Malliavin calculus for stochastic differential equations driven by subordinated Brownian motions

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Abstract Malliavin calculus is applicable to functionals of stable processes by using subordination. We prepare Malliavin calculus for stochastic differential equations driven by Brownian motions with deterministic time change, and the conditions that the existence and the regularity of the densities inherit from those of the densities of conditional probabilities. By using these, we prove regularity properties of the solutions of equations driven by subordinated Brownian motions. In [4] a similar problem is considered. In this article we consider more general cases. We also consider equations driven by rotation-invariant stable processes. We prove that the ellipticity of the equations implies the existence of the density of the solution, and we also prove that the regularity of the coefficients implies the regularity of the densities in the case when the equations are driven by one rotation-invariant stable process.

1. Introduction

Malliavin calculus is well known as a method to know regularity properties of distributions of solutions of stochastic differential equations driven by Brownian motions, and we can see that the densities of the solution have the regularity according to the smoothness of the coefficients of equations (see [3], [8]). There is a natural interest in applying Malliavin calculus the equation driven by stable processes. Consider the following N-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \sum_{k=1}^{r} \sigma_k(t, X(t-))\, dZ_k(t) + b(t, X(t))\, dt, \\ X(0) = x_0, \end{cases}$$

where $\{Z_k\}$ are independent rotation-invariant stable processes, and the coefficients are Lipschitz continuous. The indexes of the stable processes may be different. The definition of the stochastic integral can be found in [2], and the precise idea of the definition is given in Section 5. If the equation has some conditions about ellipticity, it seems that the distribution of the solution has its density function at each time.

Kyoto Journal of Mathematics, Vol. 50, No. 3 (2010), 491–520
DOI 10.1215/0023608X-2010-003,
Received September 30, 2009. Revised January 25, 2010. Accepted January 25, 2010.
Mathematics Subject Classification: 60H07
Author's work supported by Japan Society for the Promotion of Science research fellowship.

On the other hand, there is Malliavin calculus for Lévy processes (see [6]). The method works in mathematical finance very well. However, the theory is not applicable to the problem concerned. So, another idea is needed for the problem that concerns us here.

By using subordination, the classical formulation of Malliavin calculus is applicable to functionals of rotation-invariant stable processes. This method enables us to prove that the ellipticity of the stochastic differential equation driven by subordinated Brownian motions implies existence of the density of the solution. We can find Malliavin calculus for equations driven by subordinated Brownian motions in [4]. In [4] the case when the number of subordinators is one and the subordinator is an increasing Lévy process with some condition is considered.

In this article we consider the case including that the number of subordinators is more than one and the subordinators are not necessarily increasing Lévy processes. We prove our theorems in a way similar to [8] and show regularity properties of distributions of solutions of equations driven by subordinated Brownian motions. The proof consists of two parts. One is Malliavin calculus for stochastic differential equations driven by Brownian motions with deterministic time change, and the other is the inheritance of the regularity of densities from those of conditional probabilities. That is because the discussion is simplified by considering the equation under the conditional probability given by the σ-field generated by the subordinators. So we make two steps to prove it. In the last section, we consider the case of stochastic differential equations driven by stable processes. We show that the ellipticity of equations driven by stable processes implies existence of the density of the solution. Moreover, in the case $r=1$, we can also prove the regularity of the density according to the regularity of the coefficients.

In Section 2, we prepare the techniques for calculating the integrals with deterministic time change. The techniques enable us to use the deduction similar to the standard stochastic calculus. In Section 3, we discuss Malliavin calculus for stochastic differential equations with deterministic time change. In Section 4, we discuss the inheritance of regularity of densities from those of conditional probabilities. That is the reason why we consider stochastic differential equations with deterministic time change. In Section 5, we discuss the general results from Sections 3 and 4. In Section 6, we discuss the most interesting example: stochastic differential equations driven by rotation-invariant stable processes.

In the proofs in the article, we use $\{C_j; j=0,1,2,\dots\}$ as positive constants and the dependent parameters are described like $C_0(p)$.

2. Malliavin calculus for the functional of Brownian motions with deterministic time change

We fix a positive number T and let ϕ be a right-continuous increasing function on $[0,T]$ with $\phi(0)=0$. We define the inverse function ϕ^{-1} by

$$\phi^{-1}(s) := \begin{cases} \inf\{t; \phi(t) > s\} & \text{if } s \in [0, \phi(T)), \\ T & \text{if } s = \phi(T). \end{cases}$$

Set

$$\begin{aligned} W &:= C\big([0, \phi(T)]; \mathbf{R}^d\big), \\ H &:= \big\{h \in C\big([0, \phi(T)]; \mathbf{R}^d\big);\ h \text{ is absolutely continuous and} \\ &\qquad\qquad\qquad\qquad\qquad \dot{h} \in L^2\big([0, \phi(T)]; \mathbf{R}^d\big)\big\}, \end{aligned}$$

and let μ be the Wiener measure on W. Then the triplet (W, H, μ) is an abstract Wiener space. Hence, we can apply Malliavin calculus to the functionals on (W, H, μ). Let $(B(t))$ be the canonical d-dimensional Brownian motion associated to (W, H, μ), let $\mathscr{F}_t$ be the σ-field generated by $(B(s); 0 \le s \le \phi(t))$, let D be the H-derivative operator, and let D_h be the differential for direction h for each $h \in H$. Then, for all $h \in H$ we have

$$D_h B\big(\phi(t)\big) = h\big(\phi(t)\big), \quad t \in [0, T],$$

$$D_h \int_0^T f(t)\, dB\big(\phi(t)\big) = \int_0^T f(t)\, dh\big(\phi(t)\big), \quad f \in C([0, T]).$$

Here the integral of the left-hand side is in the sense of stochastic integrals by $(\mathscr{F}_t)$-martingales, and that of the right-hand side is in the sense of Stieltjes integrals. More generally, we have an analogue of [8, Proposition 6.1]. We need some lemmas and some notation before we state the analogue.

LEMMA 2.1

Let f be a right-continuous function with left limits. Then, we have

$$\int_0^T f(t-)\, d\phi(t) = \int_0^{\phi(T)} f\big(\phi^{-1}(s)-\big)\, ds.$$

Proof

Since ϕ is a function of bounded variation, the contribution of the small jumps for the integrals are sufficiently small. So we assume that the number of the jumps of ϕ is finite. Let $\{\xi_i; i = 1, 2, \dots, N-1\}$ be the discontinuous points of ϕ, $\xi_0 := 0$, $\xi_N := T$, and let $\{t_{i,j}; j = 0, 1, 2, \dots, N_i\}$ be a partition of $[\xi_{i-1}, \xi_i]$ for $i = 1, 2, \dots, N$. We denote $\max_{i,j}(t_{i,j} - t_{i,j-1})$ by Δ. Then,

$$\begin{aligned} &\int_0^T f(t-)\, d\phi(t) \\ &\quad = \lim_{\Delta \to 0} \sum_{i=1}^{N} \Big[\sum_{j=1}^{N_i - 1} f(t_{i,j-1}-)\big\{\phi(t_{i,j}) - \phi(t_{i,j-1})\big\} \\ &\qquad + f(t_{i,N_i-1}-)\big\{\phi(t_{i,N_i}-) - \phi(t_{i,N_i-1})\big\} + f(t_{i,N_i}-)\big\{\phi(t_{i,N_i}) - \phi(t_{i,N_i}-)\big\}\Big]. \end{aligned}$$

If we set $s_{i,j} := \phi(t_{i,j})$, then $\phi^{-1}(s_{i,j}) = t_{i,j}$ for $j = 0, 1, 2, \ldots, N_i - 1$. Therefore,

$$\begin{aligned}
&\int_0^T f(t-)\, d\phi(t) \\
&\quad = \lim_{\Delta \to 0} \sum_{i=1}^{N} \Bigl[\sum_{j=1}^{N_i - 1} f\bigl(\phi^{-1}(s_{i,j-1})-\bigr)\{s_{i,j} - s_{i,j-1}\} \\
&\qquad\qquad + f\bigl(\phi^{-1}(s_{i,N_i-1})-\bigr)\bigl\{\phi(t_{i,N_i}-) - s_{i,N_i-1}\bigr\} \\
&\qquad\qquad + f\bigl(\phi^{-1}(s_{i,N_i})-\bigr)\bigl\{\phi(t_{i,N_i}) - \phi(t_{i,N_i}-)\bigr\}\Bigr] \\
&\quad = \sum_{i=1}^{N} \int_{\phi(\xi_{i-1})}^{\phi(\xi_i -)} f\bigl(\phi^{-1}(s)-\bigr)\, ds + \sum_{i=1}^{N} f\bigl(\phi^{-1}(s_{i,N_i})-\bigr)\bigl\{\phi(\xi_i) - \phi(\xi_i -)\bigr\}.
\end{aligned}$$

Since $\phi^{-1}(s)$ is a constant on $s \in [\phi(\xi_i -), \phi(\xi_i)]$,

$$f\bigl(\phi^{-1}(s_{i,N_i})-\bigr)\bigl\{\phi(\xi_i) - \phi(\xi_i -)\bigr\} = \int_{\phi(\xi_i -)}^{\phi(\xi_i)} f\bigl(\phi^{-1}(s)-\bigr)\, ds, \quad i = 1, 2, \ldots, N.$$

Thus, we have

$$\int_0^T f(t-)\, d\phi(t) = \int_0^{\phi(T)} f\bigl(\phi^{-1}(s)-\bigr)\, ds.$$

□

Similarly, we also have the following lemma.

LEMMA 2.2

Let Ψ be an $(\mathscr{F}_t)$-adapted right-continuous process with left limits satisfying

$$E\Bigl[\int_0^T |\Psi(s-)|^2\, d\phi(s)\Bigr] < \infty.$$

Then, we have

$$\int_0^T \Psi(t-)\, dB\bigl(\phi(t)\bigr) = \int_0^{\phi(T)} \Psi\bigl(\phi^{-1}(s)-\bigr)\, dB(s) \quad \textit{almost surely.}$$

Here the integral of the left-hand side is in the sense of stochastic integrals by $(\mathscr{F}_t)$-martingales and that of the right-hand side is in the sense of stochastic integrals by $(\mathscr{F}_t^B)$-martingales, where $(\mathscr{F}_t^B)$ is the σ-field generated by $(B_s; 0 \le s \le t)$.

Let $A(t)$ be $[B(\phi(\cdot)), B(\phi(\cdot))](t)$ where the definition of $[\cdot,\cdot]$ is as in [7, Chapter II, Section 6]. We show the following lemma, which is a version of Burkholder's inequality (see [1, Chapter VII, Theorem 92]).

LEMMA 2.3

Let p be a positive number, and let Ψ be an $(\mathscr{F}_t)$-adapted right-continuous process *with left limits satisfying*

$$E\Big[\int_0^T |\Psi(s-)|^2\, d\phi(s)\Big] < \infty.$$

Then, we have

$$\begin{aligned}
&\Big[\Big(\int_0^T \Psi(s-)^2\, dA(s)\Big)^{p/2}\Big] \\
&\quad \le C_0(p) E\Big[\sup_{0\le t\le T}\Big|\int_0^t \Psi(s-)\, dB\big(\phi(s)\big)\Big|^p\Big] \\
&\quad \le C_1(p) E\Big[\Big(\int_0^T \Psi(s-)^2\, d\phi(s)\Big)^{p/2}\Big].
\end{aligned}$$

Proof

Theorem 92 of Chapter VII of [1] implies the first estimate. Hence, we prove the second estimate. Let

$$M(t) := \int_0^t \Psi\big(\phi^{-1}(s)-\big)\, dB(s).$$

Then M is a continuous martingale. By Burkholder's inequality, Lemma 2.1, and Lemma 2.2, with a constant $C(p)$ we have

$$\begin{aligned}
E\Big[\sup_{0\le t\le T}\Big|\int_0^t \Psi(s-)\, dB\big(\phi(s)\big)\Big|^p\Big] &= E\Big[\sup_{0\le t\le T}\big|M\big(\phi(t)\big)\big|^p\Big] \\
&\le E\Big[\sup_{0\le t\le \phi(T)} |M(t)|^p\Big] \\
&\le C(p) E\big[\langle M\rangle\big(\phi(T)\big)^{p/2}\big] \\
&= C(p) E\Big[\Big(\int_0^{\phi(T)} \Psi\big(\phi^{-1}(s)-\big)^2\, ds\Big)^{p/2}\Big] \\
&= C(p) E\Big[\Big(\int_0^T \Psi(s-)^2\, d\phi(s)\Big)^{p/2}\Big].
\end{aligned}$$

□

The next lemma is a version of Gronwall's inequality.

