m_K^{-m}/m_K^{1-m})$ (here we do not put a log pole) in $\bar{D}_m$. We have

$${}^\flat\bar{D}_m \cong \kappa[F] \otimes_\kappa \Omega^1_\kappa \otimes_\kappa m_K^{-m}/m_K^{1-m}.$$

Note that

$$\bar{D}_m/{}^\flat\bar{D}_m \cong \kappa[F] \otimes_\kappa m_K^{-m}/m_K^{1-m}, \quad F^j a \otimes d\log(t) \otimes t^{-m} \leftrightarrow F^j a \otimes t^{-m},$$

where $a \in \kappa$ and t is a prime element of K.

Let ${}^{\flat}\mathrm{fil}^F_m W_n(K) \subset \mathrm{fil}^F_m W_n(K)$ be the inverse image of ${}^{\flat}\bar{D}_m$ under $\bar{\theta}_m : \mathrm{fil}^F_m W_n(K) \to \bar{D}_m$. We have

$$ {}^{\flat}\mathrm{fil}^F_m W_n(K) = \sum_{j\geq 0} F^j\big({}^{\flat}\mathrm{fil}_m W_n(K)\big), $$

where ${}^{\flat}\mathrm{fil}_m W_n(K)$ is the subgroup of $\mathrm{fil}_m W_n(K)$ consisting of all elements $(f_{n-1},\ldots,f_0)$ which satisfy the following condition: if the p-adic order i of m is $< n$, then $p^i v_K(f_i) > -m$.

We have injections

$$ \mathrm{fil}^F_m W_n(K)/{}^{\flat}\mathrm{fil}^F_m W_n(K) \hookrightarrow \bar{D}_m/{}^{\flat}\bar{D}_m, $$

$$ {}^{\flat}\mathrm{fil}^F_m W_n(K)/\,\mathrm{fil}^F_{m-1} W_n(K) \hookrightarrow {}^{\flat}\bar{D}_m $$

induced by $\bar{\theta}_m$.

As is easily seen, we have the following.

(1) For $m \geq 1$, ${}^{\flat}\mathrm{fil}^F_m W_n(K) \supset \mathrm{fil}^F_{m-1} W_n(K)$. If m is prime to p, then ${}^{\flat}\mathrm{fil}^F_m W_n(K) = \mathrm{fil}^F_{m-1} W_n(K)$.

(2) If κ is perfect, then ${}^{\flat}\mathrm{fil}^F_m W_n(K) = \mathrm{fil}^F_{m-1} W_n(K)$.

4.8

The following relation with the refined Swan conductor in [Ka2] and [Ma] is proved easily. By Artin-Schreier-Witt theory, we have an isomorphism

$$ W_n(K)/(F-1)W_n(K) \cong H^1(K,\mathbb{Z}/p^n\mathbb{Z}) := H^1\big(\mathrm{Gal}(K^{\mathrm{sep}}/K),\mathbb{Z}/p^n\mathbb{Z}\big), $$

where K^{sep} denotes the separable closure of K. As in [Ka2], let $\mathrm{fil}_m H^1(K,\mathbb{Z}/p^n\mathbb{Z})$ be the image of $\mathrm{fil}_m W_n(K)$.

PROPOSITION 4.9

Let $\mathrm{fil}_m H^1(K,\mathbb{Z}/p^n\mathbb{Z}) \to \Omega^1_{O_K}(\log)\otimes_{O_K} m_K^{-m}/m_K^{1-m}$ $(m\geq 1)$ be the refined Swan conductor in [Ka2] whose kernel is $\mathrm{fil}_{m-1} H^1(K,\mathbb{Z}/p^n\mathbb{Z})$. Then we have a commutative diagram

$$ \begin{array}{ccc} \mathrm{fil}^F_m W_n(K) & \longrightarrow & D_m/D_{m-1} = \kappa[F]\otimes_\kappa \big(\Omega^1_{O_K}(\log)\otimes_{O_K} m_K^{-m}/m_K^{1-m}\big) \\ \downarrow & & \downarrow \\ \mathrm{fil}_m H^1(K,\mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & \Omega^1_{O_K}(\log)\otimes_{O_K} m_K^{-m}/m_K^{1-m} \end{array} $$

Here the right vertical arrow is induced from the ring homomorphism $\kappa[F]\to \kappa; \sum_i F^i a_i \mapsto \sum_i a_i$ $(a_i\in\kappa)$.

5. Homomorphisms and the filtrations

Let K be a discrete valuation field of characteristic $p>0$.

We assume here that we are given a perfect subfield k of O_K.

