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On the Uniqueness of Solutions of a System of Ordinary Differential Equations[1]

By

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The late Prof. Okamura discovered a remarkable function[2] concerning the uniqueness of the solution of Cauchy-problem of the system of differential equations. In this paper we extend it so as to fit to more general problems.

1. Extended definition. Consider a system of differential equations

$$(1) \quad \frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n) \quad (i=1.2, \cdots n),$$

where f_i are continuous for simplicity in the domain

$$G: 0 \leqq x \leqq a, \quad b_i \leqq y_i \leqq b'_i, \quad (i=1, 2, \ldots, n).$$

Let H_α be a hyperplane defined by $x=\alpha$ $(0 \leqq \alpha \leqq a)$ in G, and S_α an arbitrary sub-space in H_α. S_α may be a single point or H_α itself. Besides S_α is considered as regular property in this paper; S_α is supposed a closed set and hence among distances from a point P of H_α to any point in S_α, the minimum exists, which is called the distance from P to S_α denoted by $\overline{PS_\alpha}$ or $\overline{S_\alpha P}$.

Now let S_ξ and $S_{\xi'}$ be two sub-spaces in G such as $x=\xi$ and $x=\xi'$ $(\xi \leqq \xi')$ respectively. Divide the interval $[\xi, \xi']$ in ν parts such as

$$\xi = \xi_0 \leqq \xi_1 \leqq \cdots\cdots \leqq \xi_\nu = \xi'.$$

Take a point Q_k in G on the hyperplane H_{ξ_k} and through it draw the straight line, having the angular coefficients given by the values of f_i at the point Q_k. This line cuts the hyperplane $H_{\xi_{k+1}}$ at a point, say P_{k+1} $(k=0, 1, \ldots, \nu-1)$. Put

$$\Delta = \overline{S_\xi Q_0} + \overline{P_1 Q_1} + \cdots\cdots + \overline{P_\nu S_{\xi'}}.$$

For two given sub-spaces S_ξ and $S_{\xi'}$, consider all the possible values of $\varDelta$. Tending all the differences $\xi_k-\xi_{k-1}$ $(k=1, 2,\ldots\ldots, \nu)$ to zero, take the least of the limits of $\varDelta$. We designate it by $D(S_\xi, S_{\xi'})$. Then a very broad extension of Okamura's function is made. When S_ξ and $S_{\xi'}$ signify points P and Q respectively, it becomes Okamura's function $D(P, Q)$.

$D(S_\xi, S_{\xi'})$ has the same properties as $D(P, Q)$, which run as follows;

a) For that a solution of (1) shall exist so as to pass through S_ξ and $S_{\xi'}$ $(0\leqq\xi\leqq\xi'\leqq a)$, it is necessary and sufficient that

$$(2) \qquad D(S_\xi,\ S_{\xi'})=0.$$

The condition is necessary, for if a solution say C of (1) passes through S_ξ and $S_{\xi'}$, then $\varDelta$, formed by points Q_k on C such as $x=\xi+\dfrac{k}{\nu}(\xi'-\xi)$ $(k=0, 1,\ldots, \nu-1)$, tends to zero with $\dfrac{1}{\nu}$ by the continuity of f_i; hence $D(S_\xi, S_{\xi'})=0$. Conversely, let $D(S_\xi, S_{\xi'})=0$, then there is a sequence of values of $\varDelta$,

$$\varDelta^{(\mu)}=\overline{S_\xi Q_0^{(\mu)}}+\overline{P_1^{(\mu)}Q_1^{(\mu)}}+\ldots\ldots+\overline{P_{\nu_\mu}^{(\mu)}S_{\xi'}} \qquad (\mu=1, 2,\ldots\ldots),$$

