The homotopy types of $\mathrm{Sp}(2)$-gauge groups

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Abstract There are countably many equivalence classes of principal Sp(2)-bundles over S^4, classified by the integer value of the second Chern class. We show that the corresponding gauge groups $\mathcal{G}_k$ have the property that if there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$, then $(40,k)=(40,k')$, and we prove a partial converse by showing that if $(40,k)=(40,k')$, then $\mathcal{G}_k$ and $\mathcal{G}_{k'}$ are homotopy equivalent when localized rationally or at any prime.

1. Introduction

Let M be a simply connected, compact 4-manifold, let G be a simple, simply connected, compact Lie group, and let $P \longrightarrow S^4$ be a principal G-bundle. The *gauge group* of this bundle is the group of G-equivariant automorphisms of P which fix M. As $[M,BG]=\mathbb{Z}$, there are countably many equivalence classes of principal G-bundles over M. Each has a gauge group, so there are potentially countably many distinct gauge groups. However, in [CS] it was shown that these gauge groups have only finitely many distinct homotopy types. It has been a subject of recent interest to determine the precise number of homotopy types in special cases. Notably, it is known that there are six homotopy types of S^3-gauge groups over S^4 (see [K]); either six or four homotopy types of S^3-gauge groups over M, depending on the signature of M (see [KT]); 12 homotopy types of SU(3)-gauge groups over S^4 (see [HK]); and 12 homotopy types of SO(3)-gauge groups over S^4 (see [KKKT]).

In this article we consider the case of Sp(2)-gauge groups over S^4. To state our results, let $P \longrightarrow S^4$ be a principal Sp(2)-bundle. It is classified by an element in $[S^4, B\,\mathrm{Sp}(2)] \cong \mathbb{Z}$, where the specific integer is determined by the second Chern class. Let $\mathcal{G}_k$ be the gauge group of this principal bundle. If a,b are two integers, let (a,b) be the greatest common divisor of $|a|$ and $|b|$.

THEOREM 1.1

The following hold:

(a) *if there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$, then $(40,k)=(40,k')$;*

(b) *if $(40,k)=(40,k')$, then $\mathcal{G}_k$ and $\mathcal{G}_{k'}$ are homotopy equivalent when localized rationally or at any prime.*

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Theorem 1.1 improves on what was previously known. In [S] it was shown that if there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$, then $(10,k) = (10,k')$. Recent work in [CHM] suggested that $(40,k) = (40,k')$ implies $\mathcal{G}_k \simeq \mathcal{G}_{k'}$. In fact, this is claimed to be proved. However, the proof relies on a result of [HK] which involves a map into a target space with the property that all of its homotopy groups are finite, and this is applied to a map into the target $\Omega_0^3 \operatorname{Sp}(2)$ which has an integral summand in π_4. We conjecture that Theorem 1.1 can be improved to the following: $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(40,k) = (40,k')$. This seems to be problematic, involving a delicate application of Sullivan's arithmetic square.

The methods in this article are different from those in [K] and [HK], which have set the standard for calculating numbers of gauge groups. It is expected that our methods will also have other applications.

2. Preliminary homotopy theory

In this section we state known facts about the homotopy theory of Sp(2) and the gauge groups of principal Sp(2)-bundles. Recall that $H^*(\operatorname{Sp}(2);\mathbb{Z}) \cong \Lambda(x_3, x_7)$ and that Sp(2) can be given the CW-structure of a three-cell complex $\operatorname{Sp}(2) = S^3 \cup e^7 \cup e^{10}$. Let A be the 7-skeleton of Sp(2), and let $\imath\colon A \longrightarrow \operatorname{Sp}(2)$ be the skeletal inclusion. Then A is a two-cell complex, and there is a homotopy cofibration sequence

$$S^6 \xrightarrow{f} S^3 \longrightarrow A \xrightarrow{\pi} S^7,$$

where f is the attaching map for A and π is the pinch map to the top cell. The map f represents a generator of $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$. The following decomposition is due to Mimura [M, Lemma 2.1(ii)].

LEMMA 2.1

The map $\Sigma^2\imath$ has a left homotopy inverse, implying that there is a homotopy equivalence $\Sigma^2 \operatorname{Sp}(2) \simeq \Sigma^2 A \vee S^{12}$.

Next, consider the canonical fibration $S^3 \xrightarrow{i} \operatorname{Sp}(2) \xrightarrow{q} S^7$, where i is the inclusion of the bottom cell and q is the quotient map to $\operatorname{Sp}(2)/\operatorname{Sp}(1) = S^7$. In Lemma 2.2 we collect some information from [MT] regarding the homotopy groups of Sp(2).

LEMMA 2.2

The following hold:

(a) $\pi_6(\operatorname{Sp}(2)) = 0$;
(b) $\pi_7(\operatorname{Sp}(2)) \cong \mathbb{Z}$;
(c) $\pi_8(\operatorname{Sp}(2)) = 0$;
(d) $\pi_{10}(\operatorname{Sp}(2)) \cong \mathbb{Z}/120\mathbb{Z}$;
(e) $\pi_{13}(\operatorname{Sp}(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Now we turn our attention to gauge groups. By [AB], there is a homotopy equivalence $B\mathcal{G}_k \simeq \mathrm{Map}_k(S^4, B\,\mathrm{Sp}(2))$ between the classifying space $B\mathcal{G}_k$ of $\mathcal{G}_k$ and the component of the space of continuous maps from S^4 to $B\,\mathrm{Sp}(2)$ which contains the map inducing P. Further, there is a fibration $\mathrm{Map}^*_k(S^4, B\,\mathrm{Sp}(2)) \longrightarrow \mathrm{Map}_k(S^4, B\,\mathrm{Sp}(2)) \xrightarrow{\mathrm{ev}} B\,\mathrm{Sp}(2)$, where ev evaluates a map at the basepoint of S^4 and $\mathrm{Map}^*_k(S^4, B\,\mathrm{Sp}(2))$ is the k-th component of the space of pointed continuous maps from S^4 to $B\,\mathrm{Sp}(2)$. It is well known that there is a homotopy equivalence $\mathrm{Map}^*_k(S^4, B\,\mathrm{Sp}(2)) \simeq \mathrm{Map}^*_0(S^4, B\,\mathrm{Sp}(2))$ for every $k \in \mathbb{Z}$; the latter space is usually written as $\Omega^3_0\,\mathrm{Sp}(2)$. Putting all this together, for each k the evaluation fibration induces a homotopy fibration sequence