5.1

Let $n, n' \geq 1$, and assume that we are given a homomorphism $h : W_n \to W_{n'}$ of algebraic groups over k. Let $h_1 : \mathbb{G}_a \to \mathbb{G}_a$ be the homomorphism induced by h on the subgroups $\mathbb{G}_a \subset W_n$ (embedded via V^{n-1}) and $\mathbb{G}_a \subset W_{n'}$ (embedded via $V^{n'-1}$). Since the endomorphism ring of $\mathbb{G}_a$ over k is $k[F]$, where F acts as $\mathbb{G}_a \to \mathbb{G}_a$, $x \mapsto x^p$, we can regard h_1 as an element of $k[F]$.

The following proposition is proved easily.

PROPOSITION 5.2

(1) *The homomorphism h sends $\operatorname{fil}^F_m W_n(K)$ into $\operatorname{fil}^F_m W_{n'}(K)$.*

(2) *We have a commutative diagram*

$$\begin{array}{ccc} \operatorname{fil}^F_m W_n(K) & \xrightarrow{\theta_m} & D_m(K) \\ \downarrow & & \downarrow \\ \operatorname{fil}^F_m W_{n'}(K) & \xrightarrow{\theta_m} & D_m(K) \end{array}$$

where the left vertical arrow is induced from h and the right vertical arrow is the multiplication $x \mapsto h_1 x$ by $h_1 \in k[F]$.

Proof

Homomorphisms $W_n \to W_{n'}$ are described by F, V, and the multiplication by elements of $W(k)$. For each of them, we can check easily that the proposition holds. $\square$

THEOREM 5.3

Let $h : \bigoplus_{i=1}^{s} W_{n_i} \to \bigoplus_{j=1}^{s'} W_{n'_j}$ $(s, s' \geq 0, n_i, n'_j \geq 1)$ be an injective homomorphism defined over k. Let $m \geq 0$. Then for $x \in \bigoplus_{i=1}^{s} W_{n_i}(K)$, x belongs to $\bigoplus_{i=1}^{s} \operatorname{fil}^F_m W_{n_i}(K)$ if and only if $h(x)$ belongs to $\bigoplus_{j=1}^{s'} \operatorname{fil}^F_m W_{n'_j}(K)$.

Proof

Let $h_1 : \bigoplus_{i=1}^{s} \mathbb{G}_a \to \bigoplus_{j=1}^{s'} \mathbb{G}_a$ be the homomorphism induced by h on the subgroups $\bigoplus_{i=1}^{s} \mathbb{G}_a \subset \bigoplus_{i=1}^{s} W_{n_i}$ and $\bigoplus_{j=1}^{s'} \mathbb{G}_a \subset \bigoplus_{j=1}^{s'} W_{n'_j}$. This h_1 is understood as a matrix with entries in $k[F]$. Since h is injective, the homomorphism

$$\operatorname{Hom}_\kappa\Bigl(\mathbb{G}_a, \bigoplus_{i=1}^{s} \mathbb{G}_a\Bigr) \to \operatorname{Hom}_\kappa\Bigl(\mathbb{G}_a, \bigoplus_{j=1}^{s'} \mathbb{G}_a\Bigr), \quad g \mapsto h_1 \circ g$$

is injective, where $\operatorname{Hom}_\kappa$ means the set of homomorphisms of algebraic groups over κ. This means that the map $\bigoplus_{i=1}^{s} \kappa[F] \to \bigoplus_{j=1}^{s'} \kappa[F]$, $x \mapsto h_1 x$ is injective. Hence for $m \geq 1$, the map $\bigoplus_{i=1}^{s} \bar{D}_m \to \bigoplus_{j=1}^{s'} \bar{D}_m$, $x \mapsto h_1 x$ is injective. By Proposition 5.2(2), this proves that h induces an injective homomorphism $\bigoplus_{i=1}^{s} \operatorname{fil}^F_m W_{n_i}(K)/\operatorname{fil}^F_{m-1} W_{n_i}(K) \to \bigoplus_{j=1}^{s'} \operatorname{fil}^F_m W_{n'_j}(K)/\operatorname{fil}^F_{m-1} W_{n'_j}(K)$. $\square$

5.4 Proof of Theorem 3.3

Let $Y = \bigoplus_i W_{n_i}$. Consider another embedding $L_u \to Y' := \bigoplus_{i'} W_{n_{i'}}$. Embed the pushout Y'' of $Y \leftarrow L_u \to Y'$ into a finite direct sum $Y'' = \bigoplus_{i''} W_{n_{i''}}$. Then we have the third embedding $L_u \to Y''$ and injective homomorphisms $Y \to Y''$ and $Y' \to Y''$ which are compatible with embeddings. By Theorem 5.3, $\mathrm{mod}_v(\varphi)$ defined by the first (resp., second) embedding coincides with that defined by the third embedding. □

6. Local symbols

6.1

Let k be an algebraically closed field, let X be a normal algebraic curve over k, let G be a commutative smooth connected algebraic group over k, and let $\varphi : X \to G$ be a rational map. Then in [Se], the modulus of φ was defined by using local symbols. We show that our definition of the modulus coincides, in the curve case, with this classical definition.