tending to zero with $\dfrac{1}{\mu}$. The x-coordinates of the points $P_k^{(\mu)}$ and $Q_k^{(\mu)}$ are $\xi_k^{(\mu)}$ $(\xi=\xi_0^{(\mu)}\leqq\xi_1^{(\mu)}\leqq\ldots\ldots\leqq\xi_{\nu_\mu}^{(\mu)}=\xi')$, while $P_0^{(\mu)}$ (or $Q_{\nu_\mu}^{(\mu)}$) is the point of S_ξ (or $S_{\xi'}$) which gives the distance $\overline{S_\xi Q_0^{(\mu)}}$ (or $\overline{P_{\nu_\mu}^{(\mu)}S_{\xi'}}$). Let $y_i=Y_i^{(\mu)}(x)$ $(i=1,2,\ldots\ldots,n)$ represent the segment $\overline{Q_k^{(\mu)}P_{k+1}^{(\mu)}}$ for $\xi_k^{(\mu)}\leqq x<\xi_{k+1}^{(\mu)}$ $(k=0, 1,\ldots\ldots, \nu_\mu-1)$ and the points $P_0^{(\mu)}$ and $Q_{\nu_\mu}^{(\mu)}$ for $x=\xi$ and ξ' respectively. These functions are discontinuous at most at $x=\xi_k^{(\mu)}$ and we represent by $\sigma_i^{(\mu)}(x)$ the sum of discontinuities of $Y_i^{(\mu)}(x)$ for $[\xi, x]$. Then the differences $Y_i^{(\mu)}(x)-\sigma_i^{(\mu)}(x)$ are continuous in $\xi\leqq x\leqq\xi'$. Evidently we have $|\sigma_i^{(\mu)}(x)|\leqq\varDelta^{(\mu)}$.

Therefore we have, for $\xi\leqq x\leqq\xi'$,

$$Y_i^{(\mu)}(x)-\sigma_i^{(\mu)}(x)=Y_i^{(\mu)}(\xi)-\sigma_i^{(\mu)}(\xi)+\int_\xi^x f_i^*(s)\,ds,$$

where $\qquad f_i^*(s)=f_i[\xi_k, Y_1^{(\mu)}(\xi_k),\ldots\ldots,Y_n^{(\mu)}(\xi_k)]$ for $\xi_k\leqq s<\xi_{k+1}$.

Consequently the sequence of the functions $Y_i^{(\mu)}(x)-\sigma_i^{(\mu)}(x)$ is equally continuous. Hence we can select a uniformly convergent sequence, and we have in the limit

$$Y_i(x)=Y_i(\xi)+\int_{\xi}^{x} f_i[t,\,Y_1(t),\ldots\ldots,Y_n(t)]\,dt,$$

for $\sigma_i^{(\mu)}(x)$ tend to zero uniformly. Therefore we have obtained a solution $y_i=Y_i(x)$ of (1), passing through S_ξ and $S_{\xi'}$.

b) Consider a point P in G such as $x=\xi'$. For $\xi\leqq\xi'\leqq\xi''$, we have

$$(3)\quad D(S_\xi, S_{\xi''})\leqq D(S_\xi, P)+D(P, S_{\xi''}).$$

For three points $P(\xi, \eta_1, \eta_2,\ldots\ldots, \eta_n)$, $Q(\xi', \eta_1', \eta_2',\ldots\ldots, \eta_n')$ and $R(\xi'', \eta_1'', \eta_2'',\ldots\ldots\eta_n'')$, where $\xi\leqq\xi'\leqq\xi''$, we have

$$(4)\quad \begin{cases} D(S_\xi, Q)\leqq D(S_\xi, R)+M(\xi''-\xi')+\sqrt{(\eta_1''-\eta_1')^2+\ldots+(\eta_n''-\eta_n')^2}, \\ D(Q, S_{\xi''})\leqq D(P, S_{\xi''})+M(\xi'-\xi)+\sqrt{(\eta_1'-\eta_1)^2+\ldots+(\eta_n'-\eta_n)^2}, \end{cases}$$

M being the upper bound of $\sqrt{f_1^2+f_2^2+\ldots\ldots+f_n^2}$ in G. The proof may be done easily from the definition of $D(S_\xi, S_{\xi'})$.

c) $D(S_\xi, P)$ is a continuous function of P, and satisfies the Lipschitz condition with regard to the $(y_1, y_2,\ldots\ldots, y_n)$-coordinates of P. This is evident by b).

2. Uniqueness theorems. Consider a system of differential equations

$$(5)\quad \frac{dy_i}{dx}=f_i(x, y_1, y_2,\ldots\ldots, y_n) \quad (i=1, 2,\ldots\ldots, n),$$

where f_i are continuous in a domain

$$G:\ 0\leqq x\leqq a,\ b_i\leqq y_i\leqq b_i',\ (i=1, 2,\ldots\ldots, n,\ b_i\leqq 0,\ b_i'\geqq 0),$$

and $f_i(x, 0,\ldots\ldots, 0)=0$ for $0\leqq x\leqq a$, $(i=1, 2,\ldots\ldots, n)$,

which means that x-axis is at least a solution.