$$\mathrm{Sp}(2) \xrightarrow{\partial_k} \Omega^3_0\,\mathrm{Sp}(2) \xrightarrow{b_k} B\mathcal{G}_k \xrightarrow{\mathrm{ev}} B\,\mathrm{Sp}(2),$$

where b_k is just a name for the map from the fiber to the total space and ∂_k is the fibration connecting map.

The following lemma describes the triple adjoint of ∂_k and was proved in [L, Theorem 2.6]. Recall that $S^3 \xrightarrow{i} \mathrm{Sp}(2)$ is the inclusion of the bottom cell. Let $1:\ \mathrm{Sp}(2) \longrightarrow \mathrm{Sp}(2)$ be the identity map.

LEMMA 2.3

The adjoint of the map $\mathrm{Sp}(2) \xrightarrow{\partial_k} \Omega^3_0\,\mathrm{Sp}(2)$ *is homotopic to the Samelson product* $S^3 \wedge \mathrm{Sp}(2) \xrightarrow{\langle ki,1\rangle} \mathrm{Sp}(2)$.

The linearity of the Samelson product implies that $\langle ki, 1\rangle \simeq k\langle i, 1\rangle$. Adjointing therefore implies the following.

COROLLARY 2.4

There is a homotopy $\partial_k \simeq k \circ \partial_1$.

We also need to know how ∂_k behaves with respect to π_7. Let $\phi : S^7 \longrightarrow \mathrm{Sp}(2)$ be a generator of $\pi_7(\mathrm{Sp}(2)) \cong \mathbb{Z}$. By [MT], this has the property that the composite $S^7 \xrightarrow{\phi} \mathrm{Sp}(2) \xrightarrow{q} S^7$ has degree 12. Refining a bit, since the inclusion $A \xrightarrow{\imath} \mathrm{Sp}(2)$ is 9-connected, ϕ factors as a composite $S^7 \xrightarrow{\phi'} A \xrightarrow{\imath} \mathrm{Sp}(2)$ for some map ϕ'. The following lemma was proved in [S] and stated in terms of ϕ. We restate it in terms of ϕ'.

LEMMA 2.5

The composite $S^7 \xrightarrow{\phi'} A \xrightarrow{\imath} \mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega^3_0\,\mathrm{Sp}(2)$ *has order* 10.

Lemma 2.5 was used in [S] to prove the following.

LEMMA 2.6

There is an isomorphism $\pi_7(B\mathcal{G}_k) \cong \mathbb{Z}/12(10,k)\mathbb{Z}$, *and the map* $\pi_7(\Omega^3_0\,\mathrm{Sp}(2)) \cong$

$\mathbb{Z}/120\mathbb{Z} \xrightarrow{(b_k)_*} \pi_7(B\mathcal{G}_k)) \cong \mathbb{Z}/12(10,k)\mathbb{Z}$ *is an epimorphism. Consequently, if* $\mathcal{G}_k \simeq \mathcal{G}_{k'}$*, then* $(10,k) = (10,k')$.

3. A counting lemma

Let Y be an H-space with a homotopy inverse. Then there are power maps $Y \xrightarrow{k} Y$ for every integer k. Suppose that there is a map $f : X \longrightarrow Y$, where X is a space and f has finite order. Let F_k be the homotopy fiber of $k \circ f$. A basic problem is to determine when F_k and $F_{k'}$ are homotopy equivalent. Integrally, it seems to be difficult to give an easily checked condition for when this is true. In Lemma 3.1 we give a simple criterion for when homotopy equivalences exist after localizing rationally or at any prime.

LEMMA 3.1

Let X be a space, and let Y be an H-space with a homotopy inverse. Suppose that there is a map $X \xrightarrow{f} Y$ of order m, where m is finite. If $(m,k) = (m,k')$, then F_k and $F_{k'}$ are homotopy equivalent when localized rationally or at any prime.

Proof

Since f has order m, the homotopy class of f generates a cyclic subgroup $S = \mathbb{Z}/m\mathbb{Z}$ in $[X,Y]$. Suppose $(m,k) = (m,k') = l$. Then $k = lt$ and $k' = lt'$ for integers t and t' which are units in $\mathbb{Z}/m\mathbb{Z}$. Let s and s' be integers such that $st \equiv 1 \pmod{m}$ and $s't' \equiv 1 \pmod{m}$. Observe that $ks \equiv l \pmod{m}$ and $k's' \equiv l \pmod{m}$. Thus the composites $X \xrightarrow{f} Y \xrightarrow{ks} Y$ and $X \xrightarrow{f} Y \xrightarrow{k's'} Y$ both represent the homotopy class $[l]$ in S. That is, $ks \circ f$ is homotopic to $k's' \circ f$. Consequently, $F_{ks} \simeq F_{k's'}$. Note that this holds integrally.

