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On the Property of Riemann Surfaces and the Defect.

By

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I. Introduction.

Let $w=f(z)$ be a meromorphic function in $|z|<R\leqq\infty$ (not rational), then a is said to be exceptional (in R. Nevanlinna's sense) if the defect $\delta(a)=\varliminf_{r\to R}\dfrac{m(r,a)}{T(r,f)}$ is positive.

May we decide whether a is exceptional or not, by the local construction of Riemann surface F of its inverse function? For this, there is a well-known consequence due to Cartan and Selberg:

If there lie only schlicht discs or ones of n-sheets, having only a as the branch point, above the ρ-neighbourhood $|w-a|<\rho$, and furthermore, n is uniformly bounded, then a is not an exceptional value of $f(z)$.

In this paper we want to investigate the property of the simply connected Riemann surfaces and find some sufficient conditions in order that a given value a may be non-exceptional.

II. A property of the simply connected Riemann surfaces.

Let us project the w-plane stereographically on the Riemann sphere Σ of diameter 1 touching the w-plane at the origin.

Let $a=|a|e^{i\alpha}$ be a point on the w-plane, then the surface element of Σ is given by $d\sigma=\dfrac{|a|d|a|d\alpha}{(1+|a|^2)^2}$. We consider a circular domain D_ρ on Σ (spherical cap) obtained by the projection of the disc $|w|\leqq\rho\,(0<\rho<\infty)$. Let $I_0(D_\rho)$ denote the area of D_ρ and $I_r(D_\rho)$ the total area of common parts of the domains above D_ρ and F_r which is the Riemannian image of $|z|\leqq r$, then we have

$$(1)\qquad I_0(D_\rho)=\int_{D_\rho}d\sigma=\frac{\pi\rho^2}{1+\rho^2},$$

$$(2)\qquad I_r(D_\rho)=\int_{D_\rho} n(r,a)\,d\sigma\,,$$

where $n(r,a)$ denotes the number of a-points in $|z|\leqq r<R$. Put

$$d\mu_{D_\rho}=\frac{d\sigma}{I_0(D_\rho)}\,,$$

then μ_{D_ρ} is a continuous mass-distribution on D_ρ of total mass 1. Denoting by $S_r(D_\rho)$ the average number of sheets of F_r above D_ρ and using (2), we may write

$$(3)\quad S_r(D_\rho)\equiv\frac{I_r(D_\rho)}{I_0(D_\rho)}=\int_{D_\rho} n(r,a)\frac{d\sigma}{I_0(D_\rho)}=\int_{D_\rho} n(r,a)\,d\mu_{D_\rho}\,.$$

Here we consider the following formula

$$(4)\quad T(r)=\int_0^r\frac{A(t)}{t}dt=N(r,a)$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{k(w(re^{i\varphi}),a)}\,d\varphi-\log\frac{1}{k(w_0,a)}\,,$$

where $w_0=w(0)\neq a$ and $k(w,a)=\dfrac{|w-a|}{\sqrt{(1+|w|^2)(1+|a|^2)}}$ denotes the euclidean distance between w and a on Σ. Multiplying $d\mu_{D_\rho}(a)$ both sides of (4) and integrating on D_ρ, we have

$$(5)\quad T(r)=\int_{D_\rho} N(r,a)\,d\mu_{D_\rho}+\frac{1}{2\pi}\int_0^{2\pi}P(w(re^{i\varphi}))\,d\varphi-P(w_0),$$

where $P(w)$ denotes the spherical logarithmic potential on D_ρ of mass distribution μ_{D_ρ}:

$$P(w)=\int_{D_\rho}\log\frac{1}{k(w,a)}\,d\mu_{D_\rho}\,.$$

$P(w)$ remains finite so far as $0<\rho<\infty$. We shall next give an explicit form of it.

First, since $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\log|w-e^{i\theta}|\,d\theta=\overset{+}{\log}|w|$[5],

$$u(w)=\int_{D_\rho}\log\frac{1}{|w-a|}\,d\mu_{D_\rho}=\frac{1+\rho^2}{\rho^2}\int_0^\rho\int_0^{2\pi}\log\frac{1}{|w-a|}\cdot\frac{|a|d|a|d\alpha}{\pi(1+|a|^2)^2}$$

$$=-\frac{2(1+\rho^2)}{\rho^2}\int_0^\rho\left(\log|a|+\overset{+}{\log}\left|\frac{w}{a}\right|\right)\frac{|a|d|a|}{(1+|a|^2)^2}\,.$$