6.2

Let k, X, G, and φ be as in Section 6.1, and let K be the function field of X. For each point v of X of codimension one (that is, v is a closed point of X), the local symbol map

$$(\, , \,)_v : G(K) \times K^\times \to G(k)$$

is defined as in [Se]. It is a $\mathbb{Z}$-bilinear map and is continuous for the v-adic topology. In [Se], the modulus of φ is defined as the right-hand side of the equation in the following proposition.

PROPOSITION 6.3

Let the notation be as in Section 6.2. Then our $\mathrm{mod}_v(\varphi)$ *satisfies*

$$\mathrm{mod}_v(\varphi) = \min\{m \in \mathbb{N} \mid (\varphi, U_v^{(m)})_v = 0\}.$$

Here $U_v^{(m)}$ *is the* m*th unit group at* v*; that is,* $U_v^{(m)} = \mathrm{Ker}(\mathcal{O}_{X,v}^\times \to (\mathcal{O}_{X,v}/m_{X,v}^m)^\times)$ *where* $m_{X,v}$ *is the maximal ideal of* $\mathcal{O}_{X,v}$.

Proof

Let $0 \to L \to G \to A \to 0$ be as in Section 3. Since $(G(\mathcal{O}_{X,v}), \mathcal{O}_{X,v}^\times)_v$ vanishes and since $L(K)/L(\mathcal{O}_{X,v}) \to G(K)/G(\mathcal{O}_{X,v})$ is bijective, we are reduced to the case $G = L$. If k is of characteristic zero, we are reduced to the cases $G = \mathbb{G}_m$ and $G = \mathbb{G}_a$. If k is of characteristic $p > 0$, by embedding L_u to a finite direct sum of Witt vector groups as in Section 3, we are reduced to the cases $G = \mathbb{G}_m$ and $G = W_n$. In the case $G = \mathbb{G}_m$, the local symbol coincides with $(f, g) \mapsto (-1)^{v(f)v(g)}(g^{v(f)}/f^{v(g)})(v)$ where $v(?)$ denotes the v-adic normalized valuation and (v) denotes the value at v. By using this fact, the case $G = \mathbb{G}_m$ is proved easily. In the case $G = \mathbb{G}_a$, the local symbol map is $(f, g) \mapsto \mathrm{Res}(f d\log(g))$, where Res is the residue map. By using this fact, in the case when k is of characteristic zero, the case $G=\mathbb{G}_a$ is proved easily. In the case when k is of characteristic $p>0$ and $G=W_n$, it is sufficient to prove the following proposition.

PROPOSITION 6.4

Let $K=\kappa((t))$ with κ a perfect field of characteristic $p>0$. For $m\geq 1$, let $U^{(m)}=1+t^m\kappa[\![t]\!]\subset\kappa[\![t]\!]^\times$. Let $(,)_K : W_n(K)\times K^\times\to W_n(\kappa)$ be the local symbol for $G=W_n$.

(1) *For $m\geq 0$, we have $(\mathrm{fil}^F_m W_n(K), U_K^{(m+1)})_K=0$.*

(2) *Let $\varphi\in\mathrm{fil}^F_m W_n(K)$, and let $\sum_i F^i a_i$ be the image of φ under $\mathrm{fil}^F_m W_n(K)/\mathrm{fil}^F_{m-1} W_n(K)\to D_m/D_{m-1}\cong\kappa[F]$, where the last isomorphism is given by $F^i a\otimes d\log(t)\otimes t^{-m}\mapsto F^i a$ $(a\in\kappa)$. Then for $b\in\kappa$, the local symbol $(\varphi,1+bt^m)$ coincides with the image of $\sum_i(a_i b)^{p^{i+1-n}}\in\kappa$ under the injection $V^{n-1}:\kappa\to W_n(\kappa)$.*

(3) *If κ is an infinite field, then for any $m\geq 0$, we have*

$$\mathrm{fil}^F_m W_n(K)=\{\varphi\in W_n(K)\mid(\varphi,U_K^{(m+1)})_K=0\}.$$

6.5

For the proof of Proposition 6.4, we use the following explicit description of the local symbol map of W_n.

