J. Math. Kyoto Univ. (JMKYAZ)
49-3 (2009), 593–601

A note on strong unique continuation for normal elliptic systems with Gevrey coefficients

By

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Abstract

In this paper, we consider the strong unique continuation for normal elliptic systems whose coefficients are Gevrey class. By using Lerner's lemma, we prove the Carleman estimate with some weight function.

1. Introduction

Let Ω be an open neighbourhood of the origin in $\mathbb{R}^2$. If $v \in C^\infty(\Omega)$ satisfies

$$\partial^\alpha v(0) = 0, \quad \forall \alpha \in \mathbb{Z}^2_{\geq 0} \tag{1.1}$$

then we say that u is flat at the origin, and $C^\infty_\flat(\Omega)$ is the space of functions in $C^\infty(\Omega)$ which are flat at the origin.

We suppose that

$$P(x, \partial) = \partial_1 + N(x)\partial_2 + R(x)$$

is an elliptic differential operator in Ω where N and R are $l \times l$ matrix valued functions with the entries which are in $C^\infty(\Omega)$. For a differential operator P, we shall adopt the following

Definition 1.1. We say that P has the strong unique continuation property at the origin if whatever $Pu = 0$ in Ω and $u \in C^\infty_\flat(\Omega; \mathbb{C}^l)$, then $u \equiv 0$ in a neighbourhood of the origin.

In [1], Ōkaji proved a result of strong unique continuation property for a class of elliptic systems of normal type in two independent variables. His result is, roughly speaking, that if N is a normal elliptic matrix, and if there exists some complex number ζ such that

$$\text{all the eigenvalues of } N(0) \text{ are equal to } \zeta \text{ or } \bar{\zeta}, \tag{1.2}$$

then P has the strong unique continuation property at the origin.

Received December 16, 2008
Revised June 16, 2009

In this paper we are interested in the strong unique continuation property for elliptic systems of normal type which dose not satisfy the assumption of (1.2). As in [2], [4], studying the case in which the coefficients of P are in Gevrey class of appropriate index, we prove the strong unique continuation for elliptic systems of normal type which dose not satisfy the assumption in (1.2).

The proof is based on Carleman's method, and is based on the technique used in [2]. As in [1], there is no regularity assumptions on the eigenvalues of N. Therefore we can't use a usual smooth diagonalization approach, so to prove the Carleman inequalty for $P(x, \partial)$ with some weight function, we use the result proved in [5] which is an extention of Hörmander's pseudo convex condition in [3] to systems of partial differential operator. This method can make us compute the Gevrey index of the coefficients of P for which we can prove a Carleman estimate.

2. Statement of result

Let $G^s(\Omega)$ denote the set of all functions of Gevrey class of order $s > 1$. We set $P(x, \partial) = \partial_1 + N(x)\partial_2 + R(x)$ where N, R are $l \times l$ matrices whose entries are in $G^s(\Omega)$. Moreover we shall assume that

1. $N(x)$ is normal for any $x \in \Omega$, that is,

$$N(x)^*N(x) = N(x)N(x)^* \quad x \in \Omega, \tag{2.1}$$

where $N(x)^*$ is the conjugate transposed matrix of $N(x)$.

2. P is elliptic in Ω, that is,

$$\det P_1(x, \xi) \neq 0, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^2\backslash\{0\}, \tag{2.2}$$

where $P_1(x, \xi)$ is the principal symbol of P, that is,

$$P_1(x, \xi) = \xi_1 I_l + \xi_2 N(x), \tag{2.3}$$

where I_l is the $l \times l$ unit matrix.

Remark 1. The ellipticity of P is satisfied if and only if

$$\mathrm{Im}\lambda_j(x) \neq 0 \qquad j = 1, 2, \ldots, l, \quad x \in \Omega, \tag{2.4}$$

where $\lambda_j(x)$ are the eigenvalues of $N(x)$.

Our main theorem is the following;

Theorem 2.1. *Let $P(x, \partial) = \partial_1 + N(x)\partial_2 + R(x)$ be a differential operator with N and R in $G^s(\Omega; M_l(\mathbb{C}))$ which satisfies* (2.1), (2.2). *Set $\nu = \frac{1}{s-1}$, and $\lambda_j = \lambda_j(0)$ where $\lambda_j(x)$, $j = 1, 2, \ldots, l$ are the eigenvalues of $N(x)$. If the quadratic polynomials in $z = (z_1, z_2) \in \mathbb{R}^2$*

$$q_j(\nu, z) = (\nu + 1 - |\lambda_j|^2)z_1^2 + 2(\nu + 2)(Re\lambda_j)z_1z_2 + \{(\nu + 1)|\lambda_j|^2 - 1\}z_2^2$$

are positive definite for all $j = 1, 2, \ldots, l$, then P has the strong unique continuation property at the origin.