Now fix a prime p, and localize at p. There are two cases. First, suppose that $(m,p) = 1$. Then m and p have no common factors, so m is a unit mod-p. Thus the power map $Y \xrightarrow{m} Y$ is a homotopy equivalence, implying that f has order 1. In other words, f is null homotopic. Therefore $k \circ f$ is null homotopic for any integer k, implying that $F_k \simeq X \times \Omega Y$. Hence $F_k \simeq F_{k'}$ for any integers k and k'. Second, suppose that $(m,p) = p$. Since s is a unit in $\mathbb{Z}/m\mathbb{Z}$, we have $(m,s) = 1$. Therefore $(s,p) = 1$, which implies that s is a unit mod p. Thus the power map $Y \xrightarrow{s} Y$ is a homotopy equivalence. This implies that there is a homotopy pullback diagram

$$\begin{tikzcd}
F_k \arrow[r] \arrow[d, equal] & F_{ks} \arrow[r] \arrow[d] & * \arrow[d] \\
F_k \arrow[r] & X \arrow[r, "k\circ f"] \arrow[d, "ks\circ f"] & Y \arrow[d, "s"] \\
 & Y \arrow[r, equal] & Y
\end{tikzcd}$$

The homotopy fibration along the top row implies that $F_k \simeq F_{ks}$. Similarly, as s' is a unit in $\mathbb{Z}/m\mathbb{Z}$, we obtain $F_{k'} \simeq F_{k's'}$. Hence there is a string of homotopy equivalences $F_k \simeq F_{ks} \simeq F_{k's'} \simeq F_{k'}$.

Finally, consider the rational case. Since m is a unit in $\mathbb{Q}$, arguing as in the first case above shows that $F_k \simeq X \times \Omega \simeq F_{k'}$ for any integers k and k'. □

4. A factorization of ∂_1

Consider the homotopy cofibration $S^3 \xrightarrow{i} \mathrm{Sp}(2) \xrightarrow{c} C$ which defines the space C and the map c. Observe that the CW-structure of Sp(2) implies that C is a two-cell complex with cells in dimensions 7 and 10. Thus there is a homotopy cofibration $S^9 \xrightarrow{\theta} S^7 \longrightarrow C$ for some map θ. We claim that θ is null homotopic. To see this, observe that the homotopy decomposition of $\Sigma^2 \mathrm{Sp}(2)$ in Lemma 2.1 implies that $\Sigma^2 C \simeq \Sigma^2 S^7 \vee S^{12}$. Thus $\Sigma^2\theta$ is null homotopic. But θ is in the stable range, and so θ is null homotopic. Hence $C \simeq S^7 \vee S^{10}$.

PROPOSITION 4.1

There is a homotopy commutative square

$$\begin{tikzcd}
\mathrm{Sp}(2) \arrow[r, "\partial_1"] \arrow[d, "c"] & \Omega_0^3 \mathrm{Sp}(2) \arrow[d, equal] \\
S^7 \vee S^{10} \arrow[r, "g+h"] & \Omega_0^3 \mathrm{Sp}(2)
\end{tikzcd}$$

where the adjoint of g represents a generator of $\pi_{10}(\mathrm{Sp}(2)) \cong \mathbb{Z}/120\mathbb{Z}$ and the adjoint of h is some element of $\pi_{13}(\mathrm{Sp}(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof

By Lemma 2.3, the triple adjoint of ∂_1 is the Samelson product $S^3 \wedge \mathrm{Sp}(2) \xrightarrow{\langle i,1\rangle} \mathrm{Sp}(2)$. Consider the homotopy fibration $\Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2)) \longrightarrow \mathrm{Sp}(2) \xrightarrow{h} \mathrm{Sp}(\infty)$, where h is the canonical group homomorphism. Since h is a loop map, $h \circ \langle i, 1\rangle$ is homotopic to the Samelson product $\langle h \circ i, h\rangle$. As $\mathrm{Sp}(\infty)$ is an infinite loop space, it is homotopy commutative, and so the commutator $\langle h \circ i, h\rangle$ is null homotopic. Thus there is a lift

$$\begin{tikzcd}
S^3 \wedge \mathrm{Sp}(2) \arrow[rr, "\langle i,1\rangle"] \arrow[dr, "\lambda"'] & & \mathrm{Sp}(2) \\
& \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2)) \arrow[ur] &
\end{tikzcd}$$

for some map λ. Since $\Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2))$ is 6-connected, the composite $S^3 \wedge S^3 \xrightarrow{\Sigma^3 i} S^3 \wedge \mathrm{Sp}(2) \xrightarrow{\lambda} \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2))$ is null homotopic. Thus λ extends through $\Sigma^3 c$ to a map $\lambda' : S^{10} \vee S^{13} \longrightarrow \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2))$. Let a and b be the restrictions of λ' to S^{10} and S^{13}, respectively. The universal property for maps out of a wedge implies that $\lambda' \simeq a + b$. Thus there is a homotopy commutative diagram

$$\text{(1)} \qquad \begin{array}{ccc} S^3 \wedge \mathrm{Sp}(2) & \xrightarrow{\langle i,1\rangle} & \mathrm{Sp}(2) \\ {\scriptstyle \Sigma^3 c}\downarrow & & \uparrow \\ S^{10} \vee S^{13} & \xrightarrow{a+b} & \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2)) \end{array}$$

Let g be the triple adjoint of the composite $S^{10} \xrightarrow{a} \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2)) \longrightarrow \mathrm{Sp}(2)$, and let h be the triple adjoint of the composite $S^{13} \xrightarrow{b} \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2)) \longrightarrow \mathrm{Sp}(2)$. Then the homotopy commutative diagram asserted by the lemma is obtained by adjointing (1).