(i) For $|w|>\rho$,

$$u(w)=-\frac{2(1+\rho^2)}{\rho^2}\int_0^\rho \frac{|a|\log|w|}{(1+|a|^2)^2}\,d|a|=\log\frac{1}{|w|}\,.$$

(ii) For $|w|\leqq\rho$,

$$u(w)=-\frac{2(1+\rho^2)}{\rho^2}\left[\int_0^{|w|}\frac{|a|\log|w|}{(1+|a|^2)^2}\,d|a|+\int_{|w|}^{\rho}\frac{|a|\log|a|}{(1+|a|^2)^2}\,d|a|\right]$$

$$=\log\sqrt{1+\frac{1}{\rho^2}}+\frac{1}{\rho^2}\log\sqrt{1+\rho^2}-\left(1+\frac{1}{\rho^2}\right)\log\sqrt{1+|w|^2}\,.$$

Therefore, if we put

$$F(\rho)\equiv\int_{D_\rho}\log\sqrt{1+|a|^2}\,d\mu_{D_\rho}=\frac{1}{2}-\frac{1}{\rho^2}\log\sqrt{1+\rho^2}\,,$$

we can evaluate $P(w)$ in the following manner:

I. $\rho<|w|<\infty$;

$$P(w)=u(w)+\log\sqrt{1+|w|^2}+F(\rho)=\log\sqrt{1+\frac{1}{|w|^2}}+F(\rho).$$

Hence $F(\rho)<P(w)<\log\sqrt{1+\frac{1}{\rho^2}}+F(\rho)$.

II. $|w|\leqq\rho$;

$$P(w)=\log\sqrt{1+\frac{1}{\rho^2}}+\frac{1}{\rho^2}\log\sqrt{\frac{1+\rho^2}{1+|w|^2}}+F(\rho).$$

Hence $F(\rho)<P(w)\leqq\log\sqrt{1+\frac{1}{\rho^2}}+\frac{1}{\rho^2}\log\sqrt{1+\rho^2}+F(\rho)$.

III. $w=\infty$;

since $\log\dfrac{1}{k(\infty,a)}=\log\sqrt{1+|a|^2}$,

$$P(\infty)=\int_{D_\rho}\log\frac{1}{k(\infty,a)}\,d\mu_{D_\rho}=F(\rho).$$

Thus we have always, for any w,

$$(6)\quad F(\rho)\leqq P(w)\leqq\log\sqrt{1+\frac{1}{\rho^2}}+\frac{1}{\rho^2}\log\sqrt{1+\rho^2}+F(\rho).$$

The same result is obtained for the integral

$$(7)\quad I=\frac{1}{2\pi}\int_0^{2\pi}P(w(re^{i\varphi}))\,d\varphi\,.$$

But the equality sign now does not occur. For, otherwise $w(z)$ reduces to a constant. From (3), (5), (6) and (7), we have

$$(8)\quad \int_0^r \frac{S_t(D_\rho)}{t}\,dt < T(r) + \log\sqrt{1+\frac{1}{\rho^2}} + \frac{1}{\rho^2}\log\sqrt{1+\rho^2}.$$

$$(9)\quad T(r) < \int_0^r \frac{S_t(D_\rho)}{t}\,dt + \log\sqrt{1+\frac{1}{\rho^2}} + \frac{1}{\rho^2}\log\sqrt{1+\rho^2}.$$

That is,

$$(10)\quad \left| T(r) - \int_0^r \frac{S_t(D_\rho)}{t}\,dt \right| < \log\sqrt{1+\frac{1}{\rho^2}} + \frac{1}{\rho^2}\log\sqrt{1+\rho^2}.$$

For $0<\rho\leqq 1$ the right-hand-side can be replaced by $\log\frac{1}{\rho}+C$, where $0<C<\frac{1}{2}(1+\log 2)$, and we shall later use this form. Now, by (1), we have

$$(11)\quad \rho^2 = \frac{I_0(D_\rho)}{\pi - I_0(D_\rho)}.$$

Putting (11) into the right hand side of (10) and remarking that the quantities $A(t)$, $S_t(D_\rho)$, $I_0(D_\rho)$ appeared there are all invariant for the rotation of Riemann sphere, we have the following

Theorem 1. Suppose F the simply connected Riemann surface of the inverse function spread over Riemann sphere Σ. Let D, $I_0(D)$ and $S_r(D)$ denote respectively an arbitrary disc on Σ, its area and the average number of sheets of F_r above D. Then

$$(12)\quad \left| \int_0^r \frac{S_t(\Sigma)}{t}\,dt - \int_0^r \frac{S_t(D)}{t}\,dt \right| < \log\sqrt{\frac{\pi}{I_0(D)}} + \frac{\pi - I_0(D)}{I_0(D)}\log\sqrt{\frac{\pi}{\pi - I_0(D)}}$$

$$= \frac{1}{I_0(D)} \sum_{1,2} I_0(D_i)\log\sqrt{\frac{\pi}{I_0(D_i)}}$$

$(D_1 = D,\ D_2 = \text{complementary disc of } D)$.

