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On a property of commutators in the unitary group.

By

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In his interesting paper[1] André Weil had generalized the classical inversion theorem of Jacobi to non-abelian case. He had established namely the correspondence between the classes of representations of the Poincaré group of a Riemann surface $\boldsymbol{C}$ of genus p and the classes of divisors on $\boldsymbol{C}$. The complex variety of the divisor classes of degree n may be called the "hyperjacobian variety" since it coincides with the Jacobian variety in the case of $n=1$. The author examined more than two years ago some topological properties of the hyperjacobian variety and observed that they are based on a property of commutators in the unitary group, which can be stated as follows:

Proposition f. The analytic mapping

$$f(X, Y)=XYX^{-1}Y^{-1} \tag{1}$$

from the product $U(n)\times U(n)$[2] onto[3] ${}^{s}U(n)$ is "open at a point" $\{A, B\}$ if and only if the matrices A and B generate a subgroup of $U(n)$, which is irreducible as its own representation.

In order to "linearize" the mapping (1), we differentiate it and obtain

$$df=dX\cdot YX^{-1}Y^{-1}+XY\cdot dX^{-1}\cdot Y^{-1}$$
$$+X\cdot dY\cdot X^{-1}Y^{-1}+XYX^{-1}\cdot dY^{-1}.$$

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1) Généralisation des fonctions abéliennes, J. Math. pures et appliquées, Tome 17 (1938). Cf. Also H. Tôyoma's notes in Proc. Imp. Acad. Tokyo Vol. XIX—.

2) We shall use the same notations as in Weyl's book: Classical groups (1939).

s) This can be proved most elementary by Shoda's device: Einige Sätze über Matrizen, Jap. J. of Math., Vol. 13 (1937).

If we multiply $f^{-1}=YXY^{-1}X^{-1}$ to this equation from the left side and if we put

$$f^{-1}df=\delta f,\quad X^{-1}dX=\delta X \text{ etc.,}$$

we have

$$\begin{aligned}\delta f &= YXY^{-1}\cdot\delta X\cdot YX^{-1}Y^{-1}+Y\cdot\delta X^{-1}\cdot Y^{-1}\\ &\quad +YX\cdot\delta Y\cdot X^{-1}Y^{-1}+\delta Y^{-1}\\ &=(YX)[(Y^{-1}\cdot\delta X\cdot Y-\delta X)-(X^{-1}\cdot\delta Y\ X-\delta Y)](YX)^{-1}.\end{aligned}$$

Now the "infinitesimal matrices" δX, δY and δf generate the Lie algebras $L(n)$ and ${}^sL(n)$ of $U(n)$ and ${}^sU(n)$ respectively; and our proposition f is equivalent to the following

Proposition F. The linear mapping

$$F(X,Y)=(A^{-1}XA-X)+(B^{-1}YB-Y) \tag{2}$$

from the direct sum $L(n)+L(n)$ into ${}^sL(n)$ is an "onto-mapping" if and only if the matrices A and B generate a subgroup of $U(n)$, which is irreducible as its own representation.

In order to prove this proposition we define the "inner product" (X,Y) for X, Y in ${}^sL(n)$ by

$$(X,Y)=tr(X\bar{Y}^*),$$

then (X,Y) is an invariant of the adjoint group of ${}^sU(n)$. Now if we have

$$F(L(n),L(n))\subsetneq {}^sL(n), \tag{3}$$

there exists at least one non-zero "vector" Z in ${}^sL(n)$, which is "orthogonal" to the linear subvariety $F(L(n),L(n))$ of ${}^sL(n)$. By a suitable operation T of the adjoint group of ${}^sU(n)$, we can transform the matrix Z into the diagonal form as follows

$$TZT^{-1}=\tilde{Z}=\begin{pmatrix}\sqrt{-1}\theta_1 & & 0\\ & \ddots & \\ 0 & & \sqrt{-1}\theta_n\end{pmatrix}$$

$$\theta_1=\ldots=\theta_{n_1}>\theta_{n_1+1}=\ldots=\theta_{n_2}>\ldots>\theta_{n_{t-1}+1}=\ldots=\theta_n.$$

Since we have

$$tr\tilde{Z}=\sqrt{-1}(\theta_1+\ldots+\theta_n)=trZ=0$$

and since $\tilde{Z}\neq 0$, we must have

$$t \geqq 2.$$

Moreover if we put

$$\tilde{A} = TAT^{-1},\ \tilde{B} = TBT^{-1},$$

the statement (3) is equivalent to the equations

$$(4) \qquad (\tilde{A}^{-1}X\tilde{A} - X, \tilde{Z}) = 0,$$

$$(5) \qquad (\tilde{B}^{-1}Y\tilde{B} - Y, \tilde{Z}) = 0$$

for all X, Y in $L(n)$. Now let a_{ij} and x_{ij} $(1 \leqq i, j \leqq n)$ be the coefficients of $\tilde{A}$ and X respectively, then the first equation of the above ones can be written as

$$(6) \qquad \sum_{ijk=1}^{n} \theta_k(\bar{a}_{ik}a_{jk}x_{ij} - x_{kk}) = 0.$$

Since x_{ij} are arbitary under the unitary restriction

$$x_{ij} + \bar{x}_{ji} = 0 \quad (1 \leqq i, j \leqq n),$$

(6) is equivalent to

$$(6a) \qquad \sum_{k=1}^{n} \theta_k(|a_{hk}|^2 - \delta_{hk}) = 0,$$

$$(6b) \qquad \sum_{k=1}^{n} \theta_k \bar{a}_{ik} a_{jk} = 0$$

$$(1 \leqq i, j, h \leqq n\, ; i \neq j).$$

It follows from (6a)