Proof of Proposition 6.4

Let $A=W_n(\kappa)[\![t]\!][t^{-1}]$. We have the evident surjective ring homomorphism $A\to K$ and an injective ring homomorphism

$$\phi_n : W_n(K)\to A,\quad (a_{n-1},\dots,a_0)\mapsto\sum_{0\leq i\leq n-1}p^{n-1-i}\tilde{a}_i^{p^i}.$$

Here $\tilde{a}_i$ is any lifting of a_i to A. Note that $p^{n-1-i}\tilde{a}^{p^i}$ are independent of the choice of the lifting. The differential module Ω^1_A is a free A-module of rank 1 with basis $d\log(t)$. We have a well-defined homomorphism

$$d\log : K^\times\to\Omega^1_A/pdA;\qquad a\mapsto d\log(\tilde{a}),$$

where $\tilde{a}$ denotes any lifting of a to A. Let

$$\mathrm{Res}:\Omega^1_A\to W_n(\kappa);\quad \sum_i a_i t^i d\log(t)\mapsto a_0.$$

Then the local symbol $(\,,\,)_K$ for $G=W_n$ is expressed as

$$(f,g)_K=F^{1-n}\,\mathrm{Res}\big(\phi_n(f)d\log(\tilde{g})\big)\quad\text{for } f\in W_n(K)\text{ and } g\in K^\times. \tag{6.1}$$

Here $F^{-1}:W_n(\kappa)\to W_n(\kappa)$ is the inverse map of $F:W_n(\kappa)\to W_n(\kappa)$. In the case $n=1$, this formula coincides with the formula $(f,g)_K=\mathrm{Res}(f d\log g)$ for $G=\mathbb{G}_a$.

By using the explicit formula (6.1) of the local symbol, we obtain Proposition 6.4(1), (2). Proposition 6.4(3) follows from Proposition 6.4(1), (2).

The authors are sure that the above formula (6.1) is written in some references, but they could not find it. This (6.1) can be deduced from the formula (6.2) below.

Let $W_n\Omega_K^\bullet$ be the de Rham–Witt complex of K. Then $W_n\Omega_K^1$ is a $W_n(K)$-module, and we have a homomorphism $d\log : K^\times \to W_n\Omega_K^1$. There is a residue map

$$\mathrm{Res} : W_n\Omega_K^1 \to W_n(\kappa)$$

(see [Ka1, §2] or [Rül, §2]; see also [Rü2]) which generalizes the residue map $\Omega_K^1 \to \kappa$ (the case $n = 1$). By [KS, Chap. III, Lem. 3], we have

$$(f,g)_K = \mathrm{Res}\big(f d\log(g)\big) \quad \text{for } f \in W_n(K), g \in K^\times. \tag{6.2}$$

The above formula (6.1) follows from this formula (6.2) and from

$$F^{1-n}\,\mathrm{Res}\big(\phi_n(f)d\log(\tilde{g})\big) = \mathrm{Res}\big(f d\log(g)\big) \quad \text{for } f \in W_n(K), g \in K^\times.$$

This concludes the proof of Proposition 6.4 and hence the proof of Proposition 6.3. □

7. Higher-dimensional local fields

7.1

The above relation between modulus and local symbols for curves is generalized to the higher-dimensional cases by using local symbols for higher-dimensional local fields defined in [KS, Chap. III].

Let p be a prime number, let k_0 be a perfect field of characteristic p, and define fields k_r $(r \geq 1)$ inductively by

$$k_r = k_{r-1}((t_r)).$$

Let G be a commutative smooth connected algebraic group over k_0. Then the local symbol map

$$(\;,\;)_{k_r} : G(k_r) \times K_r^M(k_r) \to G(k_0)$$

is defined in [KS], where K_r^M denotes the rth Milnor K-group.

In the case $G = W_n$, this local symbol map is described as follows. Define rings A_r $(r \geq 0)$ inductively by $A_0 = W_n(k_0)$ and $A_r = A_{r-1}[\![t_r]\!][t_r^{-1}]$ for $r \geq 1$. Then the local symbol map of k_r for W_n is described as

$$(f,g)_{k_r} = F^{1-n}\,\mathrm{Res}\big(\phi_n(f)d\log(\tilde{g})\big) \quad \text{for } f \in W_n(k_r) \text{ and } g \in k_r^\times, \tag{7.1}$$

where Res is the map

$$\mathrm{Res} : \Omega_{A_r}^r \to W_n(k_0)$$

defined to be the composition of the evident residue maps $\Omega_{A_i}^i \to \Omega_{A_{i-1}}^{i-1}$ $(1 \leq i \leq r)$ and $\phi_n : W_n(k_r) \to A_r$ is defined in the same way as ϕ_n in the previous paragraph, respectively. This (7.1) is deduced from the description of the local symbol map (see [KS])

$$(f,g)_{k_r} = \mathrm{Res}\big(f d\log(g)\big) \quad \text{for } f \in W_n(k_r) \text{ and } g \in k_r^\times,$$

where Res is the residue map

$$\mathrm{Res} : W_n\Omega^r_{k_r} \to W_n(k_0)$$

defined in [Ka1, §2].