Remark 2. If $\mathrm{Re}\,\lambda_j = 0,\ j = 1, 2, \ldots, l$, the condition of q_j is equivalent to the following;

$$s < 1 + \frac{1}{\max_{1\le j\le l}(|\lambda_j|^2, |\lambda_j|^{-2}) - 1}. \tag{2.5}$$

If $N(x)$ satisfies Ōkaji's assumption (1.2), by the change of variables we may assume that

$$\lambda_j = i \text{ or } -i \quad \forall j.$$

Then for any Gevrey index s, our theorem holds.

3. Proof

We begin with the key lemma for the choice of the weight function used in Carleman estimate for P.

Lemma 3.1 ([4]). *If $u \in G^s(\Omega)$ is flat at zero, then there exists a function $v \in C^\infty(\Omega)$ flat at zero, such that*

$$u = \exp(-r^{-\nu})v, \quad r = |x| \tag{3.1}$$

provided $1 + \nu^{-1} > s$.

By this lemma and by Gevrey hypoellipticity of P, if $u \in C^\infty(\Omega; \mathbb{C}^l)$ is solution to

$$Pu = 0 \quad \text{in } \Omega,$$

then u satisfies

$$\partial^\alpha u(x) = o(e^{-r^{-\nu}}) \quad r \to 0, \quad \forall\alpha, \tag{3.2}$$

so we can use the weight function $\exp(r^{-\alpha})$ $(0 < \alpha < \nu)$ in the Carleman estimate of P. We now state the Carleman estimate for the operator P with respect to the singular weight

$$\varphi(x) = r^{-\alpha} \tag{3.3}$$

with $\nu_0 < \alpha < \nu$ where $\nu_0 = \inf\{\nu : q_j(\nu, x) \text{ is positive definite for all } j\}$. We denote the L^2-norm by $\|\cdot\|$.

Proposition 3.1. *There exist some constants $\tau_0 > 0$, $\varepsilon > 0$ such that for $\tau > \tau_0$ we have the estimate*

$$\tau\|e^{\tau\varphi}u\|^2 \le C\|e^{\tau\varphi}P(x,\partial)u\|^2, \qquad u \in C_0^\infty(B_\varepsilon(0)\backslash\{0\}), \tag{3.4}$$

where C is the positive constant depending only on P.

Using this proposition, Theorem 2.1 can be proved by a standard manner. In fact, let χ be a smooth function supported in $B_\epsilon(0)$ and equal to 1 in $B_{\frac{\varepsilon}{2}}(0)$ and let us put, for $j \in \mathbb{N}$, $\theta_j(x) = \theta(jx)$, where $\theta \in C^\infty(\mathbb{R}^2)$, is such that

$$\theta(x) = \begin{cases} 0 & \text{if } |x| \le \frac{1}{2} \\ 1 & \text{if } |x| \ge 1. \end{cases}$$

Let now $u \in C^\infty_\flat(\Omega)$ be a solution to

$$Pu = 0 \quad \text{in } \Omega.$$

Then we apply the Proposition 3.1 to $\theta_j \chi u$, and by (3.2), when $j \to \infty$, we have

$$\tau^{\frac{1}{2}} \|e^{\tau\varphi}\chi u\| \le C\|e^{\tau\varphi} P(x,\partial)(\chi u)\|. \tag{3.5}$$

Moreover since u is a solution to $Pu = 0$, we have

$$P(x,\partial)(\chi u) = [P(x,\partial), \chi]u, \tag{3.6}$$

where the commutator is supported in $\{x \in \mathbb{R}^2 : \frac{\varepsilon}{2} < |x| < \varepsilon\}$. Therefore we have the inequality

$$\tau^{\frac{1}{2}} \|e^{\tau\varphi}\chi u\| \le Ce^{\tau\varphi(\frac{\varepsilon}{2})}\|[P(x,\partial), \chi]u\|. \tag{3.7}$$

When $\tau \to \infty$, this shows that $u = 0$ in $B_{\frac{\varepsilon}{2}}(0)$, we have thus completed the proof of Theorem 2.1. Hence it remains to prove Proposition 3.1. For that purpose, we shall need the following Theorem which is the special case of Theorem 4.1.