It remains to show that the triple adjoint of g represents a generator of $\pi_{10}(\mathrm{Sp}(2))$. Let $\ell : S^{10} \longrightarrow \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2))$ be the inclusion of the bottom cell. Then the composite $S^{10} \xrightarrow{\ell} \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2)) \longrightarrow \mathrm{Sp}(2)$ represents a generator of $\pi_{10}(\mathrm{Sp}(2))$. By connectivity, a is homotopic to $t\ell$ for some integer t. Therefore, to show that the adjoint of g represents a generator of $\pi_{10}(\mathrm{Sp}(2))$, it is equivalent to show that $t = \pm 1$. We argue as in [T, Lemma 5.2], which factored the composite $\partial_1 \circ \imath$ rather than ∂_1 itself. Let $\phi : S^7 \longrightarrow \mathrm{Sp}(2)$ represent a generator of $\pi_7(\mathrm{Sp}(2)) \cong \mathbb{Z}$. By [MT], ϕ can be chosen so that the composite $S^7 \xrightarrow{\phi} \mathrm{Sp}(2) \xrightarrow{q} S^7$ has degree 12. Consider the diagram

$$\text{(2)} \qquad \begin{array}{ccccc} S^3 \wedge S^7 & \xrightarrow{1\wedge\phi} & S^3 \wedge \mathrm{Sp}(2) & \xrightarrow{\langle i,1\rangle} & \mathrm{Sp}(2) \\ {\scriptstyle 12}\downarrow & & \downarrow{\scriptstyle \Sigma^3 c} & & \uparrow \\ S^{10} & \xrightarrow{i_1} & S^{10} \vee S^{13} & \xrightarrow{t\ell+b} & \Omega(\mathrm{Sp}(\infty)/\mathrm{Sp}(2)) \end{array}$$

where i_1 is the inclusion. The left square homotopy commutes by connectivity and the fact that $q \circ \phi$ has degree 12. The right square is (1) with $t\ell$ substituted for a. Thus the entire diagram homotopy commutes. The composition $\langle i, 1\rangle \circ (1 \wedge \phi)$ along the top row is the Samelson product $\langle i, \phi\rangle$, which Bott [B] showed to have order a multiple of 10. On the other hand, ℓ represents a generator of $\pi_{10}(\mathrm{Sp}(2)) \cong \mathbb{Z}/120\mathbb{Z}$. So the composite $t\ell' \circ 12$ along the lower direction of the diagram has order $10/t$. The commutativity of the diagram therefore implies that $t = \pm 1$, as required. $\square$

We record a corollary of Proposition 4.1 which is useful later on. Precomposing the diagram in Proposition 4.1 with the inclusion $A \xrightarrow{\imath} \mathrm{Sp}(2)$, we obtain a homotopy commutative square

$$\begin{array}{ccccc} A & \xrightarrow{\imath} & \mathrm{Sp}(2) & \xrightarrow{\partial_1} & \Omega_0^3\,\mathrm{Sp}(2) \\ \Big\downarrow{\scriptstyle \pi} & & & & \Big\| \\ S^7 & & \xrightarrow{\qquad g \qquad} & & \Omega_0^3\,\mathrm{Sp}(2) \end{array}$$

where π is the pinch map to the top cell. By Corollary 2.4, $\partial_k \simeq k \circ \partial_1$. Since A is a co-H-space, the group structure in $[A, \Omega_0^3\,\mathrm{Sp}(2)]$ induced by the loop multiplication on $\Omega_0^3\,\mathrm{Sp}(2)$ is the same as that induced by the co-H-space structure on A. Thus $\partial_k \circ \imath \simeq k \circ \partial_1 \circ \imath \simeq \partial_1 \circ \imath \circ k$. Combining this with the previous diagram and reorienting, we obtain the following.

COROLLARY 4.2

For each $k \in \mathbb{Z}$, there is a homotopy commutative square

$$\begin{array}{ccc} A & \xrightarrow{k\pi} & S^7 \\ \Big\downarrow{\scriptstyle \imath} & & \Big\downarrow{\scriptstyle g} \\ \mathrm{Sp}(2) & \xrightarrow{\partial_k} & \Omega_0^3\,\mathrm{Sp}(2) \end{array}$$

5. Counting $\mathrm{Sp}(2)$-gauge groups

In this section we count the number of homotopy types of Sp(2)-bundles over S^4 by studying the map ∂_k in the homotopy fibration $\mathcal{G}_k \longrightarrow \mathrm{Sp}(2) \xrightarrow{\partial_k} \Omega_0^3\,\mathrm{Sp}(2)$. We begin in Proposition 5.1 by determining that the order of ∂_1 is 40. Note that Corollary 2.4 then implies that the order of $\partial_k \simeq k \circ \partial_1$ is $40/(40, k)$. It is interesting to note that in [CHM] it was shown that $[\mathrm{Sp}(2), \Omega_0^3\,\mathrm{Sp}(2)] \cong \mathbb{Z}/40\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Thus ∂_1 is an element of highest order in $[\mathrm{Sp}(2), \Omega_0^3\,\mathrm{Sp}(2)]$.

PROPOSITION 5.1

The following hold:

(a) *the composite* $A \xrightarrow{\imath} \mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3\,\mathrm{Sp}(2)$ *has order* 40*;*

(b) *the map* $\mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3\,\mathrm{Sp}(2)$ *has order* 40*.*

Proof

We first show that the order of ∂_1 divides 40. This implies that the order of $\partial_1 \circ \imath$ also divides 40. We then show that 40 divides the order of $\partial_1 \circ \imath$, implying that 40 also divides the order of ∂_1. Putting these together, both ∂_1 and $\partial_1 \circ \imath$ have order 40.

By Proposition 4.1, ∂_1 factors through the map $S^7 \vee S^{10} \xrightarrow{g+h} \Omega_0^3\,\mathrm{Sp}(2)$, where g represents a generator of $\pi_{10}(\mathrm{Sp}(2)) \cong \mathbb{Z}/120\mathbb{Z}$ and h represents some element in $\pi_{13}(\mathrm{Sp}(2)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In particular, $g + h$ has order 120. Therefore the order of ∂_1 divides $120 = 2^3 \cdot 3 \cdot 5$. On the other hand, by Lemma 2.3, the triple adjoint of ∂_1 is the Samelson product $S^3 \wedge \mathrm{Sp}(2) \xrightarrow{k\langle i,1\rangle} \mathrm{Sp}(2)$. By [Mc], $\mathrm{Sp}(2)$ is homotopy commutative at the prime 3, and so $\langle i,1\rangle$ is null homotopic when spaces and maps are localized at 3. This implies that the order of $\langle i,1\rangle$ is not divisible by 3. Adjointing, the order of ∂_1 is not divisible by 3. Thus the order of ∂_1 divides 40.