We denote the sub-space S_0, which contains the point $O(0, 0,\ldots, 0)$, by S_O; also S_a containing the point $A(a, 0,\ldots\ldots,0)$ by S_A.

Theorem 1. In order that the solution of (5), starting from a point in S_O and arriving at a point in S_A, should be unique, it is necessary and sufficient that there exist two functions $\varphi(x, y_1, y_2,\ldots\ldots, y_n)$ and $\psi(x, y_1, y_2,\ldots\ldots, y_n)$ continuous in G and

$$\varphi(x, y_1, y_2,\ldots\ldots, y_n)\geqq 0,\quad \psi(x, y_1, y_2,\ldots\ldots, y_n)\geqq 0,$$

and zero for all points of S_O respectively S_A, i. e.,

$$\varphi(S_O)=0,\qquad \psi(S_A)=0,$$

and for $0\leqq x\leqq a$,

$$\varphi(x, 0,\ldots\ldots, 0) + \psi(x, 0,\ldots\ldots, 0) = 0,$$

$$\varphi(x, y_1, y_2,\ldots\ldots, y_n) + \psi(x, y_1, y_2,\ldots\ldots, y_n) > 0$$

provided $|y_1| + |y_2| + \ldots\ldots + |y_n| \neq 0$,

and moreover both functions satisfy in G the Lipschitz condition with regard to $(y_1, y_2,\ldots\ldots, y_n)$, and for all points $(x, y_1, y_2,\ldots\ldots, y_n)$ in G, we have

$$\overline{\lim_{t\to 0}} \frac{1}{t}\{\varphi(x+t, y_1+tf_1,\ldots\ldots, y_n+tf_n) - \varphi(x, y_1,\ldots\ldots, y_n)\} \leqq 0,$$

$$\underline{\lim_{t\to 0}} \frac{1}{t}\{\psi(x+t, y_1+tf_1,\ldots\ldots, y+tf_n) - \psi(x, y_1,\ldots\ldots, y_n)\} \geqq 0.^{(3)}$$

Proof. If the solution is unique, put

$$\varphi(x, y_1, y_2,\ldots\ldots, y_n) = D(S_O, P),$$

$$\psi(x, y_1, y_2,\ldots\ldots, y_n) = D(P, S_A),$$

where P is the point $(x, y_1, y_2,\ldots\ldots, y_n)$. Then φ and ψ satisfy these conditions; e.g., $D(S_O, P)$ is a continuous function of P, non negative, and zero only when P is on a solution passing through S_O. For two points P and Q on a same hyperplane H_x, we have

$$|D(S_O, P) - D(S_O, Q)| \leqq D(P, Q) = \overline{PQ},$$

and the function $D(S_O, P)$ does not increase with x on any solution of (5). If P_1 and P_2 are two points on one solution, P_2 on the right of P_1, then

$$D(S_O, P_2) \leqq D(S_O, P_1) + D(P_1, P_2),$$

where $D(P_1, P_2) = 0$. Finally $D(S_O, P) + D(P, S_A)$ is zero when and only when P is on one solution passing through S_O and S_A, i. e. on x-axis.

Conversely, if there exist such two functions, φ and ψ, it is easy to prove that $\varphi+\psi$ is zero on a solution passing through S_O and S_A. In fact, let the solution intersect S_O at P_0 and S_A at P_1. Then $D(S_O, P_0) = 0$, since P_0 belongs to S_O. $D(S_O, P)$ being non increasing with x on the solution, $D(S_O, P)$ must always be zero on it. Similarly $D(P, S_A)$ must always be zero on the same solution. Hence the solution must be x-axis itself. Q. E. D.

Choosing S_O and S_A in Theorem 1 conveniently, we shall have the necessary and sufficient uniqueness conditions in Fukuhara's problem[4], boundary problems of the differential equation of the second order and others.