If we suppose D, as special case, be the hemi sphere, we have the following

Corollary: Let D_1 and D_2 denote respectively the north and south hemi spheres. Then we have

$$\left|\int_0^r \frac{S_t(D_1)}{t}\,dt - \int_0^r \frac{S_t(D_2)}{t}\,dt\right| < 2\log 2\,.$$

Remark 1. When $D \to \Sigma$, $S_t(D) \to S_t(\Sigma)$ and now both sides of (12) tend to zero. When $D \to$ a point, the right hand side tends to logarithmic infinity.

Remark 2. Integrating the Ahlfors' first covering theorem with respect to $\log r$, we have the same expression as the left hand side of (12), but the other side is $\dfrac{h}{I_0(D)}\displaystyle\int_0^r \frac{L(t)}{t}\,dt$. This expression depends on r, $I_0(D)$ and a constant h. While the right hand side of (12) depends only on $I_0(D)$.

III. Some Lemmas.

For our purpose we shall now give some lemmas.

Lemma 1. Let $\zeta = \zeta(\omega)$ be a regular schlicht function in $|\omega| < 1$. Suppose that $\zeta(0) \neq 0$ and ζ-image D of $|\omega| < 1$ does not contain the disc $|\zeta| \leqq |\zeta(0)|$ perfectly. Then we have

$$|\zeta'(0)| \leqq 8|\zeta(0)|.$$

Proof. Let l denote the smallest distance connecting $\zeta(0)$ to the intersection points of $|\zeta| = |\zeta(0)|$ and the boundary of D. Since D does not contain $|\zeta| \leqq |\zeta(0)|$ perfectly, such l always exists and $0 < l \leqq 2|\zeta(0)|$. By Koebe's theorem we have

$$\frac{1}{4}|\zeta'(0)| \leqq l \leqq 2|\zeta(0)|, \text{ q.e.d.}$$

Remark. The extreme case is attained by the function

$$\zeta(\omega) = a + \frac{8a\omega}{(1-\omega)^2} \quad (a\text{: arbitrary number})$$

which maps $|\omega| < 1$ to the plane with a cut $(-a, \infty)$.

Lemma 2. Suppose that $\zeta = \zeta(\omega)$ maps the n-ple disc $|\omega| < \rho$ having only $\omega = 0$ as the branch point conformally on D. Suppose further that $\zeta(0) \neq 0$ and D does not contain $|\zeta| \leqq |\zeta(0)|$ perfectly. Then we have

$$|\zeta(\omega) - \zeta(0)| < d \quad \text{in } |\omega| \leqq \left(\frac{d}{2d + 8|\zeta(0)|}\right)^n \rho,$$

where d is a real positive number.

Proof. Let $n=1$, $\rho=1$. By the "Verzerrungssatz" of schlicht functions we have

$$\left|\frac{\zeta(\omega)-\zeta(0)}{\zeta'(0)}\right|\leqq\frac{|\omega|}{(1-|\omega|)^2},$$

hence

$$\max_{|\omega|=\theta}|\zeta(\omega)-\zeta(0)|\leqq\frac{\theta}{(1-\theta)^2}|\zeta'(0)|\,(0<\theta<1).$$

Therefore, we have

$$|\zeta(\omega)-\zeta(0)|_{|\omega|\leqq\theta}\leqq d$$

for any θ which satisfies

$$\frac{\theta}{(1-\theta)^2}|\zeta'(0)|\leqq d. \tag{13}$$

Let θ_1 be a solution of (13), then we have, by lemma 1,

$$\theta_1=\frac{2d}{2d+|\zeta'(0)|+\sqrt{4d|\zeta'(0)|+|\zeta'(0)|^2}}>\frac{d}{2d+|\zeta'(0)|}$$

$$\geqq\frac{d}{2d+8|\zeta(0)|}.$$

In the other case, put $w=\sqrt[n]{\frac{\omega}{\rho}}$ and consider the mapping $w\to\omega\to\zeta$, then since $\zeta=\zeta(\omega)=\zeta(\rho w^n)\equiv\zeta_1(w)$ maps $|w|<1$ conformally on D, by the above result if $|w|\leqq\frac{d}{2d+8|\zeta_1(0)|}$ i. e.