LEMMA 2.4

Let α, β be positive constants, and let f be a right-continuous positive function on $[0, T]$ with left limits. If

$$f(t) \le \alpha + \beta \int_0^t f(u-)\, d\phi(u), \quad t \in [0, T],$$

then

$$f(t) \le \alpha e^{\beta\phi(t)}, \quad t \in [0, T].$$

Proof
Since $\phi^{-1}(\phi(t)) = t$, it follows from Lemma 2.1 that

$$f\big(\phi^{-1}(\phi(t))\big) \leq \alpha + \beta \int_0^{\phi(t)} f\big(\phi^{-1}(u)\big)\, du, \quad t \in [0,T].$$

So we have

$$f\big(\phi^{-1}(\phi(t-))\big) \leq \alpha + \beta \int_0^{\phi(t-)} f\big(\phi^{-1}(u)\big)\, du, \quad t \in [0,T].$$

If $s \in [\phi(t-), \phi(t)]$, $\phi^{-1}(s) = \phi^{-1}(\phi(t-))$. Hence we have

$$f\big(\phi^{-1}(s)\big) \leq \alpha + \beta \int_0^{s} f\big(\phi^{-1}(u)\big)\, du, \quad s \in [0,\phi(T)].$$

Applying Gronwall's inequality to $f \circ \phi^{-1}$, we have that

$$f\big(\phi^{-1}(s)\big) \leq \alpha e^{\beta s}, \quad s \in [0,\phi(T)].$$

Therefore, we have the conclusion by letting $s = \phi(t)$ and the equality $\phi^{-1}(\phi(t)) = t$. □

Next, we prepare some notation. Let $p > 1$, let n be a positive integer, let K be a Hilbert space, let $W^{n,p}(K)$ be the Sobolev space of K-valued functions associated to H-derivative with indexes n and p, and let $\mathscr{L}_2^n(H;K)$ be the total set of K-valued n-linear operators of Hilbert-Schmidt class on $\overbrace{H \times \cdots \times H}^{n}$. Now we give two classes of stochastic processes. We define $\mathcal{L}^{n,p}(dB(\phi);K)$ by the total set of $(\mathscr{F}_t)$-predictable $(\mathbf{R}^d \otimes K)$-valued functions α satisfying $\alpha(t) \in W^{n,p}(\mathbf{R}^d \otimes K)$ for all $t \in [0,T]$ and

$$\|\alpha\|_{\mathcal{L}^{n,p}(dB(\phi);K)} := E\Big[\sum_{k=0}^{n} \Big\{ \int_0^T |D^k \alpha(t-)|^2_{\mathscr{L}_2^k(H;\mathbf{R}^d \otimes K)}\, d\phi(t) \Big\}^{p/2}\Big]^{1/p} < \infty.$$

Next, we define $\mathcal{L}^{n,p}(d\phi;K)$ by the total set of $(\mathscr{F}_t)$-predictable K-valued functions β satisfying $\beta(t) \in W^{n,p}(K)$ for all $t \in [0,T]$ and

$$\|\beta\|_{\mathcal{L}^{n,p}(d\phi;K)} := \sum_{k=0}^{n} \int_0^T E[|D^k \beta(t-)|^p_{\mathscr{L}_2^k(H;K)}]^{1/p}\, d\phi(t) < \infty.$$

PROPOSITION 2.1
Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathcal{L}^{n,p}(dB(\phi);K)$, $\beta \in \mathcal{L}^{n,p}(d\phi;K)$, and $\gamma = (\gamma(t); 0 \leq t \leq T)$ be $(\mathscr{F}_t)$-adapted K-valued functions. We assume that $\gamma(t) \in W^{n,p}(K)$ for all $t \in [0,T]$, and $D^k\gamma$ is an $(\mathscr{F}_t)$-adapted $\mathscr{L}_2^k(H;K)$-valued function such that

$$\sum_{k=0}^{n} E\Big[\sup_{0 \leq t \leq T} |D^k \gamma(t)|^p_{\mathscr{L}_2^k(H;K)}\Big] < \infty.$$

Define Φ by

$$\Phi(t) := \int_0^t \alpha(s-)\, dB\big(\phi(s)\big) + \int_0^t \beta(s-)\, d\phi(s) + \gamma(t).$$

Then, $\Phi(t) \in W^{n,p}(K)$ *for all* $t \in [0,T]$, $D^k\Phi$ *are* $(\mathscr{F}_t)$*-adapted* $\mathscr{L}_2^k(H;K)$*-valued processes for* $k = 1,2,\dots,n$*, and there exists a constant* C *such that*

$$E\Big[\sup_{0\le t\le T}|D^k\Phi(t)|^p_{\mathcal{L}_2^k(H;K)}\Big]^{1/p}$$
$$\le C\Big(\|\alpha\|_{\mathcal{L}^{n,p}(dB(\phi);K)} + \|\beta\|_{\mathcal{L}^{n,p}(d\phi;K)} + \sum_{k=0}^{n} E\Big[\sup_{0\le t\le T}|D^k\gamma(t)|^p_{\mathscr{L}_2^k(H;K)}\Big]\Big).$$

Furthermore, $D\Phi(t)$ *is given by*

$$D\Phi(t)[h] = \int_0^t D\alpha(s-)[h]\,dB\big(\phi(s)\big) + \int_0^t \alpha(s-)\,dh\big(\phi(s)\big)$$
$$+ \int_0^t D\beta(s-)[h]\,d\phi(s) + D\gamma(t)[h], \quad h \in H.$$

Here the equality is in the sense of elements of $L^p(H\otimes K)$*. Therefore, if we denote one of the complete orthonormal systems of* H *by* $\{h^\lambda\}$*, then*

$$D\Phi(t) = \int_0^t D\alpha(s-)\,dB\big(\phi(s)\big) + \sum_\lambda h^\lambda \otimes \int_0^{\phi(t)} \alpha\big(\phi^{-1}(s)-\big)\dot{h}^\lambda(s)\,ds$$
$$+ \int_0^t D\beta(s-)\,d\phi(s) + D\gamma(t).$$

Proof

To prove the first assertion, we use induction on n. For $n = 0$, by Lemma 2.3 we have

$$E\Big[\sup_{0\le t\le T}\Big|\int_0^t \alpha(s-)\,dB\big(\phi(s)\big)\Big|_K^p\Big] \le C_1(p)E\Big[\Big\{\int_0^T |\alpha(s-)|^2_{\mathbf{R}^d\otimes K}\,d\phi(s)\Big\}^{p/2}\Big]$$
$$= \|\alpha\|^p_{\mathcal{L}^{0,p}(dB(\phi);K)}.$$

The other parts of the estimate are obtained easily. So we have the first assertion for $n = 0$.

We assume the result for $n-1$. We show the estimate for n. But the part of the integral with respect to $d\phi$ follows similarly, and clearly the part of γ follows. Hence, we check the part of the stochastic integral. To simplify the notation, let $d = 1$. We show it in the case when α is a step function such as

$$\alpha(t) = \alpha(t_j) \quad \text{for } t \in [t_j, t_j+1),$$

where $0 = t_0 < t_1 < \cdots < t_N = T$. The general case is obtained by taking the limit. Note that α is left-continuous.

In this case, the stochastic integral is expressed as

$$\int_0^T \alpha(t)\,dB\big(\phi(t)\big) = \sum_{j=0}^{N-1} \alpha(t_j)\big\{B\big(\phi(t_{j+1})\big) - B\big(\phi(t_j)\big)\big\}.$$

By the argument at the beginning of this section, for $h \in H$,

$$
\begin{aligned}
&D_h \int_0^T \alpha(t)\, dB\big(\phi(t)\big) \\
&= \sum_{j=0}^{N-1} D_h \alpha(t_j) \big\{ B\big(\phi(t_{j+1})\big) - B\big(\phi(t_j)\big) \big\} + \sum_{j=0}^{N-1} \alpha(t_j) \big\{ h\big(\phi(t_{j+1})\big) - h\big(\phi(t_j)\big) \big\} \\
&= \int_0^T D_h \alpha(t)\, dB\big(\phi(t)\big) + \int_0^T \alpha(t)\, dh\big(\phi(t)\big).
\end{aligned}
$$

Let $I_\alpha[h] := \int_0^T \alpha(t)\, dh(\phi(t))$. Now we show that $I_\alpha \in W^{n,p}(H \otimes K)$. By discussing it similarly to the proof of Lemma 2.1, we have

$$
\int_0^T \alpha(t)\, dh\big(\phi(t)\big) = \int_0^{\phi(T)} \alpha\big(\phi^{-1}(s)\big)\, dh(s).
$$

Hence, we can express I_α as

$$
I_\alpha = \sum_\lambda h_\lambda \otimes \int_0^{\phi(T)} \alpha\big(\phi^{-1}(s)\big) \dot{h}_\lambda(s)\, ds.
$$

Thus,

$$
D^k I_\alpha = \sum_\lambda h_\lambda \otimes \int_0^{\phi(T)} D^k \alpha\big(\phi^{-1}(s)\big) \dot{h}_\lambda(s)\, ds,
$$

and by Lemma 2.1,

$$
\begin{aligned}
|D^k I_\alpha|^2_{\mathcal{L}_2^k(H;H\otimes K)} &= \sum_\lambda \Big| \int_0^{\phi(T)} D^k \alpha\big(\phi^{-1}(s)\big) \dot{h}_\lambda(s)\, ds \Big|^2_{\mathscr{L}_2^k(H;K)} \\
&= \int_0^{\phi(T)} \big| D^k \alpha\big(\phi^{-1}(s)\big)\big|^2_{\mathscr{L}_2^k(H;K)}\, ds \\
&= \int_0^T |D^k \alpha(t)|^2_{\mathscr{L}_2^k(H;K)}\, d\phi(t).
\end{aligned}
$$

Therefore, we have

$$
\begin{gathered}
I_\alpha \in W^{n,p}(H \otimes K), \\
\|D^k I_\alpha\|_p \le \|\alpha\|_{\mathcal{L}^{k,p}(dB(\phi);K)},
\end{gathered}
$$

and

$$
\begin{aligned}
&D\Big(\int_0^T \alpha(t)\, dB(\phi(t)) \Big)[h] \\
&= \int_0^T D\alpha(t)[h]\, dB\big(\phi(t)\big) + \int_0^T \alpha(t)\, dh\big(\phi(t)\big)
\end{aligned} \tag{2.1}
$$

in the sense of elements of $L^p(H\otimes K)$. It is easy to see that equation (2.1) also holds by replacing T with $t\in[0,T]$. Hence, we have

$$\begin{aligned}D\Phi(t)[h] &= \int_0^t D\alpha(s-)[h]\,dB\big(\phi(s)\big)+\int_0^t \alpha(s-)\,dh\big(\phi(s)\big)\\ &\quad+\int_0^t D\beta(s-)[h]\,d\phi(s)+D\gamma(t)[h],\quad h\in H,\end{aligned}$$

in the sense of elements of $L^p(H\otimes K)$, and the second assertion is obtained. Now we note that $D\Phi$ satisfies the assumption of $n-1$. Indeed, the third term satisfies the assumption of γ for $n-1$. Therefore, by the assumption of induction, for $k=1,2,\dots,n-1$, we have

$$\begin{aligned}&E\Big[\sup_{0\le t\le T}|D^kD\Phi(t)|_{\mathscr{L}_2^k(H;H\otimes K)}\Big]^{1/p}\\ &\le C_2(p)\Big\{\|D\alpha\|_{\mathcal{L}^{n-1,p}(dB(\phi);H\otimes K)}+\|D\beta\|_{\mathcal{L}^{n-1,p}(dB(\phi);H\otimes K)}\\ &\qquad+E\Big[\sup_{0\le t\le T}|D^kI_\alpha(t)|^p_{\mathscr{L}_2^k(H;H\otimes K)}\Big]^{1/p}\\ &\qquad+E\Big[\sup_{0\le t\le T}|D^kD\gamma(t)|^p_{\mathscr{L}_2^k(H;H\otimes K)}\Big]^{1/p}\Big\}\\ &\le 2C_2(p)\Big\{\|\alpha\|_{\mathcal{L}^{n,p}(dB(\phi);K)}\\ &\qquad+\|\beta\|_{\mathcal{L}^{n,p}(dB(\phi);K)}+E\Big[\sup_{0\le t\le T}|D^{k+1}\gamma(t)|^p_{\mathscr{L}_2^{k+1}(H;K)}\Big]^{1/p}\Big\}.\end{aligned}$$