Now consider

$$(6)\quad \frac{d^2y}{dx^2}=f\left(x, y, \frac{dy}{dx}\right),$$

where $f(x, y, y')$ is continuous in a domain $[0\leqq x\leqq a,\ b\leqq y\leqq b',$ $(b\leqq 0,\ b'\geqq 0),\ |y'|<\infty]$, and $f(x, 0, 0)=0$ for $0\leqq x\leqq a$. Consider the solution of (6) which vanishes at $x=0$ and $x=a$. In the case of $|y'|\leqq C<\infty$, our problem becomes to search the conditions for that the solution of the system

$$\begin{cases} y'=z, \\ z'=f(x, y, z), \\ G: 0\leqq x\leqq a,\ b\leqq y\leqq b',\ |z|\leqq C, \end{cases}$$

passing through a point of the segment $(x=0,\ y=0,\ |z|\leqq C)$ and a point on the segment $(x=a,\ y=0,\ |z|\leqq C)$, shall be unique. As a special case of the Theorem 1, we have

Theorem 2. If we restrict to the solutions such as $|y'|\leqq C$, in order that the solution of (6), for which $y=0$ at $x=0$ and $x=a$, should be unique, it is necessary and sufficient that there should exist two continuous functions in G, $\varphi(x, y, z)$ and $\psi(x, y, z)$, such that

$$\begin{aligned} &\varphi(x, y, z)\geqq 0, && \psi(x, y, z)\geqq 0, \\ &\varphi(0, 0, z)=0, && \psi(a, 0, z)=0 \text{ for } |z|\leqq C, \end{aligned}$$

and for $0\leqq x\leqq a$

$$\begin{aligned} &\varphi(x, 0, 0)+\psi(x, 0, 0)=0, \\ &\varphi(x, y, z)+\psi(x, y, z)>0 \text{ provided } |y|+|z|\neq 0, \end{aligned}$$

and both functions verify in G the Lipschitz condition with regard to (y, z), and, for all points (x, y, z) in G, we have

$$\overline{\lim_{t\to 0}}\frac{1}{t}\{\varphi(x+t, y+tz, z+tf)-\varphi(x, y, z)\}\leqq 0,$$

$$\underline{\lim_{t\to 0}}\frac{1}{t}\{\psi(x+t, y+tz, z+tf)-\psi(x, y, z)\}\geqq 0.\text{[3]}$$

Theorem 3. If the restriction $|y'|\leqq C$ in Theorem 2 is taken away, we may enunciate as follows: Take an arbitrary positive number $L(0<L<\infty)$ (i. e.; C in the above) and let φ and ψ, in Theorem 2, be denoted by φ_L and ψ_L. Then for the uniqueness it is necessary and sufficient that there should exist φ_L and ψ_L, stated above, however great L may be.

Theorem 4. When $|z|\leqq C$ is replaced by $|z|<\infty$, if the functions φ and ψ with the properties stated in Theorem 2 exist, then the solution of (6) passing through $(x=0, y=0)$ and $(x=a, y=0)$ is unique (only sufficient conditions).

Analogous theorems for theorem 2, 3 and 4 may easily be concluded also, when $|y|<\infty$:

3. Further extention. In this case, to form Δ, let us divide the segment $[\alpha, \gamma]$ $(0\leqq\alpha\leqq\beta\leqq\gamma\leqq a)$ in ν parts as follow:

$$\xi_0\leqq\xi_1\leqq\xi_2\leqq\cdots\cdots\leqq\xi_\mu\leqq\xi_{\mu+1}\leqq\cdots\cdots\leqq\xi_\nu,$$

where $\xi_0=\alpha$, $\xi_\mu=\beta$ and $\xi_\nu=\gamma$ are taken into the dividing points. Now put

$$\Delta=\overline{S_\alpha Q_0}+\overline{P_1 Q_1}+\cdots\cdots+\overline{P_\mu Q_\mu}+\cdots\cdots+\overline{P_\nu S_\gamma}+\overline{P_\mu S_\beta}+\overline{S_\beta Q_\mu}.$$