7.2

By using the explicit presentation (7.1) of the local symbol, we can obtain the following generalization Proposition 7.3 of Proposition 6.4 to higher-dimensional local fields. In Proposition 7.3, for $r \geq 1$ we show that the two filtrations $\mathrm{fil}^F_\bullet W_n(k_r)$ and ${}^\flat\mathrm{fil}^F_\bullet W_n(k_r)$ (which are defined with respect to the t_r-adic valuation of k_r) are related to a certain two filtrations $U_r^{(\bullet)}$ and $V_r^{(\bullet)}$ on $K^M_r(k_r)$, respectively.

Fix $r \geq 1$. We define subgroups $U_r^{(m)}$ and $V_r^{(m)}$ of $K^M_r(k_r)$. For $m \geq 1$, let $U_r^{(m)}$ be the subgroup of $K^M_r(k_r)$ generated by all elements of the form $\{x, y_1, \ldots, y_{r-1}\}$ such that $y_i \in k_r^\times$ and $x \in 1 + t_r^m k_{r-1}[\![t_r]\!] \subset k_{r-1}[\![t_r]\!]^\times$. For $m \geq 0$, let $V_r^{(m)}$ be the subgroup of $K^M_r(k_r)$ generated by all elements of the form $\{x, y_1, \ldots, y_{r-1}\}$ such that $y_i \in k_{r-1}[\![t_r]\!]^\times$ and $x \in \mathrm{Ker}\big(k_{r-1}[\![t_r]\!]^\times \to (k_{r-1}[t_r]/(t_r^m))^\times\big)$. Then

$$V_r^{(m-1)} \supset U_r^{(m)} \supset V_r^{(m)} \quad \text{for all } m \geq 1.$$

Let $U_r^{(0)} = V_r^{(0)}$.

For $m \geq 1$, we have surjective homomorphisms

$$s_m : \Omega^{r-1}_{k_{r-1}} \to V_r^{(m)}/U_r^{(m+1)},$$

$$a\,d\log(b_1) \wedge \cdots \wedge d\log(b_{r-1}) \mapsto \{1 + at_r^m, b_1, \ldots, b_{r-1}\},$$

$$s'_m : \Omega^{r-2}_{k_{r-1}} \to U_r^{(m)}/V_r^{(m)},$$

$$a\,d\log(b_1) \wedge \cdots \wedge d\log(b_{r-2}) \mapsto \{1 + at_r^m, b_1, \ldots, b_{r-2}, t_r\}$$

$(a \in k_{r-1},\ b_j \in k_{r-1}^\times)$.

PROPOSITION 7.3

Let $r \geq 1$. Define the filtrations $\mathrm{fil}^F_m W_n(k_r)$ and ${}^\flat\mathrm{fil}_m W_n(k_r)$ by using the t_r-adic discrete valuation of k_r. Let $(\ ,\ )_{k_r} : W_n(k_r) \times K^M_r(k_r) \to W_n(k_0)$ be the local symbol map of k_r for $G = W_n$.

(1) *For $m \geq 0$, we have*

$$\big(\mathrm{fil}^F_m W_n(k_r), U_r^{(m+1)}\big)_{k_r} = 0, \qquad \big({}^\flat\mathrm{fil}^F_m W_n(k_r), V_r^{(m)}\big)_{k_r} = 0.$$

(2a) *Let $m \geq 1$. Let $\varphi \in \mathrm{fil}^F_m W_n(k_r)$, and let $\sum_i F^i a_i$ $(a_i \in k_{r-1})$ be the image of φ under $\mathrm{fil}^F_m W_n(k_r)/{}^\flat\mathrm{fil}^F_m W_n(k_r) \to \bar{D}_m/{}^\flat\bar{D}_m \cong k_{r-1}[F]$, where the last isomorphism is given by $F^i a \otimes d\log(t_r) \otimes t_r^{-m} \mapsto F^i a$ $(a \in k_{r-1})$. Then for $b \in \Omega^{r-1}_{k_{r-1}}$, the local symbol $(\varphi, s_m(b))$ coincides with the image of $\sum_i (\mathrm{Res}(a_i b))^{p^{i+1-n}} \in k_0$ under the injection $V^{n-1} : k_0 \to W_n(k_0)$. Here* Res *is the residue map $\Omega^{r-1}_{k_{r-1}} \to k_0$.*

