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A Lane-Emden-Fowler type problem with singular nonlinearity

By

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Abstract

The main purpose of this article is to establish the existence result concerning to the problem $-\Delta u(x) + c(x)u(x) = a(x)f(u(x))$, $x \in \mathbb{R}^N$, $N > 2$, $u(x) \to 0$ as $|x| \to \infty$. Similary problems have been also studied. The proofs of the existence are based on the maximum principle and sub and super solutions method.

1. Introduction and the main result

Let $f \in C^1((0,\infty),(0,\infty))$ be a singular function at 0, in the sense that $\lim_{s\searrow 0} f(s) = \infty$. In this article we consider the existence of the entire solutions for the problem

$$-\Delta u(x) + c(x)u(x) = a(x)f(u(x)), u > 0 \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $N > 2$, $a(x)$ and $c(x)$ satisfy

AC1) $a(x), c(x) \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$;

AC2) $a(x) > 0, c(x) \geq 0, \forall x \in \mathbb{R}^N$;

A3) for $\varphi(r) = \max_{|x|=r} a(x)$ we have

$$\int_0^\infty r\varphi(r)dr < \infty,$$

and the nonlinearity f, satisfies the following assumptions

F1) $\lim\limits_{u\searrow 0} \frac{f(u)}{u} = +\infty$ and $\lim\limits_{u\nearrow\infty} \frac{f(u)}{u} = 0$.

According with Callegari and Nachman [3, 4], in the case $N = 1$, the problem (1.1) arises in the study of boundary layer equations for the class of non-Newtonian fluids named pseudoplastic under the classical conditions for a steady flow over a semi-infinite flat. Considered in the context of partial differential equations this problem has been intensively studied (see [5, 6, 8, 9, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24]). The existence of entire positive

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solutions on $\mathbb{R}^N$ for $f(u) = u^{-\gamma}$, $\gamma \in (0,1)$ and under certain additional hypotheses has been established by Edelson [9] and Kusano-Swanson [14]. This result is generalized for any $\gamma > 0$ via the sub and super solutions method in Shaker [19] and by other methods by Dalmasso [8]. Lair and Shaker continued in [17] the study of (1.1) for $f(u) = u^{-\gamma}$, $\gamma > 0$ and $c(x) = 0$. Under the above conditions the authors proved the existence of a unique positive solution $u \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^N)$ vanishing at infinity to this special problem. If $\Omega \subset \mathbb{R}^N$ is an open bounded smooth domain, Shi and Yao proved in [20] that the following Dirichlet problem

$$(1.2) \qquad \begin{aligned} &-\Delta y(x) = a(x)[y^{-\gamma}(x) + y^{\delta}(x)], \text{ in } \Omega, \\ &y(x) > 0 \text{ in } \Omega, \\ &y(x)_{|\partial\Omega} = 0, \end{aligned}$$

where $a \in C^{0,\alpha}(\overline{\Omega})$ is a nonnegative function, always admits a unique solution $y \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ if $a \neq 0$ for all $x \in \Omega$ and $\gamma, \delta \in (0,1)$.

After these results, the authors Sun Yijing and Li Shujie [21] extended these results to the case when $\Omega = \mathbb{R}^N$. For more generally nonlinearity including the results in [17, 19, 20, 21] the problem (1.2) is considered by the authors Goncalves and Santos [13]. In [5] the author extended the results in [13] to the case of an equation involving the p-Laplacian, defined by

$$\Delta_p u := div(|\nabla u|^{p-2} \nabla u), 1 < p < \infty.$$

Motivated by the technique proof in [5], we give here similar results, but by a different approach, to the existence of entire solution to the problem (1.1).

The term "entire" has often been used for solutions of (1.1) in $\mathbb{R}^N$. To avoid confusion with the traditional definition for entire functions, we use the term "$C^{2,\alpha}-$ entire". By $C^{2,\alpha}-$ entire solution of (1.1) we mean a function $u(x) \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ that satisfies (1.1) pointwisely in $\mathbb{R}^N$. The method that we shall be using heavily in our proof is the so-called the sub and super solutions method due to Shangbin Cui ([7]).

Our main result is the following:

Theorem 1.1. *We suppose that hypotheses* AC1), AC2)*,* A3)*,* F1) *are satisfied. Then, the problem* (1.1) *has a* $C^{2,\alpha}-$ *entire positive solution vanishing at infinity in* $\mathbb{R}^N$*.*

To prove existence of such a solution to (1.1) we establish some preliminary results.

2. Preliminary results

We need an embedding result of Sobolev spaces in Hőlder spaces [11]:
Let U, V be Banach spaces.