Next, to show that 40 divides the order of $\partial_1 \circ \imath$, we show that both 5 and 8 divide the order of $\partial_1 \circ \imath$. By Lemma 2.5, the composite $S^7 \xrightarrow{\phi'} A \xrightarrow{\imath} \mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3 \mathrm{Sp}(2)$ has order 10. In particular, 5 divides the order of $\partial_1 \circ \imath \circ \phi'$, implying that 5 divides the order of $\partial_1 \circ \imath$. The same argument also shows that 2 divides the order of $\partial_1 \circ \imath$. However, to show that 8 divides the order of $\partial_1 \circ \imath$, we have to work harder.

Suppose that the order of $\partial_1 \circ \imath$ is m. Recall that there is a homotopy cofibration sequence $S^6 \xrightarrow{f} S^3 \longrightarrow A \xrightarrow{\pi} S^7$, where f is a stable class of order 12. Consider the diagram

$$\begin{tikzcd}
S^3 \wedge A \arrow[r, "m"] \arrow[d, "\Sigma^3\pi"] & S^3 \wedge A \arrow[r, "1\wedge\imath"] \arrow[d, "\Sigma^3\pi"] & S^3 \wedge \mathrm{Sp}(2) \arrow[r, "\langle i,1\rangle"] & \mathrm{Sp}(2) \arrow[d, equal] \\
S^{10} \arrow[r, "m"] & S^{10} \arrow[rr, "g"] & & \mathrm{Sp}(2)
\end{tikzcd}$$

Since $\Sigma^3\pi$ is a co-H-map, it commutes with degree maps, implying that the left square homotopy commutes. The right rectangle is obtained by adjointing the $k=1$ case of Corollary 4.2 and using Lemma 2.3 to identify the adjoint of ∂_1 as $\langle i,1\rangle$. Since the order of $\partial_1 \circ \imath$ is assumed to be m, its adjoint $\langle i,1\rangle \circ (1 \wedge \imath)$ also has order m. That is, the composite $\langle i,1\rangle \circ (1 \wedge \imath) \circ m$ is null homotopic. The homotopy commutativity of the previous diagram therefore implies that $g \circ m \circ \Sigma^3\pi$ is null homotopic.

The null homotopy for $g \circ m \circ \Sigma^3\pi$ implies, with respect to the homotopy cofibration $S^3 \wedge A \xrightarrow{\Sigma^3\pi} S^{10} \xrightarrow{\Sigma^4 f} S^7$, that $g \circ m$ extends through $S^{10} \xrightarrow{\Sigma^4 f} S^7$ to a map $l : S^7 \longrightarrow \mathrm{Sp}(2)$. This gives the homotopy commutativity of the square in the following diagram:

$$\begin{tikzcd}
S^{10} \arrow[r, "m"] \arrow[d, "\Sigma^4 f"] & S^{10} \arrow[d, "g"] \arrow[dr, "\overline{\nu}"] & \\
S^7 \arrow[r, "l"] & \mathrm{Sp}(2) \arrow[r, "q"] & S^7
\end{tikzcd}$$

The map $\overline{\nu}$ is defined as $q \circ g$, so the triangle homotopy commutes as well. By [MT], $\overline{\nu}$ is a class of order 8. Thus $\overline{\nu} \circ m$ has order $8/(8,m)$. The homotopy commutativity of the preceding diagram therefore implies that $q \circ l \circ \Sigma^4 f$ has order $8/(8,m)$. Suppose that $(8,m) \neq 8$. Then the order of $q \circ l \circ \Sigma^4 f$ is not 1, and so this composite is nontrivial. This implies that l is nontrivial. Let

$\phi : S^7 \longrightarrow \mathrm{Sp}(2)$ represent a generator of $\pi_7(\mathrm{Sp}(2)) \cong \mathbb{Z}$. The nontriviality of l implies that $l = n\phi$ for some nonzero integer n. By [MT], $q \circ \phi$ is of degree 12, so $q \circ l$ is of degree $12n$. Thus, as $\Sigma^4 f$ has order 12, $q \circ l \circ \Sigma^4 f$ is null homotopic. That is, $q \circ l \circ \Sigma^4 f$ has order 1, a contradiction. Hence $(8, m) = 8$. Since m is the order of $\partial_1 \circ \imath$, we have therefore shown that 8 divides the order of $\partial_1 \circ \imath$, as asserted. $\square$

Applying Lemma 3.1 to the map $\mathrm{Sp}(2) \xrightarrow{\partial_1} \Omega_0^3 \mathrm{Sp}(2)$, which we now know is of order 40, we immediately obtain the following.

PROPOSITION 5.2

If $(40, k) = (40, k')$ then $\mathcal{G}_k$ and $\mathcal{G}_{k'}$ are homotopy equivalent when localized rationally or at any prime.

REMARK

The referee has pointed out that the methods in [HKK] may potentially provide a different means of proving Proposition 5.2.

Next, we show that a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ implies that $(40, k) = (40, k')$. Lemma 2.6 gives a weaker result: if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$, then $(10, k) = (10, k')$. To improve from $(10, k) = (10, k')$ to $(40, k) = (40, k')$, we pass from the homotopy group calculation in Lemma 2.6, involving an S^6 mapping into $\mathcal{G}_k$, to a certain three-cell complex mapping into $\mathcal{G}_k$ which captures more information.