Thus, we have the conclusion for n. □

3. Malliavin calculus for stochastic differential equations with deterministic time change

We fix $T>0$. Let r be a positive integer, let $d_1,\dots,d_r$ be positive integers, and let $\phi_1,\phi_2,\dots,\phi_r$ be right-continuous increasing functions on $[0,T]$ starting at zero. Set

$$\begin{aligned}W_k &:= C\big([0,\phi_k(T)]\to\mathbf{R}^{d_k}\big),\\ H_k &:= \big\{h\in C\big([0,\phi_k(T)]\to\mathbf{R}^{d_k}\big);\ h\text{ is absolutely continuous and}\\ &\qquad\qquad \dot h\in L^2\big([0,\phi_k(T)]\to\mathbf{R}^{d_k}\big)\big\},\end{aligned}$$

and let μ_k be the Wiener measure on W_k for $k=1,2,\dots,r$. We define the probability space (W,P) by

$$\begin{aligned}W &:= W_1\times W_2\times\cdots\times W_r,\\ P &:= \mu_1\otimes\mu_2\otimes\cdots\otimes\mu_r.\end{aligned}$$

If we set

$$H := H_1 \otimes H_2 \otimes \cdots \otimes H_r,$$

then (W, H, P) is an abstract Wiener space. Let $(B_k(t))$ be the canonical d_k-dimensional Brownian motion associated to (W_k, H_k, μ_k) for $k = 1, 2, \ldots, r$. Clearly, $B_1, B_2, \ldots, B_r$ are independent under P.

Next, we consider stochastic differential equations with deterministic time change. Let $Z_k(t) := B_k(\phi_k(t))$ for $t \in [0, T]$ and $k = 1, 2, \ldots, r$, and let $(\mathscr{F}_t)$ be the filtration generated by $(Z_k(s); 0 \le s \le t, k = 1, 2, \ldots, r)$. Then Z_k is a square-integrable $(\mathscr{F}_t)$-martingale for all $k = 1, 2, \ldots, r$. We consider the next N-dimensional stochastic differential equation:

$$\begin{cases} dX(t) = \sum_{k=1}^{r} \sigma_k\big(t, X(t-)\big)\, dZ_k(t) + b\big(t, X(t)\big)\, dt, \\ X(0) = x_0, \end{cases} \tag{3.1}$$

where σ_k is an $\mathbf{R}^{d_k} \otimes \mathbf{R}^N$-valued continuous function on $[0, T] \times \mathbf{R}^N$ for $k = 1, 2, \ldots, r$, b is also an $\mathbf{R}^N$-valued continuous function on $[0, T] \times \mathbf{R}^N$, and $x_0 \in \mathbf{R}^N$. We assume that there exists a positive constant K satisfying

$$\max_k |\sigma_k(t, x) - \sigma_k(t, y)| + |b(t, x) - b(t, y)| \le K|x - y|, \quad x, y \in \mathbf{R}^N,\ t \in [0, T],$$

$$\max_k |\sigma_k(t, x)| + |b(t, x)| \le K(1 + |x|), \quad x \in \mathbf{R}^N,\ t \in [0, T].$$

Then, we have the following theorem.

THEOREM 3.1

Equation (3.1) has the unique $(\mathscr{F}_t)$-adapted solution $X = (X(t))$ satisfying that, for all $p > 1$,

$$E\Big[\sup_{0 \le t \le T} |X(t)|^p\Big] \le x_0 \exp\Big\{M\Big(T + \sum_{k=1}^{r} \phi_k(T)\Big)\Big\},$$

where M is a constant depending on r, p, and K.

Proof

It is enough to show the case $p \ge 2$. We use Picard's successive approximation. Define $(\mathscr{F}_t)$-adapted right-continuous processes $\{X_n\}$ with left limits by

$$X_0(t) := x_0,$$

$$X_{n+1}(t) := x_0 + \int_0^t \sum_{k=1}^{r} \sigma_k\big(s, X_n(s-)\big)\, dZ_k(s) + \int_0^t b\big(s, X_n(s)\big)\, ds.$$

Then, the discontinuous points of X_n correspond with the discontinuous points of ϕ for all n almost surely. Now we show that there exists a constant M depending on p and K such that

$$E\Big[\sup_{0 \le s \le t} |X_{n+1}(s) - X_n(s)|^p\Big]^{1/p} \le \frac{x_0}{2^n} \exp\Big\{M\Big(t + \sum_{k=1}^{r} \phi_k(t)\Big)\Big\} \tag{3.2}$$

by induction on n. We determine M later in this proof. It is easy to see that the inequality (3.2) holds for efficiently large M when $n=1$. We assume the inequality (3.2) for $n-1$. By Lemma 2.3,

$$
\begin{aligned}
&E\Big[\sup_{0\le s\le t}|X_{n+1}(s)-X_n(s)|^p\Big]^{1/p}\\
&\le \sum_{k=1}^r E\Big[\sup_{0\le s\le t}\Big|\int_0^s \big(\sigma_k(u,X_n(u-))-\sigma_k(u,X_{n-1}(u-))\big)\,dZ_k(u)\Big|^p\Big]^{1/p}\\
&\qquad\qquad + E\Big[\sup_{0\le s\le t}\Big|\int_0^s \big(b(u,X_n(u))-b(u,X_{n-1}(u))\big)\,du\Big|^p\Big]^{1/p}\\
&\le C_3(p)\Big\{\sum_{k=1}^r E\Big[\Big(\int_0^t \big|\sigma_k\big(u,X_n(u-)\big)-\sigma_k\big(u,X_{n-1}(u-)\big)\big|^2\,d\phi_k(u)\Big)^{p/2}\Big]^{1/p}\\
&\qquad\qquad + E\Big[\Big|\int_0^t \big(b(u,X_n(u))-b(u,X_{n-1}(u))\big)\,du\Big|^p\Big]^{1/p}\Big\}\\
&\le C_4(p,K)\Big\{\sum_{k=1}^r E\Big[\Big(\int_0^t |X_n(u-)-X_{n-1}(u-)|^2\,d\phi_k(u)\Big)^{p/2}\Big]^{1/p}\\
&\qquad\qquad + E\Big[\Big(\int_0^t |X_n(u)-X_{n-1}(u)|\,du\Big)^p\Big]^{1/p}\Big\}\\
&\le C_4(p,K)\Big\{\sum_{k=1}^r\Big(\int_0^t E[|X_n(u-)-X_{n-1}(u-)|^p]^{2/p}\,d\phi_k(u)\Big)^{1/2}\\
&\qquad\qquad + \int_0^t E[|X_n(u)-X_{n-1}(u)|^p]^{1/p}\,du\Big\}.
\end{aligned}
$$

By the assumption of induction, we have

$$
\begin{aligned}
&E\Big[\sup_{0\le s\le t}|X_{n+1}(s)-X_n(s)|^p\Big]^{1/p}\\
&\le \frac{x_0}{2^{n-1}}C_4(p,K)\Big\{\sum_{k=1}^r\Big(\int_0^t \exp\Big\{2M\Big(u+\sum_{k=1}^r\phi_k(u-)\Big)\Big\}\,d\phi_k(u)\Big)^{1/2}\\
&\qquad\qquad + \int_0^t \exp\Big\{M\Big(u+\sum_{l=1}^r\phi_l(u)\Big)\Big\}\,du\Big\}\\
&\le \frac{x_0}{2^{n-1}}C_4(p,K)\Big[\sum_{k=1}^r \exp\Big\{M\Big(t+\sum_{l\ne k}\phi_l(t-)\Big)\Big\}\\
&\qquad \times\Big(\int_0^t \exp\big(2M\phi_k(u-)\big)\,d\phi_k(u)\Big)^{1/2} + \exp\Big(M\sum_{l=1}^r\phi_l(t)\Big)\int_0^t e^{Mu}\,du\Big].
\end{aligned}
$$

Since $\phi_k(\phi_k^{-1}(s)-) \le s$, by Lemma 2.1 we have

$$\int_0^t \exp\bigl(2M\phi_k(u-)\bigr)\, d\phi_k(u) = \int_0^{\phi_k(t)} \exp\bigl(2M\phi_k(\phi_k^{-1}(s)-)\bigr)\, ds$$
$$\le \int_0^{\phi_k(t)} \exp(2Ms)\, ds \le \frac{1}{2M} \exp\bigl(2M\phi_k(t)\bigr).$$

Therefore,

$$E\Bigl[\sup_{0\le s\le t} |X_{n+1}(s) - X_n(s)|^p\Bigr]^{1/p}$$
$$\le \frac{x_0}{2^{n-1}} C_4(p,K) \Bigl[\sum_{k=1}^r \exp\Bigl\{M\Bigl(t + \sum_{l\ne k} \phi_l(t-)\Bigr)\Bigr\} \frac{1}{\sqrt{2M}} e^{M\phi_k(t)}$$
$$+ \exp\Bigl(M \sum_{l=1}^r \phi_l(t)\Bigr) \frac{1}{M} e^{Mt}\Bigr]$$
$$\le \frac{x_0}{2^{n-1}} C_4(p,K) \Bigl\{\frac{r}{\sqrt{2M}} + \frac{1}{M}\Bigr\} \exp\Bigl\{M\Bigl(t + \sum_{l=1}^r \phi_l(t)\Bigr)\Bigr\}.$$

So, if we choose M sufficiently large such that

$$\Bigl(\frac{r}{\sqrt{2M}} + \frac{1}{M}\Bigr) C_4(p,K) \le \frac{1}{2},$$

then the inequality (3.2) holds for $n+1$. Therefore, we complete the induction.

Let n and m be positive integers satisfying $n > m$. Then, by inequality (3.2),

$$E\Bigl[\sup_{0\le s\le t} |X_n(s) - X_m(s)|^p\Bigr]^{1/p} \le \frac{x_0}{2^m} \exp\Bigl\{M\Bigl(t + \sum_{k=1}^r \phi_k(t)\Bigr)\Bigr\}.$$

This inequality implies that $\{X_n\}$ is a Cauchy sequence. Hence, there exists an $(\mathscr{F}_t)$-adapted right-continuous process X with left limits satisfying the fact that the discontinuous points of X correspond with the discontinuous points of ϕ almost surely, and

$$\lim_{n\to\infty} E\Bigl[\sup_{0\le s\le t} |X(s) - X_n(s)|^p\Bigr]^{1/p} = 0.$$

We get to know that X satisfies equation (3.1) by using a similar discussion for the part of the stochastic integral. Thus, we have existence, and the estimate follows easily.

To prove the uniqueness, let both X and Y be solutions of equation (3.1). Then, by a discussion similar to that above, we have

$$E\Bigl[\sup_{0\le s\le t} |X(s) - Y(s)|^2\Bigr]$$
$$\le C_5(p,r,K) \Bigl\{\sum_{k=1}^r \int_0^t E\bigl[|X(s-) - Y(s-)|^2\bigr]\, d\phi_k(s)$$

$$+\int_0^t E\big[|X(s)-Y(s)|^2\big]\,ds\Big\}$$

$$\le C_5(p,r,K)\int_0^t E\Big[\sup_{0\le u\le s}|X(u-)-Y(u-)|^2\Big]\,d\Big(s+\sum_{k=1}^r \phi_k(s)\Big).$$

Therefore, by Lemma 2.4, we have

$$E\Big[\sup_{0\le t\le T}|X(t)-Y(t)|^2\Big]=0.$$

□

Now we apply Malliavin calculus to the solution $X=(X(t))$ of equation (3.1).