Definition 2.1. We say that U is continuously embedded in V, and write $U \hookrightarrow V$, if $U \subset V$ and there is a constant C such that $\forall u \in U$

$$\|u\|_V \leq C \|u\|_U .$$

Definition 2.2. We say that U is compactly embedded in V, written $U \hookrightarrow\hookrightarrow V$, if $U \hookrightarrow V$ and every bounded sequence in U has a subsequence which is convergent in V.

With this definitions, we have the following embedding result of Sobolev spaces in Hőlder spaces:

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary $\partial\Omega$, $m \in \mathbb{N}$, and $1 \leq p < \infty$. Under these hypotheses, for all $r \in \mathbb{N}\cup\{0\}$, $0 < \alpha < 1$ with*

$$m - N/p \geq r + \alpha$$

one has the continuous embedding

$$W^{m,p}(\Omega) \hookrightarrow C^{r,\alpha}(\overline{\Omega}).$$

More precisely, there exists a constant $C > 0$ such that for all $u \in W^{m,p}(\Omega)$ possibly after modification on a set of measure zero $u \in C^{r,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{r,\alpha}(\overline{\Omega})} \leq C\,\|u\|_{W^{m,p}(\Omega)}\,.$$

Moreover, for all $r \in \mathbb{N}\cup\{0\}$, $0 \leq \alpha \leq 1$ with

$$m - N/p > r + \alpha$$

one has the compact embedding

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C^{r,\alpha}(\overline{\Omega}).$$

We remark that for open bounded $\Omega \subset \mathbb{R}^N$ this result holds for $W_0^{m,p}(\Omega)$ instead of $W^{m,p}(\Omega)$.

The next interior estimate (interior since $\Omega' \subset\subset \Omega$) can be found in ([11, Chapter 1, p. 2]) and can be extended to a global estimate for solutions with sufficiently smooth boundary values provided the boundary $\partial\Omega$ is also sufficiently smooth.

Lemma 2.2. *Let Ω be an open bounded set in $\mathbb{R}^N$ and $u \in C^2(\Omega)$, satisfy $-\Delta u = h$ in Ω where $h \in C^\alpha(\Omega)$. Then for any domain $\Omega' \subset\subset \Omega$,*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega}')} \leq C(\sup_{\Omega} u + \|h\|_{C^\alpha(\overline{\Omega})}),$$

where C is a constant depending only on α $(0 < \alpha < 1)$, the dimension N and $dist(\Omega', \partial\Omega)$.

We have the following interior estimate given in ([11, Theorem 9.11, p. 235]), where by $W_{loc}^{m,p}(\Omega)$ we mean the space of functions which belong to $W^{m,p}(\Omega')$ for every $\Omega' \subset\subset \Omega$.

Lemma 2.3. *Let Ω be an open bounded set in $\mathbb{R}^N$ and $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, satisfies $-\Delta u = h$ in Ω where $h \in L^p(\Omega)$. Then for any domain $\Omega' \subset\subset \Omega$,*

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}),$$

where C depends on N, p, Ω', Ω.

The next lemma is useful in our proofs.

Lemma 2.4. *Let $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary $\partial\Omega$ and $a(x), c(x) \in C^{0,\alpha}(\overline{\Omega})$, $a(x) > 0, c(x) > 0$ for all $x \in \Omega$. Then, for every $c > 0$, the problem*

$$\begin{aligned} &-\Delta u(x) + c(x)u(x) + a(x)\left|\nabla u(x)\right|^s = c, \quad \text{in} \quad \Omega, \\ &u_{|\partial\Omega} = 0, \end{aligned} \tag{2.1}$$

has a unique positive solution $u(x) \in C^{2+\alpha}(\overline{\Omega})$.

Proof. We will use the sub and super solutions method due to Herbert Amann (see [2, Theorem 1.1, p. 283]). Let φ_1 be the first positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$\begin{aligned} &-\Delta u(x) = \lambda u(x), \text{ in } \quad \Omega, \\ &u_{|\partial\Omega}(x) = 0. \end{aligned} \tag{2.2}$$