(2b) *Let* $m \geq 1$. *Let* $\varphi \in {}^{\flat}\mathrm{fil}^F_m W_n(k_r)$, *and let* $\sum_i F^i a_i$ $(a_i \in \Omega^1_{k_{r-1}})$ *be the image of* φ *under* ${}^{\flat}\mathrm{fil}^F_m W_n(k_r)/\mathrm{fil}^F_{m-1} W_n(k_r) \to {}^{\flat}\bar{D}_m \cong k_{r-1}[F] \otimes_{k_{r-1}} \Omega^1_{k_{r-1}}$, *where the last isomorphism is given by* $F^i a \otimes w \otimes t_r^{-m} \mapsto F^i a \otimes w$ *for* $a \in k_{r-1}$, $w \in \Omega^1_{k_{r-1}}$. *Then for* $b \in \Omega^{r-2}_{k_{r-1}}$, *the local symbol* $(\varphi, s'_m(b))$ *coincides with the image of* $\sum_i (\mathrm{Res}(a_i \wedge b))^{p^{i+1-n}} \in k_0$ *under the injection* $V^{n-1} : k_0 \to W_n(k_0)$.

(3) *If* k_0 *is an infinite field, then for any* $m \geq 0$, *we have*

$$\mathrm{fil}^F_m W_n(k_r) = \{\varphi \in W_n(k_r) \mid (\varphi, U_r^{(m+1)})_{k_r} = 0\},$$
$${}^{\flat}\mathrm{fil}^F_m W_n(k_r) = \{\varphi \in W_n(k_r) \mid (\varphi, V_r^{(m)})_{k_r} = 0\}.$$

7.4

The following relation between modulus and higher-dimensional local fields is deduced from Proposition 7.3. Let k be an algebraically closed field, let X be a normal algebraic variety over k, let G be a commutative smooth connected algebraic group over k, and let $\varphi : X \to G$ be a rational map. Let K be the function field of X. Let v be a point of X of codimension one.

Let $r = \dim(X)$, let $k_0 = k$, and define k_i $(i \geq 1)$ as above. Assume $r \geq 1$, and assume that we are given a homomorphism of fields $K \xrightarrow{\subseteq} k_r$ such that $k_{r-1}[\![t_r]\!] \cap K = \mathcal{O}_{X,v}$, $t_r k_{r-1}[\![t_r]\!] \cap K = m_{X,v}$, k_{r-1} regarded as the residue field of $k_{r-1}[\![t_r]\!]$ is separable over the residue field of v, and the ramification index of $k_{r-1}[\![t_r]\!]$ over $\mathcal{O}_{X,v}$ is 1. (There are many such $K \to k_r$.)

PROPOSITION 7.5

(1) *For the local symbol map* $(\ ,\ )_{k_r} : G(k_r) \times K^M_r(k_r) \to G(k)$, *we have*

$$\mathrm{mod}_v(\varphi) = \min\{m \in \mathbb{N} \mid (\varphi, U_r^{(m)})_{k_r} = 1\}$$

(1 *denotes the neutral element of* G*).*

(2) *In the case* $G = W_n$, *if we endow* K *with the discrete valuation associated to* v, *we have, for any* $m \geq 0$,

$$\mathrm{fil}^F_m W_n(K) = \{f \in W_n(K) \mid (f, U_r^{(m+1)})_{k_r} = 0\},$$
$${}^{\flat}\mathrm{fil}^F_m W_n(K) = \{f \in W_n(K) \mid (f, V_r^{(m)})_{k_r} = 0\}.$$

8. Extension of local fields and the filtrations

In this section, let K be a discrete valuation field of characteristic $p > 0$, and let κ be the residue field of K.

We consider how the filtrations $\mathrm{fil}^F_\bullet W_n(K)$ and ${}^{\flat}\mathrm{fil}^F_\bullet W_n(K)$ behave when the field K extends. In Theorems 8.6 and 8.7, we show how these filtrations are characterized by using extensions of K with perfect residue fields.

The following lemma can be proved easily.

LEMMA 8.1

Let K' *be a discrete valuation field containing* K *such that* $O_{K'} \cap K = O_K$ *and*

$m_{K'} \cap K = m_K$. Let $m' = e(K'/K)m$, where $e(K'/K)$ is the ramification index of K' over K:

(1) *$\mathrm{fil}^F_m W_n(K) \subset \mathrm{fil}^F_{m'} W_n(K')$;*

(2) *for $m \geq 1$, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{fil}^F_m W_n(K) & \xrightarrow{\theta_m} & D_m(K) \\ \downarrow & & \downarrow \\ \mathrm{fil}^F_{m'} W_n(K) & \xrightarrow{\theta_{m'}} & D_{m'}(K') \end{array}$$

COROLLARY 8.2

Let $s_K(\varphi) = \min\{m \in \mathbb{N} \mid \varphi \in \mathrm{fil}^F_m W_n(K)\}$. Then $s_{K'}(\varphi) \leq e(K'/K) s_K(\varphi)$.