THEOREM 3.2

We assume that $\sigma_k\in C^{0,m}([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *and* $\nabla\sigma_k\in C_b^{0,m-1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *for* $k=1,2,\dots,r$, $b\in C^{0,m}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$, *and* $\nabla b\in C_b^{0,m-1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^N)$. *Then we have* $X(t)\in W^{m,p}(\mathbf{R}^N)$ *for* $t\in[0,T]$, *and there exists a constant* M *depending on* r,p,m *and the bounds of the spacial derivatives of* σ_k *and* b *up to order* m *such that*

$$\|X(t)\|_{m,p}\le \exp\Big\{M\Big(t+\sum_{k=1}^r\phi_k(t)\Big)\Big\},\quad t\in[0,T].$$

Proof

It is sufficient to show the case $p\ge 2$, so let $p\ge 2$, and fix p. We define X_n as in the proof of Theorem 3.1. By the proof of Theorem 3.1, we have that $X_n(t)$ converges to $X(t)$ in L^p for $t\in[0,T]$. We show that $X_n(t)$ is in $W^{m,p}$ for $t\in[0,T]$ and all n and that there exists a constant M depending on p,m and the bounds of the spatial derivatives of σ_k and b up to order m such that for $n=1,2,\dots$, $j=1,2,\dots,m$,

$$E\Big[\sup_{0\le s\le t}|D^jX_n(s)|^p_{\mathscr{L}_2^j(H;\mathbf{R}^N)}\Big]^{1/p}\le \exp\Big\{M\Big(t+\sum_{k=1}^r\phi_k(t)\Big)\Big\}. \tag{3.3}$$

We use induction on (n,j). By the proof of Theorem 3.1, we know that $X_n(t)$ is in L^p for $t\in[0,T]$ and all n, and there exists a proper M such that (3.3) holds for $j=0$. Clearly $X_0(t)$ is in $W^{m,p}$ for $t\in[0,T]$, and there exists a proper M such that (3.3) holds for $n=0$. Let $j_0\le m$. As the assumption of the induction we assume that $X_n(t)$ is in $W^{j,p}$ for "$t\in[0,T]$, all n, and $j=1,2,\dots,j_0-1$" and for "$t\in[0,T]$, $n=1,2,\dots,n_0$, and $j=1,2,\dots,j_0$", and that there exists a constant M satisfying (3.3) for "all n and $j=1,2,\dots,j_0-1$" and for "$n=1,2,\dots,n_0$ and $j=1,2,\dots,j_0$". Now we show that $X_{n_0+1}(t)$ is in $W^{j_0,p}$ for $t\in[0,T]$ and that there exists a proper constant M satisfying (3.3) for n_0+1 and j_0. By Proposition 2.1, we can express DX_{n_0+1} explicitly as

$$DX_{n_0+1}(t)=\sum_{k=1}^r\int_0^t\nabla\sigma_k\big(s,X_{n_0}(s-)\big)DX_{n_0}(s-)\,dZ_k(s)$$

$$+\int_0^t \nabla b\big(s, X_{n_0}(s)\big) DX_{n_0}(s)\, ds$$
$$+\sum_\lambda \sum_{k=1}^r h_k^\lambda \otimes \int_0^{\phi_k(t)} \sigma_k\big(\phi_k^{-1}(s), X_{n_0}(\phi_k^{-1}(s))\big) \dot{h}_k^\lambda(s)\, ds,$$

where $\{h_\lambda = (h_1^\lambda, h_2^\lambda, \dots, h_r^\lambda)\}_\lambda$ is one of the complete orthonormal normal systems of $H = H_1 \otimes H_2 \otimes \cdots \otimes H_r$. Repeating this procedure, we have

$$\begin{aligned}
&D^{j_0} X_{n_0+1}(t)\\
&= \sum_{k=1}^r \int_0^t \Big\{ \nabla \sigma_k\big(s, X_{n_0}(s-)\big) D^{j_0} X_{n_0}(s-)\\
&\qquad + \sum_{l=1}^{j_0} A_l^k\big(s, X_{n_0}(s-)\big) Q_l^k\big(DX_{n_0}(s-), \dots, D^{j_0-1} X_{n_0}(s-)\big) \Big\}\, dZ_k(s)\\
&\quad + \int_0^t \Big\{ \nabla b_k\big(s, X_{n_0}(s)\big) D^{j_0} X_{n_0}(s)\\
&\qquad + \sum_{l=1}^{j_0} \tilde{A}_l^k\big(s, X_{n_0}(s)\big) \tilde{Q}_l^k\big(DX_{n_0}(s), \dots, D^{j_0-1} X_{n_0}(s)\big) \Big\}\, ds\\
&\quad + \sum_\lambda \sum_{k=1}^r h_k^\lambda \otimes \int_0^{\phi_k(t)} \sum_{l=0}^{j_0-1} \hat{A}_l^k\big(\phi_k^{-1}(s), X_{n_0}(\phi_k^{-1}(s))\big)\\
&\quad \times \hat{Q}_l^k\big(DX_{n_0}(\phi_k^{-1}(s)), \dots, D^{j_0-1} X_{n_0}(\phi_k^{-1}(s))\big) \dot{h}_k^\lambda(s)\, ds,
\end{aligned} \tag{3.4}$$

where $A_l^k, \tilde{A}_l^k \in C_b^1([0,T] \times \mathbf{R}^N; (\mathbf{R}^N)^{\otimes l} \otimes \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$, $\hat{A}_l^k \in C^1([0,T] \times \mathbf{R}^N;$ $(\mathbf{R}^N)^{\otimes l} \otimes \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$, satisfying

$$\max_{l,k} |\hat{A}_l^k(t,x)| \le C_6(\{\|\nabla^l \sigma_k\|_\infty\}_{1 \le l \le m, 1 \le k \le r})(1 + |x|), \quad x \in \mathbf{R}^N,\ t \in [0,T],$$

and $Q_l^k, \tilde{Q}_l^k, \hat{Q}_l^k$ are $(\mathbf{R}^N)^{\otimes l} \otimes H^{j_0}$-valued functions whose components are polynomials of order l. Therefore, by Lemma 2.3, it follows that

$$\begin{aligned}
&E\Big[\sup_{0 \le s \le t} |D^{j_0} X_{n_0+1}(s)|^p_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}\Big]^{1/p}\\
&\le C_7(p) E\Big[\Big(\sum_{k=1}^r \int_0^t \Big| \nabla \sigma_k\big(s, X_{n_0}(s)\big) D^{j_0} X_{n_0}(s)\\
&\quad + \sum_{l=1}^{j_0} A_l^k\big(s, X_{n_0}(s)\big)\\
&\quad \times Q_l^k\big(DX_{n_0}(s), \dots, D^{j_0-1} X_{n_0}(s)\big)\Big|^2_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)} d\phi_k(s)\Big)^{p/2}\Big]^{1/p}\\
&\quad + \int_0^t E\Big[\Big| \nabla b_k\big(s, X_{n_0}(s)\big) D^{j_0} X_{n_0}(s)
\end{aligned}$$

$$+\sum_{l=1}^{j_0}\tilde{A}_l^k\big(s,X_{n_0}(s)\big)\tilde{Q}_l^k\big(DX_{n_0}(s),\ldots,D^{j_0-1}X_{n_0}(s)\big)\Big|^p_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}\Big]^{1/p}ds$$

$$+\sum_{k=1}^{r}E\Big[\Big(\int_0^{\phi_k(t)}\Big|\sum_{l=0}^{j_0-1}\hat{A}_l^k\big(\phi_k^{-1}(s),X_{n_0}(\phi_k^{-1}(s))\big)$$

$$\times\hat{Q}_l^k\big(DX_{n_0}(\phi_k^{-1}(s)),\ldots,D^{j_0-1}X_{n_0}(\phi_k^{-1}(s))\big)\Big|^2_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}ds\Big)^{p/2}\Big]^{1/p}.$$

By Lemma 2.1, the last term is equal to

$$\sum_{k=1}^{r}E\Big[\Big(\int_0^{t}\Big|\sum_{l=0}^{j_0-1}\hat{A}_l^k\big(s,X_{n_0}(s-)\big)$$

$$\times\hat{Q}_l^k\big(DX_{n_0}(s-),\ldots,D^{j_0-1}X_{n_0}(s-)\big)\Big|^2_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}d\phi_k(s)\Big)^{p/2}\Big]^{1/p}.$$

On the other hand, the induction assumptions tell us that

$$\begin{aligned}&E\Big[\sup_{0\le s\le t}|D^jX_{n_0}(s)|^p_{\mathscr{L}_2^{j}(H;\mathbf{R}^N)}\Big]^{1/p}\\
&\le C_8\big(m,p,\{\|\nabla^l\sigma_k\|_\infty\}_{1\le l\le m,1\le k\le r},\{\|\nabla^l b\|_\infty\}_{1\le l\le m}\big)\\
&\quad\times\exp\Big\{C_8\big(m,p,\{\|\nabla^l\sigma_k\|_\infty\}_{1\le l\le m,1\le k\le r},\{\|\nabla^l b\|_\infty\}_{1\le l\le m}\big)\\
&\quad\times\Big(t+\sum_{k=1}^{r}\phi_k(t)\Big)\Big\},\qquad j=1,2,\ldots,j_0.\end{aligned}$$

Hence, by Hölder's inequality, we have

$$\begin{aligned}&E\big[\big|A_l^k\big(s,X_{n_0}(s)\big)Q_l^k\big(DX_{n_0}(s),\ldots,D^{j_0-1}X_{n_0}(s)\big)\big|^p_{\mathscr{L}_2^{j_0}(H;\mathbf{R}^N)}\big]\\
&\le C_9\big(m,p,\{\|\nabla^l\sigma_k\|_\infty\}_{1\le l\le m,1\le k\le r},\{\|\nabla^l b\|_\infty\}_{1\le l\le m}\big)\\
&\quad\times\exp\Big\{C_9\big(m,p,\{\|\nabla^l\sigma_k\|_\infty\}_{1\le l\le m,1\le k\le r},\{\|\nabla^l b\|_\infty\}_{1\le l\le m}\big)\\
&\quad\times\Big(t+\sum_{k=1}^{r}\phi_k(t)\Big)\Big\}.\end{aligned}$$

The same estimates also hold for $\tilde{A}_l^k\tilde{Q}_l^k$ and $\hat{A}_l^k\hat{Q}_l^k$. Thus, we can make an argument similar to the proof of Theorem 3.1, and by choosing M large enough depending on r,p,m and the bounds of the spacial derivatives of σ_k and b up to order m, we have $X_{n_0+1}(t)\in W^{j_0,p}$ and (3.3). Thus, we have

$$X_n(t)\longrightarrow X(t)\quad\text{in }L^p,\ \sup_n\|X_n(t)\|_{m,p}<\infty.$$

Therefore, by [5, Lemma 1.5.3], we have the conclusion. □

Next, we consider the relation between the ellipticity of equations and the non-degeneracy of Malliavin covariance matrices.