It is well known that $\varphi_1 \in C^{2+\alpha}(\overline{\Omega})$. Now we are able to show that the function $\underline{u}(x) = \sigma_1\varphi_1$, where

$$0 < \sigma_1 \leq \min\left\{\frac{c}{2\max_{x\in\overline{\Omega}}\varphi_1[\lambda_1 + \max_{x\in\overline{\Omega}}c(x)]}, \frac{c^{1/s}}{2^{1/s}\cdot(\max_{x\in\overline{\Omega}}a(x))^{1/s}\cdot\max_{x\in\overline{\Omega}}|\nabla\varphi_1|}\right\}, \tag{2.3}$$

is a sub solution of (2.1). Indeed, by (2.3) we have

$$\begin{aligned} &-\Delta\sigma_1\varphi_1 + c(x)\sigma_1\varphi_1 + a(x)\left|\nabla\sigma_1\varphi_1\right|^s = \sigma_1\varphi_1[\lambda_1 + c(x)] + a(x)\sigma_1^s\left|\nabla\varphi_1\right|^s \\ &\quad \leq \sigma_1 \max_{x\in\overline{\Omega}}\varphi_1[\lambda_1 + \max_{x\in\overline{\Omega}}c(x)] + \max_{x\in\overline{\Omega}}a(x)\cdot\sigma_1^s\max_{x\in\overline{\Omega}}\left|\nabla\varphi_1\right|^s \\ &\quad \leq \frac{c}{2} + \frac{c}{2} = c. \end{aligned} \tag{2.4}$$

In order to provide a super solution of (2.1) we observe that the function $\overline{u}(x) = y(x)\cdot c$, where $y(x) \in C^{2+\alpha}(\overline{\Omega})$ is the unique solution of the problem

$$\begin{aligned} &-\Delta y(x) = 1 \text{ in } \quad \Omega, \\ &y_{|\partial\Omega}(x) = 0, \end{aligned} \tag{2.5}$$

satisfies

$$c + c(x)\overline{u}(x) + a(x)\left|\nabla\overline{u}(x)\right|^s = -\Delta\overline{u}(x) + c(x)\overline{u}(x) + a(x)\left|\nabla\overline{u}(x)\right|^s \geq c.$$

Clearly, $\overline{u}(x)$ is a super solution of (2.1). Now, since

$$\begin{aligned} -\Delta[\overline{u}(x) - \underline{u}(x)] &= -\lambda_1\sigma_1\varphi_1 + c \geq -\sigma_1\varphi_1[\lambda_1 + c(x)] + c \geq 0, \text{ in } \Omega, \\ \overline{u}(x) - \underline{u}(x) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{aligned}$$

it follows from the maximum principle that $\underline{u}(x) \leq \overline{u}(x), x \in \overline{\Omega}$.

We have obtained a sub solution $\underline{u}(x) \in C^{2+\alpha}(\overline{\Omega})$ and a super solution $\overline{u}(x) \in C^{2+\alpha}(\overline{\Omega})$ for the problem (2.1) such that $\underline{u}(x) \leq \overline{u}(x)$ on $\overline{\Omega}$ in the sense of *Amann* [2]. Hence, the problem (2.1) has a solution $u(x) \in C^{2+\alpha}(\overline{\Omega})$ in the ordered interval $[\underline{u}(x), \overline{u}(x)]$, that means

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x), \ x \in \overline{\Omega}. \tag{2.6}$$

These inequalities show that $u(x) > 0$ in Ω.

Uniqueness. Let us now assume that $u_1(x)$ and $u_2(x)$ are arbitrary solutions of the problem (2.1). To prove the uniqueness, it is enough to show that $u_1(x) \leq u_2(x)$ in $\overline{\Omega}$. Suppose the contrary. Denote

$$\Omega_{u_1,u_2} := \{x \in \Omega \,|\, w_0(x) := u_1(x) - u_2(x) > 0\},$$

and suppose that $\Omega_{u_1,u_2} \neq \varnothing$. Thus, we can suppose that the $\sup w_0(x)$ in Ω is positive. Then at that point, say $x_0 \in \Omega$, where the supremum is achieved we have

$$\nabla[u_1(x_0) - u_2(x_0)] = 0. \tag{2.7}$$

Using the relation (2.7) we obtain

$$\begin{aligned} 0 &\geq \Delta w_0(x_0) \\ &= -c + c(x_0)[u_1(x_0) - u_2(x_0)] + a(x_0)[|\nabla u_1(x_0)|^s - |\nabla u_2(x_0)|^s] + c \\ &= c(x_0)[u_1(x_0) - u_2(x_0)] > 0, \end{aligned}$$

a contradiction. Hence $u_1 \leq u_2$ in $\overline{\Omega}$. By symmetry we also have $u_2 \leq u_1$ in $\overline{\Omega}$ and the proof of uniqueness is now complete. □

The following result can be found in [6, 12] in a particular form and more generally in [23] and will be used here in the form:

Lemma 2.5. *Make the same assumptions as in Lemma* 2.4 *and assume that f satisfies* F1) *and $u \mapsto f(u)/u$ is decreasing on $(0, \infty)$. Then there exists a function $u \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ such that $u > 0$ for all $x \in \Omega$ and $u(x)$ satisfies*

$$\begin{aligned} &-\Delta u(x) + c(x)u(x) = a(x)[f(u(x)) + |\nabla u(x)|^s], u > 0 \text{ in } \Omega, \\ &u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.8}$$