COROLLARY 8.3

Let $m \geq 1$.

(1) *The map $\mathrm{fil}^F_m W_n(K)/{}^\flat\mathrm{fil}^F_m W_n(K) \to \mathrm{fil}^F_{m'} W_n(K')/{}^\flat\mathrm{fil}^F_{m'} W_n(K')$ is injective if $e(K'/K)$ is prime to p and is the zero map if $e(K'/K)$ is divisible by p.*

(2) *The map ${}^\flat\mathrm{fil}^F_m W_n(K)/\mathrm{fil}^F_{m-1} W_n(K) \to {}^\flat\mathrm{fil}^F_{m'} W_n(K')/\mathrm{fil}^F_{m'-1} W_n(K')$ is injective if the residue field of K' is separable over κ.*

COROLLARY 8.4

In the case when $e(K'/K)$ is prime to p and the extension of the residue field in the extension K'/K is separable, we have

$$s_{K'}(\varphi) = e(K'/K) s_K(\varphi).$$

Proof

This follows from Cor. 8.3. □

8.5

We consider what happens for extensions K' of K, which have perfect residue fields. We consider the following K'.

(1) K' is a discrete valuation field containing K such that $O_{K'} \cap K = O_K$ and $m_{K'} \cap K = m_K$, and such that the residue field of K' is perfect.

We also consider the following K'.

(2) K' is as in (1), but satisfies, furthermore, $e(K'/K) = 1$.

THEOREM 8.6

Let $\varphi \in W_n(K)$. Then

$$s_K(\varphi) = \sup\big\{e(K'/K)^{-1} s_{K'}(\varphi) \;\big|\; K' \text{ is as in Section 8.5(1)}\big\}.$$

For the filtration ${}^\flat\mathrm{fil}^F_m W_n(K)$, we have the following.

THEOREM 8.7

Let $\varphi \in W_n(K)$. Then

$$\min\{m \geq 1 \mid \varphi \in {}^{\flat}\mathrm{fil}^F_m W_n(K)\} = 1 + \max\{s_{K'}(\varphi) \mid K' \textit{ is as in Section 8.5(2)}\}.$$

We use the following lemma for the proofs of these theorems.

LEMMA 8.8

Let K' be as in Section 8.5(1). Then for $m \geq 2$, we have a commutative diagram with injective rows

$$\begin{array}{ccc} {}^{\flat}\mathrm{fil}^F_m W_n(K)/{}^{\flat}\mathrm{fil}^F_{m-1} W_n(K) & \xrightarrow{\bar{\theta}_m} & \kappa[F] \otimes_\kappa \Omega^1_\kappa \otimes_\kappa m_K^{-m}/m_K^{1-m} \\ \downarrow & & \downarrow \\ \mathrm{fil}^F_{em-1} W_n(K')/\mathrm{fil}^F_{em-2} W_n(K') & \xrightarrow{\bar{\theta}_{m-1}} & (\kappa'[F] \otimes_{\kappa'} m_{K'}^{1-m}/m_{K'}^{2-m})/N \end{array}$$

where $e = e(K'/K)$, $N = \kappa[F] \otimes_\kappa m_K^{1-m}/m_K^{2-m}$ if $e = 1$, and $N = 0$ if $e \geq 2$, and the right vertical arrow is the map induced from

$$O_K[F] \otimes_{O_K} \Omega^1_{O_K} \otimes_{O_K} m_K^{-m}/m_K^{1-m} \to O_{K'}[F] \otimes_{O_{K'}} \Omega^1_{O_{K'}} \otimes_{O_{K'}} m_{K'}^{-m}/m_{K'}^{1-m}.$$

This is proved easily.

8.9 The proofs of Theorems 8.6 and 8.7

For K' as in 8.5(1), ${}^{\flat}\mathrm{fil}^F_m W_n(K) \subset \mathrm{fil}^F_{e(K'/K)m-1} W_n(K')$ by 4.7(2). Hence by Proposition 8.1(1), it is sufficient to prove the following (1) and (2).

(1) Let $m \geq 1$, and assume that $\varphi \in \mathrm{fil}^F_m W_n(K)$, $\varphi \notin {}^{\flat}\mathrm{fil}^F_m W_n(K)$. Then for any K' as in Section 8.5(2), we have $s_K(\varphi) = s_{K'}(\varphi)$.