THEOREM 3.3

We assume that $\sigma_k \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$ *and* $\nabla \sigma_k$ *is bounded for* $k = 1, 2, \dots, r$, $b \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$, ∇b *is bounded, and that there exists a positive constant* ε *such that*

$$\sum_{k=1}^{r} \sigma_k(0, x_0)^t \sigma_k(0, x_0) \geq \varepsilon.$$

Then Malliavin covariance matrix $\Delta(t) = \big((DX^i(t), DX^j(t))_{H^*}\big)_{ij}$ *is invertible, and there exists a constant* $C = C(x_0, N, p, \varepsilon, r, \{\|\nabla \sigma_k\|_\infty\}_{1 \leq k \leq r}, \|\nabla b\|_\infty)$ *satisfying, for all* $p > 1$,

$$E\big[\det\big(\Delta(t)\big)^{-p}\big] \leq C \min\big\{\phi_i(t); i = 1, 2, \dots, r\big\}^{-Np} \times \exp\big[C\big(t + \max\big\{\phi_i(t); i = 1, 2, \dots, r\big\}\big)\big]. \tag{3.5}$$

Moreover, if there exist a positive constant ε *and* t_0 *such that*

$$\sum_{k=1}^{r} \sigma_k(t, x)^t \sigma_k(t, x) \geq \varepsilon, \quad t \in [0, t_0],\ x \in \mathbf{R}^N,$$

then we can choose a constant $C = C(t_0, N, p, \varepsilon, r, \{\|\nabla \sigma_k\|_\infty\}_{1 \leq k \leq r}, \|\nabla b\|_\infty)$ *satisfying (3.5).*

Proof

Let

$$A_k(t) := \big[B_k\big(\phi_k(\cdot)\big), B_k\big(\phi_k(\cdot)\big)\big](t), \quad t \in [0,T],\ k = 1, 2, \dots, r.$$

We define two $(N \times N)$-matrix-valued processes J_1 and J_2, respectively, by the solutions of the following stochastic differential equations:

$$\begin{cases} dJ_1(t) = \sum_{k=1}^{r} \nabla \sigma_k(t, X(t-)) J_1(t-)\, dZ_k(t) + \nabla b(t, X(t-)) J_1(t-)\, dt, \\ J_1(0) = I, \end{cases}$$

$$\begin{cases} dJ_2(t) = -\sum_{k=1}^{r} J_2(t-) \nabla \sigma_k(t, X(t-))\, dZ_k(t) - J_2(t-) \nabla b(t, X(t-))\, dt \\ \qquad\qquad + \sum_{k=1}^{r} J_2(t-) \nabla \sigma_k(t, X(t-)) \nabla \sigma_k(t, X(t-))\, dA_k(t), \\ J_2(0) = I. \end{cases}$$

By [7, Chapter II, Section 6, Corollary 2, Theorem 29], $J_1(t) J_2(t) = I$ for all $t \in [0,T]$. Therefore, it follows that $J_1(t) = J_2(t)^{-1}$. By [7, Chapter II, Section 6, Corollary 2, Theorem 29] again, we have

$$J_2(t) DX(t)[h] = \sum_{k=1}^{r} \int_0^t J_2(s-) \sigma_k\big(s, X(s-)\big)\, dh_k\big(\phi_k(s)\big),$$

where $h = (h_1, h_2, \dots, h_r) \in H$. From Lemma 2.1 we can express it as

$$J_2(t) DX(t)[h] = \sum_{k=1}^{r} \int_0^{\phi_k(t)} J_2\big(\phi_k^{-1}(u)-\big) \sigma_k\big(\phi_k^{-1}(u), X(\phi_k^{-1}(u)-)\big) \dot{h}_k(u)\, du.$$

Hence, if we denote one of the complete orthonormal normal systems of H by $\{h^\lambda\}$,

$$\begin{aligned}\Delta(t) &= J_1(t)\sum_\lambda\sum_{k=1}^r\int_0^{\phi_k(t)} J_2\big(\phi_k^{-1}(u)-\big)\sigma_k\big(\phi_k^{-1}(u),X(\phi_k^{-1}(u)-)\big)\dot h_k^\lambda(u)\,du\\ &\qquad\times\int_0^{\phi_k(t)} {}^t\big[J_2\big(\phi_k^{-1}(u)-\big)\sigma_k\big(\phi_k^{-1}(u),X(\phi_k^{-1}(u)-)\big)\big]\dot h_k^\lambda(u)\,du\,{}^tJ_1(t)\\ &= J_1(t)\sum_{k=1}^r\int_0^{\phi_k(t)} J_2\big(\phi_k^{-1}(u)-\big)\sigma_k\big(\phi_k^{-1}(u),X(\phi_k^{-1}(u)-)\big)\\ &\qquad\times{}^t\sigma_k\big(\phi_k^{-1}(u),X(\phi_k^{-1}(u)-)\big){}^tJ_2\big(\phi_k^{-1}(u)-\big)\,du{}^tJ_1(t)\\ &= J_1(t)\sum_{k=1}^r\int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big)\\ &\qquad\times{}^t\sigma_k\big(s,X(s-)\big){}^tJ_2(s-)\,d\phi_k(s){}^tJ_1(t).\end{aligned}$$

Thus, we have

$$\begin{aligned}\det\big(\Delta(t)\big) &= \det\big(J_1(t)\big)^2\det\Big(\sum_{k=1}^r\int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big)\\ &\qquad\times{}^t\sigma_k\big(s,X(s-)\big){}^tJ_2(s-)\,d\phi_k(s)\Big).\end{aligned}\tag{3.6}$$

For the estimate of $\det(J_1(t))$, the following lemma holds.

LEMMA 3.1

We have

$$\begin{aligned}E\big[\big|\det\big(J_2(t)\big)\big|^p\big] &< C_{10}\big(p,N,r,\{\|\nabla\sigma_k\|_\infty\}_{1\le k\le r},\|\nabla b\|_\infty\big)\\ &\quad\times\exp\big[C_{10}\big(p,N,r,\{\|\nabla\sigma_k\|_\infty\}_{1\le k\le r},\|\nabla b\|_\infty\big)\\ &\quad\times\big(t+\max\{\phi_i(t);i=1,2,\dots,r\}\big)\big].\end{aligned}$$

Proof

By Lemma 2.3, we can make a discussion similar to the proof of Theorem 3.1, and we have

$$\begin{aligned}&\max_{i,j}E\Big[\sup_{0\le s\le t}\big|\big(J_2(s)\big)_{ij}\big|^p\Big]^{1/p}\\ &\quad\le C_{11}\big(p,r,\{\|\nabla\sigma_k\|_\infty\}_{1\le k\le r},\|\nabla b\|_\infty\big)\\ &\qquad\times\int_0^t\max_{i,j}E\Big[\sup_{0\le u\le s}\big|\big(J_2(u)\big)_{ij}\big|^p\Big]^{1/p}d\Big(s+\sum_{k=1}^r\phi_k(s)\Big).\end{aligned}$$

From Lemma 2.4 it follows that

$$\begin{aligned}\max_{i,j} E\big[\big|\big(J_2(t)\big)_{ij}\big|^p\big]^{1/p} \le C_{11}\big(p,r,\{\|\nabla\sigma_k\|_\infty\}_{1\le k\le r},\|\nabla b\|_\infty\big) \\ \times \exp\big[C_{11}\big(p,r,\{\|\nabla\sigma_k\|_\infty\}_{1\le k\le r},\|\nabla b\|_\infty\big) \\ \times \big(t+\max\{\phi_i(t); i=1,2,\dots,r\}\big)\big].\end{aligned}$$

By Hölder's inequality, we have

$$E\big[\big|\det\big(J_2(t)\big)\big|^p\big] \le N! \max_{i,j} E\big[\big|\big(J_2(t)\big)_{ij}\big|^{Np}\big]^{1/(Np)}$$

Therefore, we have the conclusion of Lemma 3.1. □

The lemma is sufficient for the estimate of the part $\det(J_1(t))$. So we estimate the other part. Let $\xi \in S^{N-1}$, where S^{N-1} is the $(N-1)$-dimensional sphere centered at zero. From the assumption of ellipticity and the compactness of S^{N-1}, we can choose $n \in \mathbf{N}$, G_i : open sets in S^{N-1}, and $k_i = 1,2,\dots,r$, for $i = 1,2,\dots,n$ such that

$$\bigcup_{i=1}^n G_i = S^{N-1},$$

$${}^t\xi\sigma_{k_i}(0,x_0){}^t\sigma_{k_i}(0,x_0)\xi > \frac{\varepsilon}{2r}, \quad \xi \in G_i,\ i=1,2,\dots,n.$$

From continuity of $\{\sigma_k\}$, we can choose $R_i > 0$ and $t_i \in (0,T]$ satisfying

$${}^t\xi\sigma_{k_i}(s,x){}^t\sigma_{k_i}(s,x)\xi > \frac{\varepsilon}{3r}, \quad x \in B(x_0,R_i),\ s \in [0,t_i],\ \xi \in G_i,$$

for $i = 1,2,\dots,n$. Let $R := \min_i R_i$ and $t_0 := \min_i t_i$. We define a stopping time ζ by

$$\zeta := \inf\big\{t \in [0,T]; |X(t)-x_0| > R \text{ or } |J_1(t)-I| > \delta\big\} \wedge T,$$

where we choose $\delta \in (0,t_0)$ to be so small that

$$\begin{aligned}&{}^t\xi J_2(s)\sigma_{k_i}(s,x){}^t\sigma_{k_i}(s,x){}^tJ_2(s)\xi \ge \frac{\varepsilon}{4r}, \\ &\quad x \in B(x_0,R),\ s \in [0,\zeta),\ \xi \in G_i, i=1,2,\dots,n.\end{aligned}$$

To simplify the notation, we denote $\min\{\phi_i(t); i=1,2,\dots,r\}$ by $\eta(t)$. We note that η is also a right-continuous increasing function on $[0,T]$. From Lemma 2.1 we have that for $i = 1,2,\dots,n$ and $\xi \in G_i$,

$$\begin{aligned}&{}^t\xi\Big(\sum_{k=1}^r \int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big){}^t\sigma_k\big(s,X(s-)\big){}^tJ_2(s-)\,d\phi_k(s)\Big)\xi \\ &\ge \int_0^{t\wedge\zeta} {}^t\xi J_2(s-)\sigma_{k_i}\big(s,X(s-)\big){}^t\sigma_{k_i}\big(s,X(s-)\big){}^tJ_2(s-)\xi\,d\phi_{k_i}(s) \\ &\ge \frac{\varepsilon}{4r}\eta(t\wedge\zeta).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\det\Big(\sum_{k=1}^r\int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big)^t\sigma_k\big(s,X(s-)\big)^tJ_2(s-)\,d\phi_k(s)\Big)\\&\quad\ge 4^{-N}r^{-N}\varepsilon^N\eta(t\wedge\zeta)^N.\end{aligned}$$

Hence,

$$\begin{aligned}&E\Big[\det\Big(\sum_{k=1}^r\int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big)^t\sigma_k\big(s,X(s-)\big)^tJ_2(s-)\,d\phi_k(s)\Big)^{-p}\Big]\\&\le 4^{Np}r^{Np}\varepsilon^{-Np}E[\eta(t\wedge\zeta)^{-Np}]\\&=4^{Np}r^{Np}\varepsilon^{-Np}E[\eta(t)^{-Np};\zeta\ge t]+4^{Np}r^{Np}\varepsilon^{-Np}E[\eta(\zeta)^{-Np};\zeta<t].\end{aligned}$$

Since $\eta(\eta^{-1}(u)-)\le u$ and by Lemma 2.1, we have

$$\begin{aligned}\eta(\zeta)^{-Np}-\eta(t)^{-Np}&=Np\int_{\eta(\zeta)}^{\eta(t)}u^{-Np-1}\,du\\&\le Np\int_{\eta(\zeta)}^{\eta(t)}\eta\big(\eta^{-1}(u)-\big)^{-Np-1}\,du\\&=Np\int_\zeta^t\eta(s-)^{-Np-1}\,d\eta(s).\end{aligned}$$

Hence, we have

$$\begin{aligned}&E\Big[\det\Big(\sum_{k=1}^r\int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big)^t\sigma_k\big(s,X(s-)\big)^tJ_2(s-)\,d\phi_k(s)\Big)^{-p}\Big]\\&\le 4^{Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np}P(\zeta\ge t)\\&\quad+4^{Np}r^{Np}\varepsilon^{-Np}E\Big[Np\int_\zeta^t\eta(s-)^{-Np-1}\,d\eta(s)+\eta(t)^{-Np};\zeta<t\Big]\\&=4^{Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np}\\&\quad+4^{Np}r^{Np}\varepsilon^{-Np}NpE\Big[\int_0^t\mathbf{1}_{(\zeta,t]}(s)\eta(s-)^{-Np-1}\,d\eta(s);\zeta<t\Big]\qquad(3.7)\\&=4^{Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np}\\&\quad+4^{Np}r^{Np}\varepsilon^{-Np}Np\int_0^t\eta(s-)^{-Np-1}E[\mathbf{1}_{(\zeta,t]}(s);\zeta<t]\,d\eta(s)\\&=4^{Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np}\\&\quad+4^{Np}r^{Np}\varepsilon^{-Np}Np\int_0^t\eta(s-)^{-Np-1}P(\zeta<s)\,d\eta(s).\end{aligned}$$

On the other hand, we have the next estimate about ζ.