Proof. Let $\varphi_1 \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ be the normalized eigenfunction corresponding to the first eigenvalue λ_1 of the problem (2.2). We observe that the function $\underline{u} = \varepsilon_1\varphi_1$ is a subsolution of (2.8), provided that $\varepsilon_1 > 0$ is sufficiently small (see the proof of Lemma 2.6). To establish the construction of a super solution to (2.8) let $h : [0, \eta] \to [0, \infty)$ be the solution of the problem

$$
\begin{aligned}
& -h''(t) = \frac{f(h(t))}{h(t)}, 0 < t < \eta < 1, \\
& h(0) = 0, \\
& h(t) > 0, \quad 0 < t \leq \eta < 1,
\end{aligned}
\tag{2.9}
$$

which exists by the results in [1, Theorem 2.1, p. 397]. Using the results in [23] we can see that the function $\overline{u}(x) = Mh(c_0\varphi_1) \in C^2(\Omega) \cap C(\overline{\Omega})$ is a super solution of (2.8) provided for some positive constants M, c_0. With the same argument as in the following proof of (3.3) we deduce that $\underline{u}(x) \leq \overline{u}(x)$ in Ω.

Thus, by sub and super solution method (see [7]) we find at least a solution $u \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ to the problem (2.10) such that $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ in Ω. $\square$

The problem studied in the next lemma is similarly to (2.8).

Lemma 2.6. *Make the same assumptions as in Lemma* 2.5. *Then there exists a function* $u \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ *such that* $u > 0$ *for all* $x \in \Omega$ *and* $u(x)$ *satisfies*

$$
\begin{aligned}
& -\Delta u(x) + c(x)u(x) = a(x)[f(u(x)) - |\nabla u(x)|^s], u > 0 \text{ in } \Omega, \\
& u = 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{2.10}
$$

Proof. Let $\varphi_1 \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ be the first eigenfunction corresponding to the first eigenvalue λ_1 of the problem (2.2).

Since $\lim_{m \searrow 0} f(m) = +\infty$, we obtain for $\min_{x \in \overline{\Omega}} a(x)$ and c like in *Lemma* 2.4, that there exists $\delta > 0$ such that $c(\min_{x \in \overline{\Omega}} a(x))^{-1} < f(m), \forall m \in (0, \delta)$.

Now, let $\underline{u}(x) = \sigma_2\varphi_1$, where

$$
0 < \sigma_2 < \min\left\{\frac{\delta}{\max_{x \in \overline{\Omega}} \varphi_1(x)}, \sigma_1\right\},
$$

and σ_1 is the same positive constant by the proof of *Lemma* 2.4. With a similar argument as in (2.4) it follows that

$$
-\Delta\underline{u}(x) + c(x)\underline{u}(x) + a(x)\left|\nabla\underline{u}(x)\right|^s \leq c < f(\underline{u}(x)) \min_{x \in \overline{\Omega}} a(x) \leq a(x)f(\underline{u}(x)),
$$

that means, $\underline{u}(x) = \sigma_2\varphi_1$ is a lower solution to the problem (2.10). To construct a super solution, we observe that any solution of the problem

$$
\begin{aligned}
& -\Delta u(x) + c(x)u(x) = a(x)[f(u(x)) + |\nabla u(x)|^s], u > 0 \text{ in } \Omega, \\
& u = 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{2.11}
$$

is a super solution of the problem (2.10). But by Lemma 2.5, the problem (2.11) has at least a solution. Denote $\overline{u}(x)$ this solution. As in the proof of (3.3), we have $\underline{u}(x) \leq \overline{u}(x)$ in Ω. Thus, by sub and super solution method (see [7]) we find at least a solution $u \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ to the problem (2.10). $\square$

Remark 1. The existence results from Lemmas 2.4, 2.5 and 2.6 are holds and in the case $c(x) \geq 0$.

The next lemma has been proved first by [17, p. 500].

Lemma 2.7. *Suppose that A3) is satisfied. Then*

(2.12)
$$w(r) := K - \int_0^r \xi^{1-N} \int_0^\xi \sigma^{N-1}\varphi(\sigma) d\sigma d\xi, K := \int_0^\infty \xi^{1-N} \int_0^\xi \sigma^{N-1}\varphi(\sigma) d\sigma d\xi,$$

is the unique positive bounded radially symmetric solution of the problem $-\Delta w = \varphi(r)$ $(r = \mid x \mid)$ *on* $\mathbf{R}^N$ *and* $\lim_{r\to\infty} w(r) = 0$, $w(0) = K$, $w'(0) = 0$.