So we obtain a function $D(S_\alpha, S_\beta, S_\gamma)$ extending $D(P, Q)$. Further than that, we may for

$$0\leqq\alpha_1\leqq\alpha_2\leqq\cdots\cdots\leqq\alpha_\nu\leqq a$$

also define $D(S_{\alpha_1}, S_{\alpha_2},\cdots\cdots, S_{\alpha_\nu})$. These generalized functions have the same properties as $D(P, Q)$; namely

a) In order that there should exist a solution of (1), passing through all the given sub-spaces $S_{\alpha_1}, S_{\alpha_2}\cdots\cdots, S_{\alpha_\nu}$, $(\alpha_1\leqq\alpha_2\leqq\cdots\cdots\leqq\alpha_\nu)$, it is necessary and sufficient that we shall have

$$D(S_{\alpha_1}, S_{\alpha_2},\cdots\cdots, S_{\alpha_\nu})=0.$$

b) Consider a point P in G such as $x=\xi$. If $\alpha_{\mu-1}\leqq\xi\leqq\alpha_\mu$ $(2\leqq\mu\leqq\nu)$, then

$$D(S_{\alpha_1}, S_{\alpha_2},\cdots\cdots, S_{\alpha_\nu})\leqq D(S_{\alpha_1}, S_{\alpha_2},\cdots\cdots, S_{\alpha_{\mu-1}}, P)$$
$$+D(P, S_{\alpha_\mu},\cdots\cdots, S_{\alpha_\nu}).$$

For two points $P(\xi, \eta_1,\cdots\cdots, \eta_n)$ and $Q(\xi', \eta_1',\cdots\cdots, \eta_n')$, if $a_\nu\leqq\xi\leqq\xi'\leqq a$, then

$$D(S_{\alpha_1}, S_{\alpha_2},\cdots\cdots, S_{\alpha_\nu}, P)\leqq D(S_{\alpha_1}, S_{\alpha_2},\cdots\cdots, S_{\alpha_\nu}, Q)+M(\xi'-\xi)$$
$$+\sqrt{(\eta_1'-\eta_1)^2+\cdots\cdots+(\eta_n'-\eta_n)^2},$$

and also, if $0 \leqq \xi \leqq \xi' \leqq \alpha_1$, then

$$D(Q, S_{\alpha_1}, \ldots\ldots, S_{\alpha_\nu}) \leqq D(P, S_{\alpha_1}, \ldots\ldots, S_{\alpha_\nu}) + M(\xi' - \xi) + \sqrt{(\eta_1' - \eta_1)^2 + \ldots\ldots + (\eta_n' - \eta_n)^2},$$

M being the upper bound of $\sqrt{f_1^2 + f_1^2 + \ldots\ldots + f_n^2}$ in G.

c) $D(S_{\alpha_1} \ldots\ldots, S_{\alpha_\nu}, P)$, $(p \epsilon H_\xi, \alpha_\nu \leqq \xi \leqq \alpha)$ is continuous with respect to P in $[\alpha_\nu \leqq x \leqq a,\ b_i \leqq y_i \leqq b_i'\ \ (i=1, 2, \ldots\ldots, n)]$ and it satisfies the Lipschitz condition with regard to the $(y_1, y_2, \ldots\ldots, y_n)$-coordinates of P.

Remark 1. $D(S_{\alpha_1}, \ldots\ldots, S_{\alpha_{i-1}}, H_i, S_{\alpha_{i+1}}, \ldots\ldots, S_{\alpha_\nu})$

$$= D(S_{\alpha_1}, \ldots\ldots, S_{\alpha_{i-1}}, S_{\alpha_{i+1}}, \ldots\ldots, S_{\alpha_\nu}), \quad (2 \leqq i \leqq \nu - 1).$$

4. Uniqueness theorems. Theorem 5. Consider a system of differential equations (5). For the points $O(0, 0, \ldots\ldots, 0)$, $A(a, 0, \ldots\ldots, 0)$ and $A_j(a_j, 0, \ldots\ldots, 0)$ $[(j=1, 2, \ldots\ldots, \nu-1),\ 0=a_0 < a_1 < a_2 < \ldots\ldots < a_\nu = a]$ let certain sub-spaces S_0, S_{a_j} and S_a, containing the points O, A_j and A respectively, be denoted by S_O, S_{A_j} $(j=1,2,\ldots\ldots, \nu-1)$ and S_A respectively. When S_O, S_{A_j} and S_A are given, in order that the solution of (5), passing through all the sub-spaces S_O, $S_{A_j}(j=1, 2, \ldots\ldots, \nu-1)$ and S_A, should be unique, it is necessary and sufficient that there shall exist 2ν functions, $\varphi_j(x, y_1, \ldots\ldots, y_n)$ and $\psi_j(x, y_1, \ldots\ldots, y_n)$ $(j=1, 2, \ldots\ldots, \nu)$, as follows: At first let