For $k \in \mathbb{Z}$, define the space C_k and the maps s_k and i_k by the homotopy pushout

$$\begin{array}{ccc} S^6 & \xrightarrow{f} & S^3 \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i_k} \\ S^6 & \xrightarrow{s_k} & C_k \end{array}$$

where f is the attaching map for A. This induces a homotopy cofibration diagram

$$\begin{array}{ccccccc} S^6 & \xrightarrow{f} & S^3 & \longrightarrow & A & \xrightarrow{\pi} & S^7 \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i_k} & & \| & & \downarrow{\scriptstyle k} \\ S^6 & \xrightarrow{s_k} & C_k & \longrightarrow & A & \xrightarrow{k\pi} & S^7 \end{array}$$

The focus is on the homotopy cofibration along the bottom row.

We construct a map $C_k \longrightarrow \mathcal{G}_k$ which is compatible with a map $S^6 \longrightarrow \Omega_0^4\,\mathrm{Sp}(2)$ representing a generator of $\pi_6(\Omega_0^4\,\mathrm{Sp}(2)) \cong \mathbb{Z}/120\mathbb{Z}$. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "k\pi"] \arrow[d, "\imath"] & S^7 \arrow[r, "\Sigma s_k"] \arrow[d, "g"] & \Sigma C_k \arrow[r] \arrow[d, dashed, "e_k"] & \Sigma A \arrow[d, "\bar{\imath}"] \\ \mathrm{Sp}(2) \arrow[r, "\partial_k"] & \Omega_0^3\,\mathrm{Sp}(2) \arrow[r, "b_k"] & B\mathcal{G}_k \arrow[r] & B\,\mathrm{Sp}(2) \end{tikzcd} \tag{3}$$

where e_k is defined momentarily and $\bar{\imath}$ is the adjoint of $\imath$. The left square homotopy commutes by Corollary 4.2. The homotopy commutativity of this square combined with the fact that the bottom row is a homotopy fibration implies that the composite $b_k \circ g \circ k\pi$ is null homotopic. Thus there is an extension of $b_k \circ g$ through Σs_k, which defines the map e_k and makes the middle square homotopy commute. By [H, Ch. 3], the extension can be chosen so that the right square homotopy also commutes.

Since Σs_k is a suspension, the middle square in (3) can be adjointed to obtain a homotopy commutative diagram

$$\begin{tikzcd} S^6 \arrow[r, "s_k"] \arrow[d, "g^a"] & C_k \arrow[d, "e_k^a"] \\ \Omega_0^4\,\mathrm{Sp}(2) \arrow[r, "\Omega b_k"] & \mathcal{G}_k \end{tikzcd}$$

where g^a and e_k^a are the adjoints of g and e_k, respectively. Since g^a represents a generator of $\pi_6(\Omega_0^4\,\mathrm{Sp}(2))$, the adjoint of Lemma 2.6 implies the following.

LEMMA 5.3

The composite $S^6 \xrightarrow{s_k} C_k \xrightarrow{e_k^a} \mathcal{G}_k$ *represents a generator of* $\pi_6(\mathcal{G}_k) \cong \mathbb{Z}/12(10,k)\mathbb{Z}$.

We need to establish some additional properties of the map e_k^a.

LEMMA 5.4

The composite $S^3 \xrightarrow{i_k} C_k \xrightarrow{e_k^a} \mathcal{G}_k \longrightarrow \mathrm{Sp}(2)$ *is homotopic to the inclusion of the bottom cell.*

Proof

It is equivalent to adjoint and show that the composite $S^4 \longrightarrow \Sigma C_k \xrightarrow{e_k} B\mathcal{G}_k \longrightarrow B\,\mathrm{Sp}(2)$ is the inclusion of the bottom cell. The right square in (3) implies that this composite is homotopic to the composite $S^4 \xrightarrow{j} \Sigma A \xrightarrow{\bar{\imath}} B\,\mathrm{Sp}(2)$, where j is the inclusion of the bottom cell. Since $\imath$ is a skeletal inclusion, it is the inclusion on the bottom cell, and therefore its adjoint $\bar{\imath}$ is also an inclusion on the bottom cell. Hence $\bar{\imath} \circ j$ is the inclusion of the bottom cell. $\square$

Consider the map $\mathcal{G}_k \longrightarrow \mathrm{Sp}(2)$. Lemma 5.4 implies that the bottom cell of $\mathrm{Sp}(2)$ lifts to $\mathcal{G}_k$, and one choice of a lift is given by $e^a_k \circ i_k$. We now use this lift to give a choice of a low-dimensional homotopy decomposition of $\mathcal{G}_k$. Let μ be the loop multiplication on $\mathcal{G}_k$.

LEMMA 5.5

The composite

$$S^3 \times \Omega^4_0 \mathrm{Sp}(2) \xrightarrow{(e^a_k \circ i_k) \times \Omega b_k} \mathcal{G}_k \times \mathcal{G}_k \xrightarrow{\quad \mu \quad} \mathcal{G}_k$$

is a homotopy equivalence in dimensions ≤ 5.

Proof

The 6-skeleton of $\mathrm{Sp}(2)$ is S^3. Therefore taking 6-skeletons in the homotopy fibration $\Omega^4_0 \mathrm{Sp}(2) \xrightarrow{\Omega b_k} \mathcal{G}_k \longrightarrow \mathrm{Sp}(2)$ we obtain a sequence $(\Omega^4_0 \mathrm{Sp}(2))_6 \xrightarrow{\Omega b_k} (\mathcal{G}_k)_6 \longrightarrow S^3$ which induces a long exact sequence in homotopy groups in dimensions ≤ 5. Thus, as $e^a_k \circ i_k$ is a right homotopy inverse of the map $(\mathcal{G}_k)_6 \longrightarrow S^3$, the composite $\mu \circ ((e^a_k \circ i_k) \times \Omega b_k)$ in the statement of the lemma induces an isomorphism in homotopy groups in dimensions ≤ 5. The lemma follows. $\square$