The following result establishes an existence of a super solution to (1.1) decaying to zero.

Lemma 2.8 (in [5, 13]). *Make the same assumptions as in Theorem* 1.1 *on* a *and as in Lemma* 2.5 *on* f. *By a suitable application of the Implicit Function Theorem, there is* $v(r) := \Gamma^{-1}(Cw(r))$ *a radially symmetric function such that*

$$-\Delta v(r) \geq \varphi(r) f(v(r)), v(r) > 0, 0 < r < \infty, \lim_{r\to\infty} v(r) = 0,$$

where $\Gamma(r) = \int_0^r \frac{s}{\overline{f}(s)} ds$, $r \geq 0$, $\overline{f}(s) = s^2 / \int_0^s \frac{t}{f(t)} dt$, $r \geq 0$, Γ^{-1} *denotes the inverse function of* Γ *on* $[0, \infty)$, C *is a positive constant with*

$$KC \leq \Gamma(C) = \int_0^C \frac{s}{\overline{f}(s)} ds.$$

A useful observation is given in the following.

Lemma 2.9. *Assume that* $f \in C^1((0,\infty),(0,\infty))$ *is singular at* 0. *The following statements are equivalent*

i) $\lim\limits_{s\searrow 0} \frac{f(s)}{s} = \infty$ *and* $\lim\limits_{s\nearrow\infty} \frac{f(s)}{s} = 0$;

ii) *exists* $\varepsilon > 0$ *such that* $\lim\limits_{s\searrow 0} \frac{f(s)}{s+\varepsilon} = \infty$ *and* $\lim\limits_{s\nearrow\infty} \frac{f(s)}{s+\varepsilon} = 0$.

Proof. "i)$\Longrightarrow$ii)" It is important to observe that

$$\lim_{s\searrow 0} \frac{f(s)}{s+\varepsilon} = \lim_{s\searrow 0} f(s) \frac{1}{s+\varepsilon} = +\infty$$

and

$$\lim_{s\nearrow\infty} \frac{f(s)}{s+\varepsilon} = \lim_{s\nearrow\infty} \frac{f(s)}{s} \frac{s}{s+\varepsilon} = 0.$$

"ii)$\Longrightarrow$i)" We proceed as in the first step. More exactly,

$$\lim_{s\searrow 0} \frac{f(s)}{s} = \lim_{s\searrow 0} \frac{f(s)}{s+\varepsilon} \frac{s+\varepsilon}{s} = +\infty$$

and

$$\lim_{s\nearrow\infty}\frac{f(s)}{s}=\lim_{s\nearrow\infty}\frac{f(s)}{s+\varepsilon}\frac{s+\varepsilon}{s}=0.$$

□

The following two lemmas are due to [10, 24] in a particular form. We give here the generalizations in order to obtain the main result.

Lemma 2.10. *Let* $\varepsilon \geq 0$. *If* $f \in C^1((0,\infty),(0,\infty))$ *is singular at* 0 *and* i) *or* ii) *by Lemma* 2.9 *hold, then, there exists the functions* $\underline{f}_\varepsilon \in C^1((0,\infty),(0,\infty))$ *such that*

(1)$\underline{f}_\varepsilon$ *is non-increasing in* $(0,\infty)$ *and* $\underline{f}_\varepsilon(s) \leq \frac{f(s)}{s+\varepsilon}$, $\forall s > 0$;
(2) $\lim_{s\to\infty}\underline{f}_\varepsilon(s)=0$ *and* $\lim_{s\to+0}\underline{f}_\varepsilon(s)=\infty$.

Proof. Due to Lemma 2.9 we can define

$$\underline{f}_\varepsilon(s)=\inf_{s\geq t>0}\frac{f(t)}{t+\varepsilon}$$

We observe that

$$0<\underline{f}_\varepsilon(s)\leq\frac{f(s)}{s+\varepsilon},\forall s>0;$$

and that $\underline{f}_\varepsilon(s)$ are non-increasing functions in $(0,\infty)$. Hence,

$$\lim_{s\to\infty}\underline{f}_\varepsilon(s)=0 \text{ and } \lim_{s\to+0}\underline{f}_\varepsilon(s)=\infty.$$

Moreover, we can suppose that $\underline{f}_\varepsilon \in C^1(0,\infty)$. On the contrary, we can replace this functions by

$$\underline{f}^1_\varepsilon(s)=\int_s^{s+1}\underline{f}_\varepsilon(t)dt, s>0.$$