LEMMA 3.2
We have

$$P(\zeta \le t) \le 2Nr \exp\big\{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(t)^{-1}\big\}.$$

Proof
Note that it is sufficient to prove the estimate for small t. Set

$$\zeta_1 = \inf\big\{t \in [0,T]; |X(t) - x_0| > R\big\} \wedge T,$$
$$\zeta_2 = \inf\big\{t \in [0,T]; |J_1(t) - I| > \delta\big\} \wedge T.$$

Then, it is sufficient to prove the same estimate for ζ_1 and ζ_2. Since the proofs are similar, we prove the estimate only for ζ_2. We define continuous martingales $(M_k(t))$ by

$$M_k(t) := \int_0^t \nabla\sigma_k\big(\phi_k^{-1}(s), X(\phi_k^{-1}(s)-)\big) J_1\big(\phi_k^{-1}(s)-\big)\, dB_k(s), \quad k = 1,2,\dots,r.$$

Denote $\sum_{i,j=1}^N \langle (M_k)_{ij}\rangle$ by $\langle M_k\rangle$ for $k = 1,2,\dots,r$. Then we have

$$\begin{aligned}\langle M_k\rangle\big(\phi_k(t\wedge\zeta_2)\big) &= \int_0^{\phi_k(t\wedge\zeta_2)} \big|\nabla\sigma_k\big(\phi_k^{-1}(s), X(\phi_k^{-1}(s)-)\big) J_1\big(\phi_k^{-1}(s)-\big)\big|^2\, ds\\ &\le C_{13}(\delta)\phi_k(t\wedge\zeta_2).\end{aligned}$$

By Lemma 2.2, it follows that

$$\sup_{s\in[0,t]} |J_1(s\wedge\zeta_2) - I| \le \sum_{k=1}^r \sup_{s\in[0,\phi_k(t\wedge\zeta_2)]} |M_k(s)| + C_{14}(\|\nabla b\|_\infty)t.$$

Therefore, if $t \le \delta/(2C_{14}(\|\nabla b\|_\infty))$, then by [8, Proposition 6.8], we have

$$\begin{aligned}&P(\zeta_2 \le t)\\ &\quad\le P\Big(\sup_{s\in[0,t]} |J_1(s\wedge\zeta_2) - I| \ge \delta\Big)\\ &\quad\le P\Big(\sum_{k=1}^r \sup_{s\in[0,\phi_k(t\wedge\zeta_2)]} |M_k(s)| \ge \frac{\delta}{2}\Big)\\ &\quad\le \sum_{k=1}^r P\Big(\sup_{s\in[0,\phi_k(t\wedge\zeta_2)]} |M_k(s)| \ge \frac{\delta}{2r}\Big)\\ &\quad= \sum_{k=1}^r P\Big(\sup_{s\in[0,\phi_k(t\wedge\zeta_2)]} |M_k(s)| \ge \frac{\delta}{2r},\ \langle M_k\rangle\big(\phi_k(t\wedge\zeta_2)\big) \le C_{13}(\delta)\phi_k(t\wedge\zeta_2)\Big)\\ &\quad= 2N\sum_{k=1}^r \exp\Big(-\frac{\delta^2}{8N^2r^2C_{13}(\delta)\phi_k(t)}\Big).\end{aligned}$$

This completes the proof of Lemma 3.2. □

By the lemma, it holds that

$$P(\zeta < t) \le 2Nr \exp\{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(t-)^{-1}\}.$$

Therefore, by (3.7) we have

$$\begin{aligned}
&E\Big[\det\Big(\sum_{k=1}^{r}\int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big){}^t\sigma_k\big(s,X(s-)\big){}^tJ_2(s-)\,d\phi_k(s)\Big)^{-p}\Big] \\
&\quad\le 2^{2Np}r^{Np}\varepsilon^{-Np}\eta(t)^{-Np} + 2^{2Np+1}r^{Np+1}\varepsilon^{-Np}N^2p \\
&\qquad\times\int_0^t \eta(s-)^{-Np-1}\exp\{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(s-)^{-1}\}\,d\eta(s).
\end{aligned}$$

Now we estimate the second term. Let

$$f(x) := x^{-Np-1}e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)x^{-1}}.$$

Then, f is a positive, bounded, and concave function on $(0,\infty)$, and the maximum is marked at $C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)/(Np+1)$. We denote the maximum jump of $(\eta(t); t\in[0,T])$ by J. Since $u-J\le\eta(\eta^{-1}(u)-)\le u$, by Lemma 2.1,

$$\begin{aligned}
&\int_0^t \eta(s-)^{-Np-1}e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(s-)^{-1}}\,d\eta(s) \\
&\quad= \int_0^{\eta(t)} \eta\big(\eta^{-1}(u)-\big)^{-Np-1}e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(\eta^{-1}(u)-)^{-1}}\,du \\
&\quad\le \int_0^{J+C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)/(Np+1)} \eta\big(\eta^{-1}(u)-\big)^{-Np-1} \\
&\qquad\times e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(\eta^{-1}(u)-)^{-1}}\,du \\
&\qquad+ \int_{J+C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)/(Np+1)}^{\infty} \eta\big(\eta^{-1}(u)-\big)^{-Np-1} \\
&\qquad\times e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(\eta^{-1}(u)-)^{-1}}\,du \\
&\quad\le \|f\|_\infty\Big(J+\frac{C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)}{Np+1}\Big) \\
&\qquad+ \int_{J+C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)/(Np+1)}^{\infty} (u-J)^{-Np-1} \\
&\qquad\times e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)(u-J)^{-1}}\,du \\
&\quad\le \|f\|_\infty\Big(J+\frac{C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)}{Np+1}\Big) \\
&\qquad+ \int_0^{\infty} u^{-Np-1}e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)u^{-1}}\,du.
\end{aligned}$$

If we denote the gamma function by Γ, then by changing variables we have

$$\int_0^\infty u^{-Np-1}e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)u^{-1}}\,du = C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)^{-Np}\Gamma(Np).$$

So we have

$$\int_0^t \eta(s-)^{-Np-1} e^{-C_{12}(N,r,R,\delta,\|\nabla b\|_\infty)\eta(s-)^{-1}}\, d\eta(s)$$
$$\leq C_{15}(N,p,r,R,\delta,\|\nabla b\|_\infty)(1+J).$$

Therefore,

$$E\Big[\det\Big(\sum_{k=1}^r \int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big){}^t\sigma_k\big(s,X(s-)\big){}^tJ_2(s-)\,d\phi_k(s)\Big)^{-p}\Big]$$
$$\leq 2^{2Np} r^{Np} \varepsilon^{-Np} \eta(t)^{-Np}$$
$$+\, 2^{2Np+1} r^{Np+1} \varepsilon^{-Np} N^2 p C_{15}(N,p,r,R,\delta,\|\nabla b\|_\infty)(1+J).$$

Thus, we can conclude that for all $t \in [0,T]$,

$$E\Big[\det\Big(\sum_{k=1}^r \int_0^t J_2(s-)\sigma_k\big(s,X(s-)\big){}^t\sigma_k\big(s,X(s-)\big){}^tJ_2(s-)\,d\phi_k(s)\Big)^{-p}\Big]$$
$$\leq C_{16}(N,p,r,\varepsilon,R,\delta,\|\nabla b\|_\infty)\big(1+J+\eta(t)^{-Np}\big).$$

Note that R and δ are determined by $\{\|\nabla\sigma_k\|_\infty\}_{1\leq k\leq r}$, x_0, and ε. By (3.6), this estimate, and Lemma 3.1, we have the first assertion. Since the condition of the second assertion implies that the constants for the estimates can be chosen independently from x_0 but dependently on t_0, the second assertion follows.

This completes the proof of Theorem 3.3. □

Thus, we can apply Sobolev's inequality with respect to the H-derivative, and we have the following theorem.

THEOREM 3.4

We consider the stochastic differential equation (3.1) and we assume that $\sigma_k \in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *and* $\nabla\sigma_k \in C_b^{0,m+1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ *for* $k=1,2,\dots,r$, $b\in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$, *and* $\nabla b\in C_b^{0,m+1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^N)$ *and that there exists a positive constant* ε *such that*

$$\sum_{k=1}^r \sigma_k(0,x_0){}^t\sigma_k(0,x_0) \geq \varepsilon.$$

Then the law $P(t,x_0,dy)$ *of* $X(t,x_0)$ *is absolutely continuous with respect to the Lebesgue measure, and for its density function* $p(t,x_0,y)$, *there exist positive constants* c_1,c_2,c_3 *such that*

$$\begin{aligned}(3.8)\qquad &\max_{0\leq l\leq m}\sup_{y\in\mathbf{R}^d}|\nabla_y^l p(t,x_0,y)|\\ &\quad\leq c_1\min\{\phi_i(t);i=1,2,\dots,r\}^{-c_3}\exp\Big\{c_2\Big(t+\sum_{k=1}^r\phi_k(t)\Big)\Big\}.\end{aligned}$$

Moreover, if there exist a positive constant ε and t_0 such that

$$\sum_{k=1}^{r} \sigma_k(t,x)^t\sigma_k(t,x) \geq \varepsilon, \quad t \in [0,t_0],\ x \in \mathbf{R}^N,$$

then we can choose constants c_1, c_2, c_3 satisfying (3.8) independently from x_0 but dependent on t_0.

Proof
The conclusion follows from Theorems 3.3, 3.2, and [8, Theorem 5.9]. □

4. Regularity of conditional probabilities

In this section, we consider the inheritance of regularity of densities from those of conditional probabilities.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $\mathscr{G}$ be a sub-σ-field of $\mathscr{F}$. We assume that there exists a regular conditional probability of P with respect to $\mathscr{G}$, and we denote it by $p(\omega, d\omega')$. To begin, we consider the absolute continuity.

THEOREM 4.1
If the regular conditional probability $p(\omega, d\omega')$ is absolutely continuous with respect to a measure ν on $(\Omega, \mathscr{F})$ for almost all ω, then P is also absolutely continuous with respect to the measure ν.

Proof
Let $A \in \mathscr{F}$ be a ν-null set. Since A is also a $p(\omega, d\omega')$-null set for almost all ω,

$$\int_\Omega \mathbf{1}_A(\omega) P(d\omega) = \int_\Omega \int_\Omega \mathbf{1}_A(\omega') p(\omega, d\omega') P(d\omega) = 0.$$

Thus, we have the conclusion. □

Next, we consider the regularity. Assume that the regular conditional probability $p(\omega, d\omega')$ has the density function $p(\omega, y)$ for almost all ω.

THEOREM 4.2
We assume that $p(\omega, \cdot) \in C_b^n(\mathbf{R}^N)$ for almost all ω and that there exists a positive random variable Y such that $E[Y] < \infty$ and for almost all ω,

$$\|\partial^\alpha p(\omega, \cdot)\|_\infty \leq Y(\omega), \quad |\alpha| \leq n.$$

Then P has its density function q and $q \in C_b^n(\mathbf{R}^N)$.

Proof
By Theorem 4.1, P has its density function. We denote it by q. For $f \in C_0^\infty(\mathbf{R}^N)$ and a multiindex α satisfying $|\alpha| \leq n$,

$$\Big|\int_{\mathbf{R}^N} (\partial^\alpha f)(x) q(x)\, dx\Big| = \Big|\int_{\mathbf{R}^N} \Big(\int_{\mathbf{R}^N} (\partial^\alpha f)(y) p(\omega, y)\, dy\Big) P(d\omega)\Big|$$

$$
\begin{aligned}
&= \Big| \int_{\mathbf{R}^N} \Big(\int_{\mathbf{R}^N} f(y)(\partial_y^\alpha p)(\omega, y)\, dy \Big) P(d\omega) \Big| \\
&\le \int_{\mathbf{R}^N} \Big(\int_{\mathbf{R}^N} |f(y)||(\partial_y^\alpha p)(\omega, y)|\, dy \Big) P(d\omega) \\
&\le \int_{\mathbf{R}^N} |f(y)|\, dy \int_{\mathbf{R}^N} Y(\omega) P(d\omega).
\end{aligned}
$$

Therefore, $q \in W^{n,\infty}(\mathbf{R}^N, dx)$, and by [9, Chapter V, Theorem 2], we have the conclusion. $\square$

5. Regularities of solutions of stochastic differential equations driven by subordinated Brownian motions

In this section, we join the results of Sections 3 and 4. First, we give settings.