By Corollary 2.4, $\partial_k \simeq k \circ \partial_1$, and by Proposition 5.1, ∂_1 has finite order. Thus ∂_k has finite order. Therefore in the homotopy fibration $\Omega^4_0 \mathrm{Sp}(2) \xrightarrow{\Omega b_k} \mathcal{G}_k \longrightarrow \mathrm{Sp}(2)$, we have $\pi_3(\mathcal{G}_k) \cong \mathbb{Z} \oplus \mathbb{Z}$ with one summand coming from $\pi_3(\Omega^4_0 \mathrm{Sp}(2)) \cong \mathbb{Z}$ and the other coming from $\pi_3(\mathrm{Sp}(2)) \cong \mathbb{Z}$. If there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$, it may be possible that the induced isomorphism in π_3 interchanges the $\mathbb{Z}$ summands, so that the composite $S^3 \xrightarrow{e^a_k \circ i_k} \mathcal{G}_k \xrightarrow{e} \mathcal{G}_{k'} \longrightarrow \mathrm{Sp}(2)$ is null homotopic. In the following lemma we show that this cannot occur. Let $\mathbb{Z}_{(2)}$ be the 2-local integers.

LEMMA 5.6

Suppose that there is a homotopy equivalence $\mathcal{G}_k \xrightarrow{e} \mathcal{G}_{k'}$. *Then* 2*-locally, the composite* $S^3 \xrightarrow{e^a_k \circ i_k} \mathcal{G}_k \xrightarrow{e} \mathcal{G}_{k'} \longrightarrow \mathrm{Sp}(2)$ *has degree* u, *where* u *is a unit in* $\mathbb{Z}_{(2)}$.

Proof

By Lemma 5.5, there is an isomorphism $\pi_4(\mathcal{G}_k) \cong \pi_4(S^3) \oplus \pi_4(\Omega^4_0 \mathrm{Sp}(2))$. It is well known that $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, and by Lemma 2.2, $\pi_4(\Omega^4_0 \mathrm{Sp}(2)) = 0$. Thus $\pi_4(\mathcal{G}_k) \cong \pi_4(S^3)$. Further, the homotopy decomposition in Lemma 5.5 implies that the maps $S^3 \xrightarrow{e^a_k \circ i_k} \mathcal{G}_k$ and $\mathcal{G}_k \longrightarrow \mathrm{Sp}(2)$ induce isomorphisms on π_4. Similarly, $\pi_4(\mathcal{G}_{k'}) \cong \pi_4(S^3)$, and the map $\mathcal{G}_{k'} \longrightarrow \mathrm{Sp}(2)$ induces an isomorphism on π_4. Since e is a homtopy equivalence, it too induces an isomorphism on π_4. Thus the composite $S^3 \xrightarrow{e^a_k \circ i_k} \mathcal{G}_k \xrightarrow{e} \mathcal{G}_{k'} \longrightarrow \mathrm{Sp}(2)$ is an isomorphism on π_4. Restricting to 5-skeletons, we obtain a self-map $\gamma : S^3 \longrightarrow S^3$ which induces an isomorphism on $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore the degree of γ cannot be divisible by 2. In other words, 2-locally γ has degree u, where u is a unit in $\mathbb{Z}_{(2)}$. Hence 2-locally $e\circ e_k^a\circ i_k$ has degree u. $\square$

Now suppose that there is a homotopy equivalence $e:\mathcal{G}_k\longrightarrow\mathcal{G}_{k'}$. Consider the composite $C_k\xrightarrow{e_k^a}\mathcal{G}_k\xrightarrow{e}\mathcal{G}_{k'}$. By Lemma 5.3, the composite $S^6\xrightarrow{s_k}C_k\xrightarrow{e_k^a}\mathcal{G}_k$ represents a generator of $\pi_6(\mathcal{G}_k)$. Since e is a homotopy equivalence, it induces an isomorphism in homotopy groups. Thus $e\circ e_k^a\circ s_k$ represents a generator of $\pi_6(\mathcal{G}_{k'})$. Adjointing the map in Lemma 2.6, there is an epimorphism $\pi_6(\Omega_0^4\,\mathrm{Sp}(2))\xrightarrow{(\Omega b_{k'})_*}\pi_6(\mathcal{G}_{k'})$. Thus $e\circ e_k^a\circ s_k$ lifts through Ωb_k, giving a homotopy commutative square

$$\begin{tikzcd}
S^6 \arrow[r, "s_k"] \arrow[d, "g'"] & C_k \arrow[d, "e\circ e_k^a"] \\
\Omega_0^4\,\mathrm{Sp}(2) \arrow[r, "\Omega b_{k'}"] & \mathcal{G}_{k'}
\end{tikzcd}$$

for some map g' which represents a generator of $\pi_6(\Omega_0^4\,\mathrm{Sp}(2))\cong\mathbb{Z}/120\mathbb{Z}$. Arguing as for (3), this square induces a homotopy commutative diagram

$$\begin{tikzcd}
S^6 \arrow[r, "s_k"] \arrow[d, "g'"] & C_k \arrow[r] \arrow[d, "e\circ e_k^a"] & A \arrow[r, "k\pi"] \arrow[d, "\lambda"] & S^7 \arrow[d, "\bar{g}'"] \\
\Omega_0^4\,\mathrm{Sp}(2) \arrow[r, "\Omega b_{k'}"] & \mathcal{G}_{k'} \arrow[r] & \mathrm{Sp}(2) \arrow[r, "\partial_{k'}"] & \Omega_0^3\,\mathrm{Sp}(2)
\end{tikzcd} \tag{4}$$

for some map λ, where $\bar{g}'$ is the adjoint of g'. Since A has dimension 7, λ factors through the 7-skeleton of $\mathrm{Sp}(2)$, which is A. Thus λ is homotopic to a composite $A\xrightarrow{\lambda'}A\xrightarrow{\imath}\mathrm{Sp}(2)$ for some map λ'.

LEMMA 5.7

The map $A\xrightarrow{\lambda'}A$ is a homotopy equivalence when localized at 2.