Obviously,

$$\underline{f}_\varepsilon(s+1)\leq\underline{f}^1_\varepsilon(s)\leq\underline{f}_\varepsilon(s)$$

and

$$[\underline{f}^1_\varepsilon(s)]'=\underline{f}_\varepsilon(s+1)-\underline{f}_\varepsilon(s)\leq 0,\forall s>0$$

i.e., $\underline{f}^1_\varepsilon(s)\in C^1((0,\infty),(0,\infty))$ are non-increasing functions. □

Lemma 2.11. *Let* $\varepsilon \geq 0$. *If* $f \in C^1((0,\infty),(0,\infty))$ *is singular at* 0 *and* i) *or* ii) *from Lemma* 2.9 *is satisfied, then there exists the function* $\overline{f}^\varepsilon \in C^1((0,\infty),(0,\infty))$ *such that*

(1) $\overline{f}^\varepsilon$ *is non-increasing on* $(0,\infty)$ *and* $\frac{f(s)}{s+\varepsilon}\leq\overline{f}^\varepsilon(s)$, $\forall s>0$;
(2) $\lim\limits_{s\to\infty}\overline{f}^\varepsilon(s)=0$ *and* $\lim_{s\to+0}\overline{f}^\varepsilon(s)=\infty$;

Proof. Due to Lemma 2.9 we can define

$$\overline{f}^{\varepsilon}(s) = \sup_{t \geq s > 0} \frac{f(t)}{t+\varepsilon}.$$

We observe that

$$\overline{f}^{\varepsilon}(s) \geq \frac{f(t)}{t+\varepsilon}, \forall s > 0 \text{ and } t \geq s;$$

and that $\overline{f}^{\varepsilon}(s)$ is a non-increasing function in $(0,\infty)$. Hence

$$\lim_{s\to\infty} \overline{f}^{\varepsilon}(s) = 0 \text{ and } \lim_{s\to+0} \overline{f}^{\varepsilon}(s) = \infty.$$

Moreover, we can assume that $\overline{f}^{\varepsilon} \in C^1(0,\infty)$. On the contrary, we can replace this functions by

$$\overline{\overline{f}}^{\varepsilon}(s) = \frac{2}{s}\int_{s/2}^{s} \overline{f}^{\varepsilon}(t)dt, s > 0.$$

Obviously,

$$\overline{f}^{\varepsilon}(s) \leq \overline{\overline{f}}^{\varepsilon}(s) \leq \overline{f}^{\varepsilon}(s/2),\ \forall s > 0;$$

and, for $s > 0$,

$$\begin{aligned}
[\overline{\overline{f}}^{\varepsilon}(s)]' &= \frac{2}{s}\left(\overline{f}^{\varepsilon}(s) - \frac{1}{2}\overline{f}^{\varepsilon}(s/2)\right) - \frac{2}{s^2}\int_{s/2}^{s} \overline{f}^{\varepsilon}(t)dt \\
&\leq \frac{2}{s}\left(\overline{f}^{\varepsilon}(s) - \frac{1}{2}\overline{f}^{\varepsilon}(s/2)\right) - \frac{2}{s^2}\frac{s}{2}\overline{f}^{\varepsilon}(s) \\
&= \frac{1}{s}[\overline{f}^{\varepsilon}(s) - \overline{f}^{\varepsilon}(s/2)] \leq 0,
\end{aligned}$$

respectively $\overline{\overline{f}}^{\varepsilon}(s) \in C^1((0,\infty),(0,\infty))$ are non-increasing functions. □

3. Proof of the Theorem 1.1

Consider the following boundary value problem

$$\begin{aligned}
&-\Delta u(x) + c(x)u(x) = a(x)f(u(x)), u > 0 \text{ in } B_k, \\
&u = 0 \text{ on } \partial B_k,
\end{aligned} \tag{3.1}$$

where $B_k := \{x \in \mathbb{R}^N\, ||x| < k\}$ is a ball of center 0 and radius k. Let $\varepsilon > 0$. We will prove that the problem (3.1) has at least one solution. For this we observe that any solution to

$$\begin{aligned}
&-\Delta u(x) + c(x)u(x) = a(x)[(u(x)+\varepsilon)\underline{f}_{\varepsilon}(u(x)) - |\nabla u(x)|^{s}] \text{ in } B_k, \\
&u = 0 \text{ on } \partial B_k,
\end{aligned}$$

which exists by Lemma 2.6, is a sub solution to (3.1). To construct a super solution, we see by Lemma 2.5 that the following problem

$$-\Delta u(x)+c(x)u(x)=a(x)[(u(x)+\varepsilon)(\overline{f}^{\varepsilon}(u(x))+\tfrac{1}{u(x)+\varepsilon})+|\nabla u(x)|^{s}] \text{ in } B_k,$$
$$u=0 \text{ on } \partial B_k,$$

has a solution which is a super solution (3.1). Denote $\underline{u_\varepsilon}$ (resp. $\overline{u^\varepsilon}$) the sub solution (resp. super solution) to (3.1). Clearly, by the below proof $\underline{u_\varepsilon}\le\overline{u^\varepsilon}$ in B_k. Thus, by sub and super solution method and elliptic regularity theory we find at least a solution $u_k\in C(\overline{B}_k)\cap C^{2+\alpha}(B_k)$ to the problem (3.1), which satisfies

$$\underline{u_\varepsilon}\le u_k(x)\le\overline{u^\varepsilon} \textit{ in } B_k. \tag{3.2}$$