Let r be a positive integer, let $d_1, \dots, d_k$ be positive integers, let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $Z_k(t)$ be a d_k-dimensional right continuous process on $[0,T]$ with left limits for $k = 1, 2, \dots, r$, where $\{Z_k\}$ are totally independent for $k = 1, 2, \dots, r$.

We assume that $(Z_k(t))$ can be expressed as $(B_k(\tau_k(t)))$ for $k = 1, 2, \dots, r$, where $(B_k(t))$ is a d_k-dimensional Brownian motion for $k = 1, 2, \dots, r$, and $\{\tau_k; k = 1, 2, \dots, r\}$ are one-dimensional right continuous increasing processes starting at zero, and $\{B_k; k = 1, 2, \dots, r\}$ and $\{\tau_k; k = 1, 2, \dots, r\}$ are totally independent. We define a Poisson point process p_k by

$$
p_k(t) := Z_k(t) - Z_k(t-)
$$

and decompose the counting measure $N_{p_k}(dt\, dx)$ on $[0,T] \times \mathbf{R}^{d_k}$ of p_k as

$$
N_{p_k}(dt\, dx) = \mathbf{1}_D(x) N_{p_k}(dt\, dx) + \mathbf{1}_{D^c}(x) N_{p_k}(dt\, dx),
$$

where D is a unit ball centered at zero in $\mathbf{R}^{d_k}$. Then we have the jump part of $Z_k(t)$:

$$
\int_0^{t+} \int_{\mathbf{R}^{d_k}} x \mathbf{1}_D(x) N_{p_k}(ds\, dx) + \int_0^{t+} \int_{\mathbf{R}^{d_k}} x \mathbf{1}_{D^c}(x) N_{p_k}(ds\, dx).
$$

Now, as an additional assumption, we assume that the first term of the right-hand side is a square integrable martingale and that the second term is a function of bounded variation with respect to t. The assumption implies that we can define the stochastic integrals by $\{Z_k\}$. The precise definition can be found in [2].

Let $(\mathscr{F}_t)$ be the filtration generated by $\{Z_k(s); 0 \le s \le t, k = 1, 2, \dots, r\}$. We consider the next N-dimensional stochastic differential equation:

$$
\begin{cases} dX(t) = \sum_{k=1}^r \sigma_k(t, X(t-))\, dZ_k(t) + b(t, X(t))\, dt, \\ X(0) = x_0, \end{cases} \tag{5.1}
$$

where $\sigma_k \in C([0,T] \times \mathbf{R}^N; \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$ for $k = 1, 2, \dots, r$ and $b \in C([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$.

It is known that the stochastic differential equation has pathwise uniqueness when the coefficients are Lipschitz continuous (see [2, Chapter IV, Section 9]).

We denote the σ-field generated by $\{\tau_k; k=1,2,\dots,r\}$ by $\mathscr{F}^\tau$. Then, the argument of Section 3 is available when we take equation (5.1) on $(\Omega, \mathscr{F}, P(\cdot|\mathscr{F}^\tau))$, and by the argument of Section 4, we have the next theorem.

THEOREM 5.1

Assume that $\sigma_k \in C^{0,1}([0,T]\times \mathbf{R}^N; \mathbf{R}^{d_k}\otimes \mathbf{R}^N)$ and that $\nabla\sigma_k$ is bounded for $k=1,2,\dots,r$; assume that $b \in C^{0,1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$, ∇b is bounded, and there exists a positive constant ε such that

$$\sum_{k=1}^r \sigma_k(0,x_0)^t\sigma_k(0,x_0) \geq \varepsilon.$$

Then equation (5.1) has the unique solution $(X(t))$, and the distribution of $X(t)$ has its density for $t\in(0,T]$.

Proof

Under the probability $P(\cdot|\mathscr{F}^\tau)$, we can use Theorem 3.2, Theorem 3.3, and [9, Chapter VIII, Theorem 1]. Therefore, we have that $P(\cdot|\mathscr{F}^\tau)$ is absolutely continuous with respect to the N-dimensional Lebesgue measure almost surely. Thus, the conclusion follows by Theorem 4.1. $\square$

For the regularity of the density function of the solution, we have the following theorem.

THEOREM 5.2

Assume that $\sigma_k \in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ and $\nabla\sigma_k \in C_b^{0,m+1}([0,T]\times \mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^{d_k}\otimes\mathbf{R}^N)$ for $k=1,2,\dots,r$, $b\in C^{0,m+2}([0,T]\times\mathbf{R}^N;\mathbf{R}^N)$, and $\nabla b \in C_b^{0,m+1}([0,T]\times\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^N)$, and there exists a positive constant ε such that

$$\sigma(0,x_0)^t\sigma(0,x_0)\geq\varepsilon.$$

Moreover, we assume that

$$\sum_{k=1}^r E\big[\big(\tau_k(T)\big)^{-A}\exp\big(A\tau_k(T)\big)\big]<\infty \quad \text{for all } A\in[0,\infty).$$

Let $(X(t))$ be the solution of the stochastic differential equation (5.1). Then, the distribution of $X(T)$ has its density $q(x)$, and $q\in C_b^m(\mathbf{R}^N)$.

Proof

Under the probability $P(\cdot|\mathscr{F}^\tau)$, we can use Theorem 3.4. Therefore, we have the conclusion by Theorem 4.2. $\square$

6. Regularities of solutions of stochastic differential equations driven by stable processes

In this section, we consider a special case that is the most interesting of the above results, stochastic differential equations driven by stable processes. First, we make settings.

Let r be a positive integer, let $d_1, \dots, d_k$ be positive integers, let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $Z_k(t)$ be a d_k-dimensional rotation-invariant α_k-stable process for $k = 1, 2, \dots, r$, where Z_k are independent for $k = 1, 2, \dots, r$. Set $(\mathscr{F}_t)$ be the filtration generated by $\{Z_k(s); 0 \le s \le t, k = 1, 2, \dots, r\}$. We consider the next N-dimensional stochastic differential equation,

$$\begin{cases} dX(t) = \sum_{k=1}^{r} \sigma_k(t, X(t-))\, dZ_k(t) + b(t, X(t))\, dt, \\ X(0) = x_0, \end{cases} \tag{6.1}$$

where $\sigma_k \in C([0,T] \times \mathbf{R}^N; \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$ for $k = 1, 2, \dots, r$, and $b \in C([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$. The stochastic integrals are defined as in Section 5.

Now we use subordination: $(Z_k(t))$ can be expressed as $(B_k(\tau_k(t)))$ for $k = 1, 2, \dots, r$, where $\{B_k; k = 1, 2, \dots, r\}$ is a d_k-dimensional Brownian motion and $\{\tau_k; k = 1, 2, \dots, r\}$ is a one-sided $(\alpha_k/2)$-stable process for $k = 1, 2, \dots, r$, and $\{B_k; k = 1, 2, \dots, r\}$ and $\{\tau_k; k = 1, 2, \dots, r\}$ are totally independent. If necessary, we extend the probability space $(\Omega, \mathscr{F}, P)$. So the assumptions of Section 5 are satisfied. We denote the σ-field generated by $\{\tau_k; k = 1, 2, \dots, r\}$ by $\mathscr{F}^\tau$. Then, by Theorem 5.1, we have the next theorem.

THEOREM 6.1

Assume that $\sigma_k \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^{d_k} \otimes \mathbf{R}^N)$ and that $\nabla \sigma_k$ is bounded for $k = 1, 2, \dots, r$; assume that $b \in C^{0,1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$, ∇b is bounded, and there exists a positive constant ε such that

$$\sum_{k=1}^{r} \sigma_k(0, x_0)^t \sigma_k(0, x_0) \ge \varepsilon.$$

Then equation (6.1) has the unique solution $(X(t))$, and the distribution of $X(t)$ has its density for $t \in (0, T]$.

Finally, we consider the regularity of the density function of the solution. But Theorem 5.2 is not available because the condition about the expectation of exponential function does not hold. However, in the case when $r = 1$, we can conclude the next theorem.

THEOREM 6.2

Assume that $\sigma \in C^{0,m+2}([0,T] \times \mathbf{R}^N; \mathbf{R}^d \otimes \mathbf{R}^N)$ and $\nabla \sigma_k \in C_b^{0,m+1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N \otimes \mathbf{R}^d \otimes \mathbf{R}^N)$, $b \in C^{0,m+2}([0,T] \times \mathbf{R}^N; \mathbf{R}^N)$, and $\nabla b \in C_b^{0,m+1}([0,T] \times \mathbf{R}^N; \mathbf{R}^N \otimes \mathbf{R}^N)$, and there exists a positive constant ε such that

$$\sigma(t, x)^t \sigma(t, x) \ge \varepsilon, \quad t \in [0, T],\ x \in \mathbf{R}^N.$$

Let $(X(t))$ be the solution of the stochastic differential equation (6.1). Then, the distribution of $X(T)$ has its density $q(x)$, and $q \in C_b^m(\mathbf{R}^N)$.

Proof
Fix $T_0 > 0$. We define an $\mathscr{F}^\tau$-measurable random time ρ by

$$\rho := \sup\{t > 0; \tau(T) - \tau(t) > T_0\}$$

and let $\mathscr{F}^{\tau,T_0}$ be the σ-field generated by τ and $(B(t); 0 \le t \le (\tau(T) - T_0) \vee 0)$. Consider the next stochastic differential equation on $[0, T-\rho]$ under $(\Omega, \mathscr{F}, P(\cdot|\mathscr{F}^{\tau,T_0}))$:

$$\begin{cases} d\tilde{X}(t) = \sigma\big(\rho + t, \tilde{X}(t-)\big)\, d\tilde{Z}(t) + b\big(\rho + t, \tilde{X}(t)\big)\, dt, \\ \tilde{X}(0) = \xi_0 + \xi, \end{cases} \tag{6.2}$$

where $\tilde{Z}(t) := B(\tau(\rho + t)) - B(\tau(\rho))$,

$$\begin{aligned} \xi_0 &:= X(\rho-) + \sigma\big(\rho, X(\rho-)\big)\big(B(\tau(T) - T_0) - B(\tau(\rho-))\big) \\ &\qquad + b\big(\rho, X(\rho-)\big)\big(\tau(\rho) - \tau(\rho-)\big), \\ \xi &:= \sigma\big(\rho, X(\rho-)\big)\big(B(\tau(\rho)) - B(\tau(T) - T_0)\big). \end{aligned}$$

Note that $(\tilde{Z}(t))$ is a Brownian motion with deterministic time change, and note that ξ_0 is a constant under $P(\cdot|\mathscr{F}^{\tau,T_0})$. By Theorem 3.1, equation (6.2) has the unique solution $\tilde{X}$ on $(\Omega, \mathscr{F}, P(\cdot|\mathscr{F}^{\tau,T_0}))$, and it holds that

$$\tilde{X}(t) = X(\rho + t) \quad \text{for } t \in [0, T-\rho],\ P(\cdot|\mathscr{F}^{\tau,T_0})\text{-a.s.} \tag{6.3}$$

On the other hand, if (W, H, μ) is the Wiener space generated by $(B(t); \tau(T) - T_0 \le t \le \tau(T))$, then the Malliavin calculus is available for ξ and $(\tilde{X}(t))$ under $P(\cdot|\mathscr{F}^{\tau,T_0})$. It is easy to see that $|D\xi|_H \le \|\sigma\|_\infty$ and that $D^k\xi = 0$ for $k \ge 2$. By a discussion similar to the proof of Theorem 3.2, for all $p > 0$ there exists a constant M such that

$$\sum_{k=1}^{m+2} E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\big[|D^k\tilde{X}(\rho)|^p_{\mathscr{L}_2^k(H;\mathbf{R}^N)}\big] \le M \exp\big\{M\big(T + \tau(T) - \tau(\rho)\big)\big\}.$$