$$|\omega|\leqq\left(\frac{d}{2d+8|\zeta(0)|}\right)^n\cdot\rho,\quad\text{we have } |\zeta(\omega)-\zeta(0)|<d,\quad\text{q.e.d.}$$

To make the expression simple, we write $N(r)$, $n(r)$ instead of $N(r,a)$, $n(r,a)$ respectively.
Let $w=f(z)$ be a meromorphic function in $|z|<\infty$.
Since $N(r)$ is the convex function with respect to $\log r$, we have for $r<\rho<\rho'$

$$N(\rho)-N(r)\leqq\frac{\log\frac{\rho}{r}}{\log\frac{\rho'}{r}}(N(\rho')-N(r))\leqq\frac{\log\frac{\rho}{r}}{\log\frac{\rho'}{r}}(T(\rho')+O(1))$$

$$\leqq \frac{\rho'}{r} \cdot \frac{\rho - r}{\rho' - r} (T(\rho') + O(1)).^{(4)}$$

Therefore if ρ is defined as

$$\rho - r = \frac{r}{\rho'} \cdot \frac{\rho' - r}{T(\rho')} \quad (< \rho' - r) \tag{14}$$

it follows

$$N(\rho) - N(r) \leqq O(1). \tag{15}$$

Here for our later purpose we adopt $\rho' = r + \dfrac{1}{\log T(r)}$. Then we have by (14), (15) and easy calculation,

$$\begin{cases} \text{if} \quad \rho = r + 1/T\left(r + \dfrac{1}{\log T(r)}\right)^{\alpha} & (\alpha > 1), \\ N(\rho) \leqq N(r) + O(1) & (r \geqq r_0). \end{cases} \tag{16}$$

IV. Theorems.

Consider $w = f(z)$ which is meromorphic in $|z| < R \leqq \infty$ (not rational). Let F_r denote the Riemannian image of $|z| \leqq r$ and $\rho = \rho(r, a)$ be taken so small that all the discs above $\rho(r, a)$-neighbourhood $|w - a| < \rho$* having common part with F_r are only schlicht discs or those with n-sheets having only a as the branch point. Let $\lambda(r)$ be a maximum number of n, then we have

Theorem 2. Let F be an open Riemann surface of the parabolic type. Suppose that $\lambda(r) \leqq \Lambda$ (bounded) and $\lim\limits_{r \to \infty} \dfrac{\log 1/\rho(r, a)}{T(r)} = 0$*, then a is not exceptional.

Proof. For simplicity we assume $a = 0$. The other case can be reduced to this case, if we bring a to the origin by a certain rotation of the Riemann sphere Σ. Now consider the functions

$$d(r) = 1/T\left(r + \frac{1}{\log T(r)}\right)^{\alpha} \quad (\alpha > 1)$$

and

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* When $a = \infty$, we may consider $|w| > \rho(r, \infty)$ for the neighbourhood and $\lim\limits_{r \to \infty} \dfrac{\log \rho(r, \infty)}{T(r)} = 0$ as the condition in Theorem 2.

(17) $\bar{\rho}(r) \equiv \bar{\rho}(r, 0) = \left\{ \dfrac{d(r)}{k(2d(r)+r)} \right\}^{\Lambda} \cdot \rho_1(r, 0) \qquad (r \geqq r_0),$

where $\rho_1(r, 0) = \dfrac{\rho(r, 0)}{2}$ and k is a numerical constant $\geqq 8$. Next, we describe a circle $|w| = \bar{\rho}(r)$ in w-plane and let us map every domain above this disc to z-plane by the inverse function of $f(z)$. Now by the definition of $\rho_1(r, 0)$, the images of $|w| \leqq \rho_1(r, 0)$ having common parts with $|z| \leqq r$ are all simply connected and have no common part one another and moreover, for $|w| \leqq \bar{\rho}(r)$, by lemma 2, they are either contained in circles of radius $d(r)$ around zero-points except at most a domain containing the origin, or have no common part with $|z| \leqq r$. Namely according as the modulus of the zero-point is less than $r + d(r)$ or equal to $r + l\,(l > d)$, each domain containing it belongs respectively to the former or to the latter, since

$$\frac{d}{k(2d+r)} < \frac{d}{2d+8(r+d)} < 1, \quad \frac{d}{k(2d+r)} < \frac{l}{2l+8(r+l)} < 1, \qquad (k \geqq 8)$$

and $\lambda(r) \leqq \Lambda$.