In outside of B_k we put $u_k=0$. We now observe that Lemma 2.8 implies that there exists a positive smooth function v_ε that satisfies

$$-\Delta v_\varepsilon(r)+c(x)v_\varepsilon(r)\ge a(x)(v_\varepsilon(r)+\varepsilon)\left(\overline{f}^{\varepsilon}(v_\varepsilon(r))+\frac{1}{v_\varepsilon(r)+\varepsilon}\right)\ge a(x)f(v_\varepsilon(r))$$
$$r:=|x|\,,\quad x\in\mathbb{R}^N,$$

$0<v_\varepsilon(r)<C_\varepsilon$, with $C_\varepsilon>0$ suitable constant and $v_\varepsilon(r)\to0$ as $|x|\to\infty$. We claim that

$$u_k\le v_\varepsilon,\;\; x\in R^N,\, k=1,2,3,... \tag{3.3}$$

or, equivalently

$$\ln(u_k(x)+\varepsilon)\le\ln(v_\varepsilon+\varepsilon),\; x\in R^N,\, k=1,2,3,...$$

To prove the latter assertion, suppose the contrary, that

$$\Omega_{u_k,v_\varepsilon}:=\{x\in\mathbb{R}^N\,|w_1(x):=\ln(u_k(x)+\varepsilon)-\ln(v_\varepsilon(x)+\varepsilon)>0\,\}\ne\varnothing.$$

Thus, we can suppose that the $\sup_{\mathbb{R}^N}w_1(x)$ is positive. In this case, the point where the supremum occurs must lies in $\mathbb{R}^N$ since we have

$$\lim_{|x|\to\infty}[\ln(u_k(x)+\varepsilon)-\ln(v_\varepsilon(x)+\varepsilon)]=0.$$

Then at that point, say $x_0\in\mathbb{R}^N$, where the supremum is achieved we have

$$\nabla[\ln(u_k(x_0)+\varepsilon)-\ln(v_\varepsilon(x_0)+\varepsilon)]=0,$$

or, equivalently

$$\frac{1}{u_k(x_0)+\varepsilon}\cdot\nabla u_k(x_0)=\frac{1}{v_\varepsilon(x_0)+\varepsilon}\cdot\nabla v_\varepsilon(x_0). \tag{3.4}$$

So, we see that

$$\begin{aligned}
0 \geq \Delta w_1(x_0) &= \frac{\Delta u_k(x_0)}{u_k(x_0)+\varepsilon} - \frac{|\nabla u_k(x_0)|^2}{[u_k(x_0)+\varepsilon]^2} - \frac{\Delta v_\varepsilon(x_0)}{v_\varepsilon(x_0)+\varepsilon} + \frac{|\nabla v_\varepsilon(x_0)|^2}{[v_\varepsilon(x_0)+\varepsilon]^2} \\
&= \frac{\Delta u_k(x_0)}{u_k(x_0)+\varepsilon} - \frac{\Delta v_\varepsilon(x_0)}{v_\varepsilon(x_0)+\varepsilon} \\
&\geq \frac{+c(x_0)u_k(x_0) - a(x_0)f(u_k(x_0))}{u_k(x_0)+\varepsilon} \\
&\quad + a(x_0)\left(\overline{f}^{\varepsilon}(v_\varepsilon(x_0)) + \frac{1}{v_\varepsilon(x_0)+\varepsilon}\right) - \frac{c(x_0)v_\varepsilon(x_0)}{v_\varepsilon(x_0)+\varepsilon} \\
&= c(x_0)\left[\frac{u_k(x_0)}{u_k(x_0)+\varepsilon} - \frac{v_\varepsilon(x_0)}{v_\varepsilon(x_0)+\varepsilon}\right] \\
&\quad - a(x_0)\left[\frac{f(u_k(x_0))}{u_k(x_0)+\varepsilon} - \left(\overline{f}^{\varepsilon}(v_\varepsilon(x_0)) + \frac{1}{v_\varepsilon(x_0)+\varepsilon}\right)\right] \\
&> 0,
\end{aligned}$$

which is a contradiction. Hence $\Omega_{u_k,v_\varepsilon} = \varnothing$. In conclusion, (3.3) holds.