Since $\tau(T) - \tau(\rho) \le T_0$, we have

$$\sum_{k=1}^{m+2} E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\big[|D^k\tilde{X}(\rho)|^p_{\mathscr{L}_2^k(H;\mathbf{R}^N)}\big] \le M \exp\big\{M(T + T_0)\big\}. \tag{6.4}$$

Now we consider the case $\tau(T) > T_0$. By equation (6.2) and Proposition 2.1, for $h \in H$,

$$\begin{aligned} &D\tilde{X}(T-\rho)[h] \\ &\quad = D\xi[h] + \int_0^{T-\rho} \nabla\sigma\big(\rho + t, \tilde{X}(t-)\big) D\tilde{X}(t-)[h]\, d\tilde{Z}(t) \\ &\qquad + \int_0^{T-\rho} \sigma\big(\rho + t, \tilde{X}(t-)\big)\, dh\big(\tau(\rho + t)\big) + \int_0^{T-\rho} \nabla b\big(\rho + t, \tilde{X}(t)\big) D\tilde{X}(t)[h]\, dt. \end{aligned}$$

We use a discussion similar to the proof of Theorem 3.3. Set

$$\tilde{A}(t) := [\tilde{Z}, \tilde{Z}](t), \quad t \in [0, T-\rho].$$

We define two $(N \times N)$-matrix-valued processes $\tilde{J}_1$ and $\tilde{J}_2$, respectively, on $[0, T-\rho]$ by the solutions of the following stochastic differential equations:

$$\begin{cases} d\tilde{J}_1(t) = \nabla\sigma(\rho+t, \tilde{X}(t-))\tilde{J}_1(t-)\, d\tilde{Z}(t) + \nabla b(\rho+t, \tilde{X}(t))\tilde{J}_1(t-)\, dt, \\ \tilde{J}_1(0) = I, \end{cases}$$

$$\begin{cases} d\tilde{J}_2(t) = -\tilde{J}_2(t-)\nabla\sigma(\rho+t, \tilde{X}(t-))\, d\tilde{Z}(t) - \tilde{J}_2(t-)\nabla b(\rho+t, \tilde{X}(t))\, dt \\ \qquad\qquad\qquad + \tilde{J}_2(t-)\nabla\sigma(\rho+t, \tilde{X}(t-))\nabla\sigma(\rho+t, \tilde{X}(t-))\, d\tilde{A}(t), \\ \tilde{J}_2(0) = I. \end{cases}$$

By [7, Chapter II, Section 6, Corollary 2, Theorem 29], $\tilde{J}_1(t)\tilde{J}_2(t) = I$ for all $t \in [0, T-\rho]$. Therefore, it follows that $\tilde{J}_1(t) = \tilde{J}_2(t)^{-1}$. To simplify the notation, let $\tilde{J}_2(t) = I$ for $t < 0$. By [7, Chapter II, Section 6, Corollary 2, Theorem 29] again, we have

$$\tilde{J}_2(T-\rho)D\tilde{X}(T-\rho)[h] = D\xi[h] + \int_0^{T-\rho} \tilde{J}_2(t-)\sigma\big(\rho+t, \tilde{X}(t-)\big)\, dh\big(\tau(\rho+t)\big).$$

Therefore, by (6.3) we have

$$\tilde{J}_2(T-\rho)D\tilde{X}(T-\rho)[h] = D\xi[h] + \int_\rho^{T} \tilde{J}_2\big((t-\rho)-\big)\sigma\big(t, X(t-)\big)\, dh\big(\tau(t)\big).$$

From Lemma 2.1 and the definition of ξ, we have

$$\begin{aligned} &\tilde{J}_2(T-\rho)D\tilde{X}(T-\rho)[h] \\ &= \sigma\big(\rho, X(\rho-)\big)\big(h(\tau(\rho)) - h(\tau(T)-T_0)\big) \\ &\quad + \int_{\tau(\rho)}^{\tau(T)} \tilde{J}_2\big((\tau^{-1}(t)-\rho)-\big)\sigma\big(\tau^{-1}(t), X(\tau^{-1}(t)-)\big)\dot{h}(t)\, dt \\ &= \int_{\tau(T)-T_0}^{\tau(\rho)} \sigma\big(\rho, X(\rho-)\big)\, dh(t) \\ &\quad + \int_{\tau(\rho)}^{\tau(T)} \tilde{J}_2\big((\tau^{-1}(t)-\rho)-\big)\sigma\big(\tau^{-1}(t), X(\tau^{-1}(t)-)\big)\dot{h}\big(\tau^{-1}(t)\big)\, dt \\ &= \int_{\tau(T)-T_0}^{\tau(T)} \tilde{J}_2\big((\tau^{-1}(t)-\rho)-\big)\sigma\big(\tau^{-1}(t), X(\tau^{-1}(t)-)\big)\dot{h}\big(\tau^{-1}(t)\big)\, dt. \end{aligned}$$

Hence, if we denote the Malliavin covariance matrix $((D\tilde{X}^i(t), D\tilde{X}^j(t))_{H^*})_{ij}$ by $\tilde{\Delta}(t)$, then

$$\begin{aligned} \tilde{\Delta}(T-\rho) = \tilde{J}_1(T-\rho) &\int_{\tau(T)-T_0}^{\tau(T)} \tilde{J}_2\big((\tau^{-1}(t)-\rho)-\big)\sigma\big(\tau^{-1}(t), X(\tau^{-1}(t)-)\big) \\ &\times {}^t\sigma\big(\tau^{-1}(t), X(\tau^{-1}(t)-)\big){}^t\tilde{J}_2\big((\tau^{-1}(t)-\rho)-\big)\, dt\, {}^t\tilde{J}_1(T-\rho). \end{aligned}$$

Thus, we have

$$\begin{aligned}
&\det\bigl(\tilde{\Delta}(T-\rho)\bigr)\\
&\quad=\det\bigl(\tilde{J}_1(T-\rho)\bigr)^2\det\Bigl(\int_{\tau(T)-T_0}^{\tau(T)}\tilde{J}_2\bigl((\tau^{-1}(t)-\rho)-\bigr)\sigma\bigl(\tau^{-1}(t),X(\tau^{-1}(t)-)\bigr)\\
&\qquad\times{}^t\sigma\bigl(\tau^{-1}(t),X(\tau^{-1}(t)-)\bigr){}^t\tilde{J}_2\bigl((\tau^{-1}(t)-\rho)-\bigr)\,dt\Bigr).
\end{aligned}$$

Similarly to the proof of Theorem 3.3, we have, for $p\geq 1$ and $\tau(T)>T_0$,

$$\begin{aligned}
&E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\bigl[\det\bigl(\tilde{\Delta}(T-\rho)\bigr)^{-p}\bigr]\\
&\quad\leq C_{16}\bigl(N,p,\varepsilon,r,\bigl\{\|\nabla\sigma_k\|_\infty\bigr\}_{1\leq k\leq r},\|\nabla b\|_\infty\bigr)T_0^{-Np}\\
&\qquad\times\exp\bigl[C_{16}\bigl(N,p,\varepsilon,r,\bigl\{\|\nabla\sigma_k\|_\infty\bigr\}_{1\leq k\leq r},\|\nabla b\|_\infty\bigr)(T+T_0)\bigr].
\end{aligned}$$

In the case $\tau(T)\leq T_0$, $\rho=0$. Theorem 3.3 implies that for $p\geq 1$ and $\tau(T)\leq T_0$,

$$\begin{aligned}
&E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\bigl[\det\bigl(\tilde{\Delta}(T)\bigr)^{-p}\bigr]\\
&\quad\leq C_{17}\bigl(N,p,\varepsilon,r,\bigl\{\|\nabla\sigma_k\|_\infty\bigr\}_{1\leq k\leq r},\|\nabla b\|_\infty\bigr)\tau(T)^{-Np}\\
&\qquad\times\exp\bigl[C_{17}\bigl(N,p,\varepsilon,r,\bigl\{\|\nabla\sigma_k\|_\infty\bigr\}_{1\leq k\leq r},\|\nabla b\|_\infty\bigr)(T+T_0)\bigr].
\end{aligned}$$

So, for $p\geq 1$ and for all τ, we have

$$\begin{aligned}
&E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\bigl[\det\bigl(\tilde{\Delta}(T)\bigr)^{-p}\bigr]\\
&\quad\leq C_{18}\bigl(N,p,\varepsilon,r,\bigl\{\|\nabla\sigma_k\|_\infty\bigr\}_{1\leq k\leq r},\|\nabla b\|_\infty\bigr)\bigl(T_0\wedge\tau(T)\bigr)^{-Np}\\
&\qquad\times\exp\bigl[C_{18}\bigl(N,p,\varepsilon,r,\bigl\{\|\nabla\sigma_k\|_\infty\bigr\}_{1\leq k\leq r},\|\nabla b\|_\infty\bigr)(T+T_0)\bigr].
\end{aligned}$$

Therefore, by (6.4) and [8, Theorem 5.9], the law of $\tilde{X}(\rho)$ under $P(\cdot|\mathscr{F}^{\tau,T_0})$ has its density function p_{τ,T_0} belonging to $C_b^m(\mathbf{R}^N)$ P-almost surely, and it holds that there exist positive constants c_1,c_2,c_3 independent from ξ and satisfying

$$\max_{0\leq l\leq m}\sup_{y\in\mathbf{R}^d}|\nabla_y^l p_{\tau,T_0}(T-\rho,\xi_0+\xi,y)|\leq c_1\bigl(T_0\wedge\tau(T)\bigr)^{-c_3}\exp\bigl\{c_2(T+T_0)\bigr\}. \tag{6.5}$$

Finally, we consider the law of $(X(t))$. By (6.3), (6.5), and the Markov property of X under $P(\cdot|\mathscr{F}^{\tau,T_0})$, we have that for $f\in C_b^\infty(\mathbf{R}^N)$ and a multi-index $\beta=(\beta_1,\dots,\beta_d)$ satisfying $|\beta|\leq m$,

$$\begin{aligned}
\bigl|E^P\bigl[\partial^\beta f\bigl(X(T)\bigr)\bigr]\bigr|&=\bigl|E^P\bigl[E^P\bigl[\partial^\beta f\bigl(X(T)\bigr)\big|\mathscr{F}^{\tau,T_0}\bigr]\bigr]\bigr|\\
&=\bigl|E^P\bigl[E^{P(\cdot|\mathscr{F}^{\tau,T_0})}\bigl[\partial^\beta f\bigl(\tilde{X}^{\tau,T_0}(T-\rho)\bigr)\bigr]\bigr]\bigr|\\
&=\Bigl|E^P\Bigl[\int_{\mathbf{R}^N}\partial^\beta f(y)p_{\tau,T_0}(T-\rho,\xi_0+\xi,y)\,dy\Bigr]\Bigr|\\
&\leq E^P\Bigl[\int_{\mathbf{R}^N}|f(y)|\;|\partial_y^\beta p_{\tau,T_0}(T-\rho,\xi_0+\xi,y)|\,dy\Bigr]\\
&\leq c_1E^p\bigl[\bigl(T_0\wedge\tau(T)\bigr)^{-c_3}\bigr]\|f\|_{L^1(\mathbf{R}^N)}\exp\bigl\{c_2(T+T_0)\bigr\}.
\end{aligned}$$

Since τ is a one-sided $(\alpha/2)$-stable process, by the equality

$$E^P[\tau(T)^{-n}] = \int_0^\infty \int_{\eta_n}^\infty \cdots \int_{\eta_2}^\infty E^P\left[\exp\left(-\eta_1 \tau(T)\right)\right] d\eta_1 \cdots d\eta_n,$$

we have

$$E^P\left[\left(\tau(T)\right)^{-c_3}\right] < \infty.$$

Thus, we have the conclusion. □

Acknowledgment. I thank Professor Yozo Tamura for helpful discussions and his careful reading of an earlier version.

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