Here we adopt $\bar{\rho}(r)$ for ρ in (10) and vary the basic domain with r. Since $S_t(D_{\bar{\rho}}) = \dfrac{I_t(D_{\bar{\rho}})}{I_0(D_{\bar{\rho}})}$ and all the zero-points of the above mentioned image-domains which have common parts with $|z| \leqq t$ are contained at most in $|z| \leqq t + d(r)$ for any $t \leqq r$, we have

(18) $S_t(D_{\bar{\rho}(r)}) \leqq n(t + d(r)) \qquad (r \geqq r_1),$

where r_1 denotes the smallest modulus of zero-points.

I. In case $w(0) \neq 0$, for any given $\varepsilon > 0$, we can choose r_0 so large that $\bar{\rho}(r)$ $(r > r_0)$ becomes very small. Then we have

$$\int_0^r \frac{S_t(D_{\bar{\rho}})}{t}\, dt = \int_0^{r_0} + \int_{r_0}^r \leqq \int_{r_0}^r \frac{n(t+d(r))}{t}\, dt + O(1)$$

$$= (1+\varepsilon)\{N(r+d(r)) - N(r_0)\} + O(1)$$

$$= (1+\varepsilon) N(r) + O(1) \qquad \text{by (16).}$$

II. In case $w(0)$ is λ-ple zero, we can also choose r_0 so large that $\bar{\rho}(r)$ $(r > r_0)$ becomes very small. Then.

$$\int_0^{\bar{\rho}} \frac{S_t(D_{\bar{\rho}})}{t}\, dt = O\left(\frac{1+\bar{\rho}^2}{\pi\bar{\rho}^2}\int_0^{\bar{\rho}} \frac{dt}{t}\int_0^t\int_0^{2\pi} \frac{|w'|^2}{(1+|w|^2)^2}\,\tau d\tau d\theta\right) = O(1)$$

$$\int_{\bar{\rho}}^{r} \frac{S_t(D_{\bar{\rho}})}{t}\, dt = \int_{\bar{\rho}}^{r} \frac{S_t(D_{\bar{\rho}})-\lambda}{t}\, dt + \lambda \log r + \lambda \log \frac{1}{\bar{\rho}}\,.$$

Therefore

$$\int_0^{r} \frac{S_t(D_{\bar{\rho}})}{t}\, dt \leqq \int_{r_0}^{r+d(r)} \frac{n(\tau)-\lambda}{\tau}\, d\tau + \lambda \log(r+d) + \lambda \log \frac{1}{\bar{\rho}} + O(1)$$

$$\leqq (1+\varepsilon)N(r) + \lambda \log \frac{1}{\bar{\rho}} + O(1). \quad \text{by (16).}$$

Thus, by (9), we have for any $r \geqq r_0$

$$(19) \qquad m(r, 0) \leqq \varepsilon N(r) + \mu \log \frac{1}{\bar{\rho}(r)} + O(1) \quad (\mu = \lambda + 1)$$

As $\qquad N(r) \leqq T(r) + O(1),$

$$(20) \qquad \delta(0) = \varlimsup_{r\to\infty} \frac{m(r, 0)}{T(r)} \leqq \varepsilon + \mu \varlimsup_{r\to\infty} \frac{\log \frac{1}{\bar{\rho}(r)}}{T(r)}\,.$$

Under the condition $\displaystyle \varlimsup_{r\to\infty} \frac{\log \frac{1}{\rho(r, 0)}}{T(r)} = 0$,

$$\varlimsup_{r\to\infty} \frac{\log \frac{1}{\bar{\rho}(r)}}{T(r)} = N\alpha \varlimsup_{r\to\infty} \frac{\log T\left(r + \frac{1}{\log T(r)}\right)}{T(r)}\,.$$

While by Borel's Lemma $T(r)$ satisfies a relation

$$T\left(r + \frac{1}{\log T(r)}\right) < T(r)^2$$

except at most the suit of intervals that the total linear mass is finite. Therefore we have

$$\varlimsup_{r\to\infty} \frac{\log T\left(r + \frac{1}{\log T(r)}\right)}{T(r)} = 0 \text{ and } \delta(0) \leqq \varepsilon\,.$$

As $\varepsilon > 0$ is arbitrary, we can conclude that $\delta(0) = 0$, q.e.d.