Let Ω_0 be a bounded open subset in $\mathbb{R}^2$ with $\bar{\Omega}_0 \subset \Omega$. We suppose that $Q(x,\partial) = \partial_1 + N(x)\partial_2 + R(x)$ is a partial differential operator with N and R in $C^\infty(\bar{\Omega}_0; M_l(\mathbb{C}))$. Set $Q_1(x,\xi) = \xi_1 I_l + \xi_2 N(x)$.

Theorem 3.1. *Let $\psi \in C^\infty(\bar{\Omega}_0; \mathbb{R})$ with $\psi'(x) \neq 0$ on $\bar{\Omega}_0$. If $Q_1(x,\xi)$ satisfies*

1. det $Q_1(x,\xi) \neq 0, \qquad (x,\xi) \in \bar{\Omega}_0 \times (\mathbb{R}^2\backslash\{0\})$.
2. *For any $(x,\xi) \in \bar{\Omega}_0 \times (\mathbb{R}^2)$, the Hermitian matrix*

$$\frac{1}{2i}[\{Q^*_{1,\psi}, Q_{1,\psi}\} - \{Q_{1,\psi}, Q^*_{1,\psi}\}](x,\xi) \tag{3.8}$$

is positive definite on ker $Q_{1,\psi}(x,\xi)$, where

$$Q_{1,\psi}(x,\xi) = Q_1(x, \xi + i\psi^{'}(x)) \tag{3.9}$$

and $\{\ ,\ \}$ denotes the Poisson bracket for matrix valued functions.

3. $N(x)N(x)^* = N(x)^*N(x), \qquad x \in \bar{\Omega}_0,$

then there exists $\tau_0 > 0$ such that for any $\tau > \tau_0$ we have

$$\sum_{|\gamma|\le 1} \tau^{2(1-|\gamma|)-1}\|e^{\tau\psi}\partial^\gamma v\|^2 \le C\|e^{\tau\psi}Q(x,\partial)v\|^2, \qquad v \in C^\infty_0(\overline{\Omega}_0; \mathbb{C}^l), \tag{3.10}$$

where C is a positive constant independent of v, τ.

However, because the weight function φ is singular at 0, Theorem 3.1 can't be applicable to the proof of Proposition 3.1 directly. So following [2], we use a dyadic decomposition for $\mathbb{R}^2$,

$$\cup_{k\in\mathbb{Z}} A_k = \mathbb{R}^2\setminus\{0\}, \tag{3.11}$$

where

$$A_k = \{2^{k-1} < |x| < 2^{k+1}\}. \tag{3.12}$$

And we take the smooth partition of unity

$$1 = \sum_{k\in\mathbb{Z}} \chi_k \qquad \text{on } \mathbb{R}^2\setminus\{0\} \tag{3.13}$$

with $\chi_k \in C_0^\infty(A_k)$.

In order to prove Proposition 3.1 we shall derive a Carleman estimate on A_k. Then we use the following lemma.

Lemma 3.2. *There exists some negative integer k_0 such that for any negative integer $k < k_0$ and any $\tau > 1$ we have*

$$\sum_{|\gamma|\le 1} \sigma^{1-2|\gamma|}\|e^{\sigma\varphi}\partial^\gamma v\|^2 \le C\|e^{\sigma\varphi}P_1(2^k y,\partial)v\|^2, \qquad v \in C_0^\infty(A_0), \tag{3.14}$$

where $\sigma = \tau 2^{-\alpha k}$ and C is a constant independent of v, k, τ.

To prove this lemma, the following lemma is useful.

Lemma 3.3. *Suppose that φ and $P_1(\partial) = P_1(0,\partial)$ satisfy the conditions of Theorem 3.1 in $\overline{A_0}$. Then there exists some $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, φ and $P(\varepsilon x, D)$ satisfy the same conditions in $\overline{A_0}$.*