(2) Let $m \geq 2$, and assume that $\varphi \in \mathrm{fil}^F_{m-1} {}^{\flat}W_n(K)$, $\varphi \notin \mathrm{fil}^F_{m-1} W_n(K)$. Then for any integer $e \geq 1$, there is K' as in Section 8.5(1) such that $e = e(K'/K)$ and such that $s_{K'}(\varphi) = em - 1$.

We prove (1) and (2).

Item (1) follows from Proposition 8.1(2) easily by looking at the coefficient of $d\log(t) \otimes t^{-m}$ in the image of φ under $\bar{\theta}_m$. (Here t denotes any prime element of K.)

We prove (2). Take a lifting $(\tilde{b}_i)_{i\in I}$ of a p-base $(b_i)_{i\in I}$ of κ to O_K. Let

$$\kappa' = \bigcup_{r\geq 0} \kappa(T_i; i \in I)^{1/p^r},$$

where T_i $(i \in I)$ are indeterminates. Let t be another indeterminate. Let π be a prime element of K. Then there is a unique homomorphism of fields $K \to K' := \kappa'((t))$ which sends O_K into $O_{K'}$, m_K into $m_{K'}$, $\tilde{b}_i$ $(i \in I)$ to $b_i + T_i t$, and π to t^e. The right vertical arrow in the diagram in Lemma 8.8 sends $F^j a \otimes db_i$ $(a \in \kappa)$ to $F^j a T_i$, and sends $F^j a \otimes d\pi$ $(a \in \kappa)$ to $F^j a$ if $e = 1$ and to 0 if $e \geq 2$. From this, we see that in the case $e = 1$, the map

$${}^{\flat}\mathrm{fil}^F_m W_n(K)/{}^{\flat}\mathrm{fil}^F_{m-1} W_n(K) \to \mathrm{fil}^F_{m-1} W_n(K')/\mathrm{fil}^F_{m-2} W_n(K')$$

is injective, and in the case $e \geq 2$, the map

$$^{\flat}\mathrm{fil}^F_m W_n(K)/\mathrm{fil}^F_{m-1} W_n(K) \to \mathrm{fil}^F_{m-1} W_n(K')/\mathrm{fil}^F_{m-2} W_n(K')$$

is injective. This proves (2).

This concludes the proofs of Theorems 8.6 and 8.7. □

References

[Br] J.-L. Brylinski, *Théorie du corps de classes de Kato et revêtements abéliens de surfaces*, Ann. Inst. Fourier (Grenoble) **33** (1983), 23–38.

[Ka1] K. Kato, *A generalization of local class field theory by using K-groups, II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 603–683.

[Ka2] ———, "Swan conductors for characters of degree one in the imperfect residue field case" in *Algebraic K-Theory and Algebraic Number Theory (Honolulu, 1987)*, Contemp. Math. **83**, Amer. Math. Soc., Providence, 1989, 101–131.

[KR] K. Kato and H. Russell, *Albanese varieties with modulus and Hodge theory*, preprint, arXiv:0906.0047v1 [math.AG].

[KS] K. Kato and S. Saito, "Two-dimensional class field theory" in *Galois Groups and Their Representations (Nagoya, 1981)*, Adv. Stud. Pure Math. **2**, North-Holland, Amsterdam, 1983, 103–152.

[Ma] S. Matsuda, *On the Swan conductor in positive characteristic*, Amer. J. Math. **119** (1997), 705–739.

[Ön] H. Önsiper, *Generalized Albanese varieties for surfaces in characteristic $p > 0$*, Duke Math. J. **59** (1989), 359–364.

[Ru1] H. Russell, *Generalized Albanese and its dual*, J. Math. Kyoto Univ. **48** (2008), 907–949.

[Ru2] ———, *Albanese varieties with modulus over a perfect field*, preprint, arXiv:0902.2533v2 [math.AG].

[Rül] K. Rülling, *The generalized de Rham–Witt complex over a field is a complex of zero-cycles*, J. Algebraic Geom. **16** (2007), 109–169.

[Rü2] ———, *Erratum to "The generalized de Rham–Witt complex over a field is a complex of zero-cycles*," J. Algebraic Geom. **16** (2007), 793–795.

[Se] J.-P. Serre, Groupes algébriques et corps de classes, Publ. Inst. Math. Univ. Nancago **7**, Hermann, Paris, 1959.

Kato: Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, Illinois 60637, USA; kzkt@math.uchicago.edu

Russell: Universität Duisburg-Essen, Fachbereich Mathematik, Campus Essen, 45117 Essen, Germany; henrik.russell@uni-due.de