Remark. Cartan-Selberg's theorem is the special case—— $\rho(r, a) = \text{const.}$——of our Theorem 2.

Theorem 3. Let F be an open Riemann surface of the hyperbolic type. Suppose that $\lambda(r) \leqq \Lambda$ (bounded),

$$\lim_{r\to 1} \frac{\log 1/\rho(r, a)}{T(r)} = 0^* \quad \text{and} \quad \lim_{r\to 1} \frac{\log \frac{1}{1-r}}{T(r)} = 0$$

(i.e. the case $\Sigma\delta(a) \leqq 2$),

then a is not an exceptional value.

Proof. We can prove this by taking $d(r)$ in the above proof as follows. i.e. Here we adopt ρ' defined by

$$\frac{1}{1-\rho'} = \frac{1}{1-r} + \frac{1}{\log T(r)} \quad \text{as } \rho \text{ in (14)}.$$

Hence we have

$$(21) \quad \begin{cases} \text{If } \rho = r + \dfrac{(1-r)^2}{T\left(r + \dfrac{(1-r)^2}{(1-r) + \log T(r)}\right)^{\alpha}} & (\alpha > 1), \\ N(\rho) \leqq N(r) + O(1) \quad (r \geqq r_0). \end{cases}$$

In this case also we obtain

$$\delta(0) \leqq \varepsilon + \mu N\alpha \varliminf_{r\to 1} \frac{\log T(\rho')}{T(r)},$$

If we take $d(r) = \rho - r$ (ρ in (21)). While by Borel's Lemma we have

$$T(\rho') < T(r)^2$$

except at most the suit of intervals such that the total variation $\int d\left(\frac{1}{1-r}\right)$ is finite. Hence $\varliminf_{r\to 1} \frac{\log T(\rho')}{T(r)} = 0$, thus our proof is completed.

V. Remarks.

1. If we reflect the above proofs, it will be found that $\lambda(r)$ does not need to be bounded. i.e.

If $\lambda(r) \leqq \Lambda$ or $\lambda(r) \log r = O(T(r)^{1-\delta})$ (for Theorem 2.),

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* Cf. the foot-note of Theorem 2.

$$\lambda(r)\log\frac{1}{1-r}=O(T(r)^{1-\delta}) \quad \text{(for Theorem 3.)} \quad 0<\delta<1,$$

and $\lim\limits_{r\to R}\dfrac{\log 1/\rho(r,a)}{T(r)}=0$, we have also $\delta(0)=0$.

2. We shall find, by the slight modification of the definition of $\rho(r,a)$, that there may lie logarithmic singular points above a. i.e. Let $\tilde{\rho}(r,a)$ be taken so small that each $\tilde{\rho}(r,a)$-neighbourhood $|w-a|<\tilde{\rho}(r,a)$ of the logarithmic singular points has no common part with F_r, then Theorems 2 and 3 hold good even when $\rho(r,a)$ is replaced by the function min. $(\tilde{\rho}(r,a),\ \rho(r,a))$.

At the end I wish to express my hearty thanks to Professors Toshizô Matsumoto and Akira Kobori for their kind guidance during my researches.

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References.

(1) Collingwood, Sur les values exceptionnelles des fonctions entière d'ordre fini, C.R. 179.
(2) Cartan, Sur les values exceptionnelles d'une fonction mèromorphe dans tout le plan, C.R. 190.
(3) Kakutani, On the exceptional value of meromorphic functions, Proc. Phy.-Math. 17. (1935).
(4) R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions mèromorphes. 1929.
(5) R. Nevanlinna, Eindeutige analytische Funktionen. 1936.
(6) Teichmüller, Eine Umkehrung des zweiten Hauptsatzes der Wertverteilungslehre. Deut. Math. 2. (1937).

Added during the proof. We can always choose $\rho(r,a)$ so that all the discs above $|w-a|<\rho(r,a)$ intersecting with F_r belong either to K_1-class or to K_2-class, where K_1 and K_2 consist respectively of n-ple discs having only a as the branch point $(1\leq n\leq\lambda(r))$, and of infinite sheets of discs S, such as $a\bar{\in}S\cap F_r$. Let $q(t)$ denote the number of sheets which $\tilde{F}_t$, the Riemannian image of $|z|<t\leq r$, penetrates in K_2 above $|w-a|<\bar{\rho}(r)$, then the results of § V. 1 hold good, if $\int^r\frac{q(t)}{t}dt=o(T(r))$.