Proof. We shall prove this lemma by contradiction. Assume that there exists some sequence $\{\varepsilon_j\} \subset [0,\infty)$ with $\varepsilon_j \to 0$ such that for any j, φ and $P(\varepsilon_j x,\partial)$ don't satisfy the condition 2 in Theorem 3.1. When we define

$$P_{1,\varepsilon,\varphi}(x,\xi) = P_1(\varepsilon x, \xi + i\varphi'(x)), \tag{3.15}$$

then we can take $(x_j,\xi_j) \in \overline{A_0}\times\mathbb{R}^2$ such that

$$\det P_{1,\varepsilon_j,\varphi}(x_j,\xi_j) = 0, \tag{3.16}$$

and

$$M_{\varepsilon_j}(x_j,\xi_j) \text{ is not positive definite in } \ker P_1(\varepsilon_j x_j, \xi_j + i\varphi'(x_j)) \tag{3.17}$$

where

(3.18)
$$M_{\varepsilon_j}(x,\xi) = \frac{1}{2i}[\{P_{1,\varphi}(\varepsilon_j\cdot,\cdot)^*, P_{1,\varphi}(\varepsilon_j\cdot,\cdot)\} - \{P_{1,\varphi}(\varepsilon_j\cdot,\cdot), P_{1,\varphi}(\varepsilon_j\cdot,\cdot)^*\}](x,\xi).$$

By (3.17) there exists $\{w_j\} \subset \ker P_1(\varepsilon_j x_j, \xi_j + i\varphi'(x_j))$ with $|w_j| = 1$ such that

$$(M_{\varepsilon_j}(x_j, \xi_j)w_j, w_j) \leq 0, \qquad j = 1, 2, \dots. \tag{3.19}$$

Now we may assume that the sequences $\{x_j\}, \{w_j\}$ are convergent to $\tilde{x}, \tilde{w}$ where $\tilde{x} \in \overline{A_0}$, $\tilde{w} \in \mathbb{R}^2$ with $|\tilde{w}| = 1$. If $\{\xi_j\}$ has no subsequence which is convergent, then $\{\xi_j\}$ is not bounded, and we can choose the subsequence of $\{\xi_{j_k}\}$ which satisfy

$$\lim_{k\to\infty} \xi_{j_k} = \infty \text{ and } \frac{\xi_{j_k}}{|\xi_{j_k}|} \text{ is convergent to } \tilde{\eta}.$$

Then, by (3.16), we have

$$\begin{aligned}\det P_1(0, \tilde{\eta}) &= \lim_{k\to\infty} \det P_1\left(\varepsilon_{j_k} x_{j_k}, \frac{\xi_{j_k}}{|\xi_{j_k}|} + i\frac{\varphi'(x_{j_k})}{|\xi_{j_k}|}\right) \\ &= \lim |\xi_{j_k}|^{-l} \det P_1(\varepsilon_{j_k} x_{j_k}, \xi_{j_k} + i\varphi'(x_{j_k})) \\ &= 0.\end{aligned}$$

But this contradicts with the ellipticity of P_1, so we may assume that $\{\xi_j\}$ is bounded, and convergent to the $\tilde{\xi}$. Then $\tilde{x}$, $\tilde{\xi}$, and $\tilde{w}$ satisfy

$$\begin{gathered}\det P_1(0, \tilde{\xi} + i\varphi'(\tilde{x})) = 0, \\ (M_0(\tilde{x}, \tilde{\xi})\tilde{w}, \tilde{w}) \leq 0, \\ \tilde{w} \in \ker P_1(0, \tilde{\xi} + i\varphi'(\tilde{x})).\end{gathered}$$

But this contradicts with the assumption that $P_1(\partial) = P_1(0, \partial)$ and φ satisfy the condtion of Theorem 3.1 in $\bar{A}_0$. We have thus proved the Lemma 3.3. □

Proof of Lemma 3.2. By Lemma 3.3, it remains that we check the condition (2) in Theorem 3.1 for $P_1(\partial)$ in order to prove Lemma 3.2. Put $p_j(\xi) = \xi_1 + \lambda_j \xi_2$. Because we may assume that $N(0)$ is a diagonal matrix, we have

$$P(0, \xi + i\varphi(x)) = \operatorname{diag}[p_{1,\varphi}(x, \xi), \dots, p_{l,\varphi}(x, \xi)],$$

where $p_{j,\varphi}(x, \xi) = p_j(\xi + i\varphi'(x))$. Therefore φ and $P_1(D)$ satisfy the condition (2) of Theorem 3.1 in $\overline{A_1}$ if and only if for all j

$$\frac{1}{i}\{\overline{p_{j,\varphi}}, p_{j,\varphi}\}(x, \xi) > 0, \qquad \text{if } p_{j,\varphi}(x, \xi) = 0. \tag{3.20}$$