Proof

The homotopy pushout defining C_k implies that the composite $i_A:S^3\xrightarrow{i_k}C_k\longrightarrow A$ is the inclusion of the bottom cell. Thus Lemma 5.6 and the homotopy commutativity of the middle square in (4) imply that the composite $S^3\xrightarrow{i_A}A\xrightarrow{\lambda}\mathrm{Sp}(2)$ has degree u, where u is a unit in $\mathbb{Z}_{(2)}$. Therefore the composite $S^3\xrightarrow{i_A}A\xrightarrow{\lambda'}A$

also has degree u. From this we obtain a homotopy fibration diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\theta"] & S^3 \arrow[r, "i_A"] \arrow[d, "u"] & A \arrow[d, "\lambda'"] \\ F \arrow[r] & S^3 \arrow[r, "i_A"] & A \end{tikzcd}$$

which defines the space F and the map θ. A Serre spectral sequence calculation implies that the 8-skeleton of F is S^6. Note that the composite $S^6 \longrightarrow F \longrightarrow S^3$ is f, the attaching map for A. By the Serre exact sequence, when the homotopy fibration $F \longrightarrow S^3 \longrightarrow A$ is restricted to 8-skeletons, we obtain a homotopy cofibration $S^6 \xrightarrow{f} S^3 \longrightarrow A$. Moreover, when the map of fibration sequences in the previous diagram is restricted to 8-skeletons, we obtain a map of cofibration sequences. That is, there is a homotopy cofibration diagram

$$\begin{tikzcd} S^6 \arrow[r, "f"] \arrow[d, "\theta'"] & S^3 \arrow[r, "i_A"] \arrow[d, "u"] & A \arrow[d, "\lambda'"] \\ S^6 \arrow[r, "f"] & S^3 \arrow[r, "i_A"] & A \end{tikzcd} \tag{5}$$

where θ' is the restriction of θ to F_8.

Observe that as S^3 is an H-space, the group structure in $[S^6, S^3]$ induced by the co-H-structure on S^6 is the same as that induced by the group structure on S^3. Thus $u \circ f \simeq f \circ u$. Therefore $f \circ (u - \theta')$ is null homotopic. Let G be the homotopy fiber of f. The null homotopy for $f \circ (u - \theta')$ implies that there is a lift

$$\begin{tikzcd} & S^6 \arrow[dl, "\psi"'] \arrow[d, "u-\theta'"] & \\ G \arrow[r, "\gamma"] & S^6 \arrow[r, "f"] & S^3 \end{tikzcd}$$

for some map ψ. Observe that in dimensions ≤ 10 the homotopy fibration $G \xrightarrow{\gamma} S^6 \xrightarrow{f} S^3$ is identical to the homotopy fibration $\Omega\,\mathrm{Sp}(2) \xrightarrow{\Omega q} \Omega S^7 \longrightarrow S^3$ because the fibration connecting map $\Omega S^7 \longrightarrow S^3$ is f through dimension 11. In particular, $\pi_6(G) \cong \pi_6(\Omega\,\mathrm{Sp}(2)) \cong \mathbb{Z}$. Let ϕ be a generator of $\pi_6(\Omega\,\mathrm{Sp}(2))$. By [MT], $\Omega q \circ \phi$ has degree 4 (2-locally). As $\psi \simeq n\phi$ for some integer n, we have $\gamma \circ \psi \simeq 4n$. That is, $u - \theta' \simeq 4n$. This implies that in mod-2 homology, θ' and u induce the same map. Since u is a unit in $\mathbb{Z}_{(2)}$, it induces the degree 1 map in mod-2 homology, and hence so does θ'. Thus when mod-2 homology is applied to the homotopy cofibration diagram in (5), the 5-lemma implies that $(\lambda')_*$ is an isomorphism. Hence the 2-local version of Whitehead's theorem implies that λ' is a homotopy equivalence. $\square$

PROPOSITION 5.8

If $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $(40,k)=(40,k')$.

Proof

By Lemma 2.6, if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $(10,k)=(10,k')$. In particular, $(5,k)=(5,k')$. We will show that such a homotopy equivalence also implies that $(8,k)=(8,k')$. Hence $(40,k)=(40,k')$.

Consider the right square in (4),

$$\begin{tikzcd} A \arrow[r, "k\pi"] \arrow[d, "\lambda"] & S^7 \arrow[d, "\bar{g}'"] \\ \mathrm{Sp}(2) \arrow[r, "\partial_{k'}"] & \Omega_0^3\,\mathrm{Sp}(2) \end{tikzcd}$$

By Lemma 5.7, λ is homotopic to $\imath$ up to a self-equivalence of A. Therefore Proposition 5.1 implies that $\partial_1 \circ \lambda$ has order 8 (2-locally). By Corollary 2.4, $\partial_{k'} \simeq k' \circ \partial_1$. Thus $\partial_{k'} \circ \lambda$ has order $8/(8,k')$. On the other hand, by Corollary 4.2, the composite $A \xrightarrow{k\pi} S^7 \xrightarrow{\bar{g}'} \Omega_0^3\,\mathrm{Sp}(2)$ is homotopic to the composite $A \xrightarrow{\imath} \mathrm{Sp}(2) \xrightarrow{\partial_k} \Omega_0^3\,\mathrm{Sp}(2)$. Since $\partial_k \simeq k \circ \partial_1$ and $\partial_1 \circ \imath$ has order 8, $\partial_k \circ \imath$ has order $8/(8,k)$. Thus $\bar{g}' \circ kq$ has order $8/(8,k)$. Since $\partial_{k'} \circ \lambda$ and $\bar{g}' \circ kq$ are homotopic, they have the same order. Hence $8/(8,k')=8/(8,k)$, implying that $(8,k')=(8,k)$, as asserted. $\square$

Proof of Theorem 1.1

Combine Propositions 5.2 and 5.8. $\square$

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