$$\left(1 - \sum_{k=1}^{n_1} |a_{hk}|^2\right)\theta_{n_1} = \sum_{k > n_1} \theta_k |a_{hk}|^2 \leqq \theta_{n_1+1}\left(1 - \sum_{k=1}^{n_1} |a_{hk}|^2\right)$$

for $1 \leqq h \leqq n_1$; since $\theta_{n_1} > \theta_{n_1+1}$, we must have

$$\sum_{k=1}^{n_1} |a_{hk}|^2 = 1,$$

and therefore

$$a_{hk} = a_{kh} = 0 \quad (1 \leqq h \leqq n_1,\ k > n_1).$$

By the continuation of this process we conclude that $\tilde{A}$ must be of the following form

$$(7) \qquad \begin{pmatrix} \left.\underbrace{\boxed{}}_{n_1}\right\} n_1 & & 0 \\ & \left.\underbrace{\boxed{}}_{n_2}\right\} n_2 & \\ 0 & & \ddots \end{pmatrix}.$$

The equations (6b) are then automatically satisfied; and conversely if $\tilde{A}$ is of this form, (4) holds for every X in $L(n)$. In the same way (5) is equivalent to the fact that $\tilde{B}$ is of the form (7), which completes our proof.

Now let $\mathfrak{G}$ be any discrete group, then the set $G=\{\Phi\}$ of all unitary representations of degree n of this group constitute a compact space, if we introduce in G a "weak topology" by the following set of operators on G:

$$\dot{a}(\Phi)=\Phi(a) \qquad (a\epsilon\mathfrak{G}).$$

In particular, if $\mathfrak{G}$ admits a finite number of generators $a_i(1\leqq i\leqq m)$ with defining relations

$$\pi:\ \rho(a)=1$$

the points of G are in a one-to-one correspondence with the set of m matrices $X_i(1\leqq i\leqq m)$ satisfying

$$P:\rho(X)=I_n \qquad \text{(unit matrix of degree } n),$$

$$X_i\bar{X}_i^*=I_n \qquad (1\leqq i\leqq m).$$

Furthermore let $x_{ijk}+\sqrt{-1}\,y_{ijk}$ $(1\leqq j,\ k\leqq n)$ be the coefficients of X_i for $1\leqq i\leqq m$, then the above equations are polynomial relations in the $2m\cdot n^2$ real parameters x_{ijk} and y_{ijk} with integral rational coefficients, so that they define a compact algebraic variety $G(a,\pi)$ in the $2mn^2$ dimensional real affine space, which is homeomorphic with G. The "geometric structure" of $G(a,\pi)$ depends neither on the choice of the generators a nor on the choice of the defining relations π, when $\mathfrak{G}$ is given, so that we may speak of the geometric structure of G

In particular let $\mathfrak{G}$ be the Poincaré group of the Riemann surface C; $\mathfrak{G}$ has $2p$ generators $\{\alpha_i,\beta_i\}(1\leqq i\leqq p)$ with the defining relation

$$\Pi_{i=1}^{p}\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}=1.$$

We conclude from our proposition F the following

Corollary 1. If $p\geqq 2$, a point $\{A_i,\ B_i\}(1\leqq i\leqq p)$ of the variety G is multiple on G if and only if the representation

$$\Phi:\ \alpha_i\rightarrow A_i,\ \beta_i\rightarrow B_i\ \ (1\leqq i\leqq p)$$

is reducible. Moreover it holds

$$\dim G=(2p-1)n^2+1.$$

By a similar argument as in the proof of proposition F, we can also prove the

Corollary 2. If $p=1$, a point $\{A, B\}$ is multiple if and only if the representation

$$\Phi: \alpha \rightarrow A, \quad \beta \rightarrow B$$

contains the same irreducible representation at least two times. Moreover it holds

$$\dim G=n^2+n.$$

We note that our propositions are false if we take $GL(n)$ instead of $U(n)$; and the two corollaries are also false if we consider the representations by $GL(n)$.

Now the variety G admits the transformation

$$\{A_i, B_i\} \rightarrow \{TA_iT^{-1}, TB_iT^{-1}\} \quad (1 \leqq i \leqq p)$$

for every T in $U(n)$; they form a group $\mathfrak{F}$ which is locally isomorphic with ${}^sU(n)$. We construct a "conjugate space" V of G by identifying the "equivalent" points of G under this transformation group. It holds

$$\dim V=\begin{cases} 2[(p-1)n^2+1] & (p \geqq 2) \\ 2n & (p=1) \end{cases}$$

and V is a "base" of the "fibre-space" G, where the "fibre" is the manifold of the group $\mathfrak{F}$ in general for $p \geqq 2$. In view of Weil's result[4] V gives the topological structure of the hyperjacobian variety (which is invariant under the "birational correspondence"). Now we can prove that the Poincaré group of V is a free abelian group with $2p$ generators. Moreover C is a subvariety of V and the "canonical paths" $\{\alpha_i, \beta_i\}$ $(1 \leqq i \leqq p)$ on C generate exactly the Poincaré group of V, which is well-known in the case of the Jacobin variety.

4) Cf. loc. cit 1)