By the caluculation, we have

$$\frac{1}{i}\{\overline{p_{j,\varphi}}, p_{j,\varphi}\}(x, \xi) = 2\alpha|x|^{-\alpha-4} q_j(\alpha, x), \tag{3.21}$$

therefore the condition (3.20) is equivalent to

$$2\alpha|x|^{-\alpha-4} q_j(\alpha, x) > 0,\ x \in \overline{A_0}, \tag{3.22}$$

thus we have completed the proof of Lemma 3.2. □

Set $\varepsilon_0 = 2^{k_0-2}$. For $u \in C_0^\infty(B_\epsilon(0)\backslash\{0\})$ with $0 < \varepsilon < \varepsilon_0$, we define

$$u_k = \chi_k u.$$

Then we have

$$u = \sum_{k\le k_0} u_k, \tag{3.23}$$

$$C_1\|u\|^2 \le \sum_{k\le k_0} \|u_k\|^2 \le C_2\|u\|^2. \tag{3.24}$$

Let us apply Lemma 3.2 to $v = u_k(2^k x) \in C_0^\infty(A_0)$, then for any negative integer $k < k_0$ and any $\tau > 1$ we have

$$\sum_{|\gamma|\le 1} \sigma^{1-2|\gamma|}\|e^{\sigma\varphi}\partial^\gamma v\|^2 \le C\|e^{\sigma\varphi}P_1(2^k y,\partial)v\|^2, \qquad v \in C_0^\infty(A_0). \tag{3.25}$$

$$\begin{aligned}
\text{LHS of (3.25)} &= \sum_{|\gamma|\le 1} \sigma^{1-2|\gamma|} \int e^{2\sigma\varphi(x)}|\partial_x^\gamma v(x)|^2 dx \\
&= \sum_{|\gamma|\le 1} (\tau 2^{-\alpha k})^{1-2|\gamma|} 2^{k(2|\gamma|-2)} \int e^{2\tau\varphi(y)}|\partial_y^\gamma u_k(y)|^2 dy \\
&= \sum_{|\gamma|\le 1} \tau^{1-2|\gamma|} 2^{k\{(2\alpha+1)|\gamma|-\alpha-2\}} \|e^{\tau\varphi}\partial^\gamma u_k\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\text{RHS of (3.25)} &= C\|e^{\tau\varphi}P_1(x,\partial)u_k\|^2 \\
&\le C\{\|\chi_k e^{\tau\varphi}P_1(x,\partial)u_k\|^2 + \|e^{\tau\varphi}[P_1(x,\partial),\chi_k]u\|^2\} \\
&\le C\|\chi_k e^{\tau\varphi}P_1(x,\partial)u_k\|^2 \\
&\quad + 2^{-2k}C\left(\|e^{\tau\varphi}u_{k-1}\|^2 + \|e^{\tau\varphi}u_k\|^2 + \|e^{\tau\varphi}u_{k+1}\|^2\right)
\end{aligned}$$

since $[P_1(\cdot,D),\chi_k]$ is a function belonging to $C_0^\infty(A_k)$, whose L^∞-norm is $O(2^{-k})$. Therefore we have the following inequality:

$$\begin{aligned}
2^{-\alpha k}\tau\|e^{\tau\varphi}u_k\|^2 &\le 2^{-k}C\|\chi_k e^{\tau\varphi}P_1 u\|^2 \\
&\quad + C\left(\|e^{\tau\varphi}u_{k-1}\|^2 + \|e^{\tau\varphi}u_k\|^2 + \|e^{\tau\varphi}u_{k+1}\|^2\right), \qquad k \le k_0.
\end{aligned}$$

So, summing up over k, if supp $u \subset \{|x| < 2^{k_0-1}\}$ and τ is large, we obtain

$$\tau\|e^{\tau\varphi}u\|^2 \le C\|e^{\tau\varphi}P_1 u\|^2. \tag{3.26}$$

If τ is so large, we can substitute P for P_1 in (3.26), therefore we have thus proved Proposition 3.1.

4. Appendix

In this section, we give a general theorem of the Carleman estimates for systems of partial differential operators . The following theorem is considered as the extension of the Hörmander's theorem about the Carleman estimates (Theorem 8.3.1 in [3]) to systems of elliptic partial differential operators.