To conclude the proof, it is sufficient to estimate $\{u_k\}$. For any open bounded $C^{2+\alpha}$-smooth domain $\Omega' \subset \mathbb{R}^N$, take Ω_1 and Ω_2 with $C^{2+\alpha}$-smooth boundaries, and K_1 large enough, such that

$$\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset B_k, k \geq K_1$$

Note that

$$u_k(x) \geq \underline{u_\varepsilon} > 0, \forall x \in B_{K_1}, \tag{3.5}$$

when B_{K_1} is the substitution for B_k in (3.2).

Let

$$h_k(x) = a(x)f(u_k(x)) - c(x)u_k(x), x \in \overline{B}_{K_1}.$$

It follows by AC1)-AC2), (3.3) and (3.5) that $\{h_k\}_{K_1}^{\infty}$ is uniformly bounded on $\overline{\Omega}_2$ and hence $h_k \in L^p(\Omega_2)$ for any $p > 1$. Since

$$-\Delta u_k(x) = h_k(x), x \in \Omega_2,$$

we see by Lemma 2.3, that there exists a positive constant C_1 independent of k such that

$$\|u_k\|_{W^{2,p}(\Omega_1)} \leq C_1(\|h_k(x)\|_{L^p(\Omega_2)} + \|u_k\|_{L^p(\Omega_2)}), \forall k \geq K_1,$$

i.e., $\{\|u_k\|_{W^{2,p}(\Omega_1)}\}_{K_1}^{\infty}$ is uniformly bounded. Now take p such that $p > N$ and $p > N(1-\alpha)^{-1}$. Then by applying Sobolev embedding, *Lemma* 2.1, we conclude that

$$\left\{\|u_k\|_{C^{1,\alpha}(\overline{\Omega}_1)} | k \geq K_1\right\}$$

is uniformly bounded by a constant independent of k, which furthermore implies that

$$\{\|h_k\|_{C^{\alpha}(\overline{\Omega}_1)}\}_{K_1}^{\infty},$$

is uniformly bounded. Then an application of the interior Schauder estimates Lemma 2.2 for solutions of elliptic equations we have that there exists a positive constant C_2 independent of k such that

$$\|u_k\|_{C^{2+\alpha}(\overline{\Omega}')} \leq C_2 \left(\|h_k\|_{C^{\alpha}(\overline{\Omega}_1)} + \sup_{\overline{\Omega}_1} u_k \right), \forall k \geq K_1,$$

i.e.,

$$\left\{ \|u_k\|_{C^{2,\alpha}(\overline{\Omega}')} \, | k \geq K_1 \right\} \tag{3.6}$$

is uniformly bounded. Then we can use (3.6) and the Ascoli-Arzela theorems to construct the subsequences $\{u_{k_n}\}$ of $\{u_k\}$, denoted also by $\{u_k\}$, such that it converges uniformly in the $C^2\left(\overline{\Omega}'\right)$ norm to a function $u \in C^2\left(\overline{\Omega}'\right)$ satisfying the equation

$$-\Delta u = a(x)f(u(x)) - c(x)u(x), \text{ in } \overline{\Omega}'.$$

By (3.5), we obtain that $u > 0, \forall x \in \overline{\Omega}'$. Applying the interior Schauder's estimates in (3.1) we see that $u \in C^{2,\alpha}(\overline{\Omega}')$. Since Ω' is arbitrary, we also see that $u \in C^{2,\alpha}_{loc}(\mathbb{R}^N)$.

Thus, we obtain that $u_k \to u$ (pointwise) in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ and $u_k \leq u \leq v_\varepsilon$ in $\mathbb{R}^N$. Since $v_\varepsilon(x) \to 0$ as $|x| \to \infty$, we deduce that $u(x) \to 0$ as $|x| \to \infty$. Hence u is a $C^{2,\alpha}$-entire solution of the problem (1.1). □

In the end of this paper, let us point out that, in the same way as above we can easily deduce the following:

Remark 2. Make the same assumptions as in Theorem 1.1. Then the Lane, Emden and Fowler problem with mixed nonlinear gradient term

$$-\Delta u(x)+c(x)u(x) = a(x)[f(u(x))-|\nabla u(x)|^q+|\nabla u(x)|^s], u > 0 \text{ in } \mathbb{R}^N, \tag{3.7}$$

has at least one solution vanishing at infinity, provided for $s \in (0,1)$, $q \in (0,2]$.

We mention here that the similar problems like (3.7) are proposed by the authors Kusano, Swanson and Usami in [15, p. 396].

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