Let U be a bounded open set in $\mathbb{R}^n$. We suppose that $Q(y,D) = \sum_{|\alpha|\le m} A_\alpha(y)D_y^\alpha$ is a partial differential operator whose coefficients A_α are in $C^\infty(\overline{U}; M_l(\mathbb{C}))$. $Q_m(y,\eta)$ is the principal symbol of $Q(y,D)$, that is,

$$Q_m(y,\eta) = \sum_{|\alpha|=m} A_\alpha(y)\eta^\alpha. \tag{4.1}$$

Theorem 4.1. *Let ψ be in $C^2(\overline{U};\mathbb{R})$ with $\psi'(x) \neq 0$ on $\overline{U}$. If $Q_m(y,\eta)$ satisfies*

1. det $Q_m(y,\eta) \neq 0$, $\quad y \in \overline{U}, \eta \in \mathbb{R}^n \backslash \{0\}$,
2. *For any $(y,\eta) \in \overline{U} \times \mathbb{R}^n$, the Hermitian matrix*

$$\frac{1}{2i}[\{Q^*_{m,\psi}, Q_{m,\psi}\} - \{Q_{m,\psi}, Q^*_{m,\psi}\}](y,\eta) \tag{4.2}$$

is positive definite on $\ker Q_{m,\psi}(y,\eta)$*, where $Q_{m,\psi}(y,\eta) = Q_m(y, \eta + i\psi'(y))$.*

3. $A_\alpha(y)^* A_\beta(y) = A_\beta(y)A_\alpha(y)^*$, $\quad y \in \overline{U}$, $|\alpha| = |\beta| = m$

then there exist some $C > 0$, $\tau_0 > 0$ such that for $\tau > \tau_0$

$$\sum_{|\alpha|\le m} \tau^{2(m-|\alpha|)-1} \|e^{\tau\psi} D^\alpha v\|^2 \le C\|e^{\tau\psi}Q(y,D)v\|^2, \qquad v \in C_0^\infty(\overline{U}), \tag{4.3}$$

where C is independent of v, τ.

Remark 3. For $P(y,\eta) = (p_{i,j}(y,\eta))_{1\le i,j\le l}$, $Q(y,\eta) = (q_{i,j}(y,\eta))_{1\le i,j\le l} \in C^\infty(\overline{U} \times \mathbb{R}^n; M_l(\mathbb{C}))$, $\{P,Q\}(y,\eta)$ is the Poisson bracket of P and Q, that is,

$$\{P,Q\}(y,\eta) = \sum_{j=1}^n \left(\partial_{\eta_j} P \partial_{y_j} Q - \partial_{y_j} P \partial_{\eta_j} Q\right)(y,\eta), \tag{4.4}$$

where $\partial P = (\partial p_{i,j})_{1\le i,j\le l}$, $\partial Q = (\partial q_{i,j})_{1\le i,j\le l}$.

Remark 4. The constant C in (4.3) depends on $\|\partial^\gamma A_\alpha\|_\infty$ with $|\gamma| \le m$.

This theorem can be proved by the following facts.

Lemma 4.1 (Lemma 8.3.1 in [3]). *Let Q be a differential operator with coefficients in $L^\infty(\Omega; M_l(\mathbb{C}))$, and assume that every $x \in \overline{U}$ has an open neighbourhood U_x such that* (4.3) *is valid for some constant K_x when $v \in C_0^\infty(\overline{U} \cap U_x; \mathbb{C}^l)$ and τ is sufficiently large. Then one can find a constant K such that* (4.3) *is valid for all $v \in C_0^\infty(\overline{U}; \mathbb{C}^l)$ when τ is large enough.*

Theorem 4.2 ([5]). *If $Q_m(y,\eta)$ satisfies*

1. $\det Q_m(0,\eta) \neq 0, \qquad \eta \in \mathbb{R}^n \backslash \{0\}$,
2. $\frac{1}{2i}[\{Q^*_{m,\psi}, Q_{m,\psi}\} - \{Q_{m,\psi}, Q^*_{m,\psi}\}](0,\eta)$ *is positive definite on* $\ker Q_{m,\psi}(0,\eta)$,
3. $A_\alpha(y)^* A_\beta(y) = A_\beta(y) A_\alpha(y)^*, \qquad x \in \overline{U}, |\alpha| = |\beta| = m$,

then there exist some $C > 0$, $\tau_0 > 0$, and the neighborhood of the origin, $V \subset U$ such that for $\tau > \tau_0$

$$\sum_{|\alpha| \le m} \tau^{2(m-|\alpha|)-1} \|e^{\tau\varphi} D^\alpha v\|^2 \le C \|e^{\tau\varphi} P(x,D) v\|^2, \qquad v \in C_0^\infty(V; \mathbb{C}^l), \tag{4.5}$$

where C is independent of v, τ.

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