$$G_j : a_{j-1} \leqq x \leqq a_j,\ b_i \leqq y_i \leqq b_i'\ \ (i=1, 2, \ldots\ldots, n),$$
$$[j=1,\ 2, \cdots\cdots,\ \nu].$$

$$L : x\text{-axis},$$

and $$P : (x, y_1, y_2, \ldots\ldots, y_n).$$

Then $\varphi_j(P)$ and $\psi_j(P)$ are continuous functions defined in G_j and always

$$\varphi_j(P) \geqq 0, \quad \psi_j(P) \geqq 0 \quad (j=1, 2, \ldots\ldots, \nu),$$
$$\varphi_1(P) = 0 \quad \text{for} \quad P \epsilon L + S_O,$$
$$\varphi_j(P) = 0 \quad \text{for} \quad P \epsilon L \quad (j=2, 3, \ldots\ldots, \nu)$$

and, for the point P such as $\varphi_j(P)=0$ $(P \epsilon S_{A_j})$, $\varphi_{j+1}(P) = 0 (j=1, 2, \ldots\ldots, \nu-1)$ and also

$$\psi_\nu(P) = 0. \quad \text{for} \quad P \epsilon L + S_A,$$
$$\psi_j(P) = 0 \quad \text{for} \quad P \epsilon L \quad (j=1, 2, \ldots\ldots, \nu-1)$$

and, for the point P such as $\psi_j(P)=0$ $(P\epsilon S_{A_{j-1}})$, $\psi_{j-1}(P)=0$ $(j=2,3,\ldots\ldots,\nu)$ and, for $P\epsilon G_j$,

$$\varphi_j(P)+\psi_j(P)=0 \qquad (P\epsilon L),$$

$$\varphi_j(P)+\psi_j(P)>0 \qquad (P\bar{\epsilon} L),$$

both functions φ_j and ψ_j satisfying in G_j the Lipschitz condition with regard to $(y_1, y_2,\ldots\ldots, y_n)$, and, for all points $(x, y_1, y_2\ldots\ldots, y_n)$ in G_j, we have

$$\overline{\lim_{t\to 0}}\frac{1}{t}\{\varphi_j(x+t, y_1+tf_1,\ldots\ldots, y_n+tf_n)-\varphi_j(x, y_1,\ldots\ldots, y_n)\}\leqq 0$$

$$\underline{\lim_{t\to 0}}\frac{1}{t}\{\psi_j(x+t, y_1+tf_1,\ldots\ldots, y_n+tf_n)-\psi_j(x, y_1,\ldots\ldots, y_n)\}\geqq 0^{(3)}$$

$$(j=1, 2,\ldots\ldots, \nu)$$

The proof is omitted.

According to the suitable choice of S_o, $S_{A_j}(j=1, 2,\ldots\ldots, \nu-1)$ and S_A in Theorem 5, we may obtain the necessary and sufficient conditions for the uniqueness in Fukuhara's problem (loc. cit.), boundary problems of a differential equation of n-th order, generalized Fukuhara's problem and various other kinds of problems.

Let us e.g., consider the differential equation

$$(7)\qquad \frac{d^3y}{dx^3}=f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right),$$

where $f(x, y, y', y'')$ is continuous in the domain $[0\leqq x\leqq b,\ |y|\leqq C,\ |y'|<\infty,\ |y''|<\infty]$ and $f(x, 0, 0, 0)=0$ for $0\leqq x\leqq b$. Consider the solution of (7) for which $y=0$ at $x=0$, $x=a$ and $x=b$, where $0<a<b$. Now we consider only the solution such as $|y'|, |y''|\leqq d<\infty$. Then our problem is reduced to search the uniqueness condition for the solution of the system

$$\begin{cases} y'=u, \\ u'=v, \\ v'=f(x, y, u, v) \\ G:\ 0\leqq x\leqq b,\ |y|\leqq C,\ |u|\leqq d,\ |v|\leqq d, \end{cases}$$

which passes through a point of $S_O(x=0,\ y=0,\ |u|\leqq d,\ |v|\leqq d)$, a point of $S_A(x=a,\ y=0,\ |u|\leqq d,\ |v|\leqq d)$ and a point of $S_B(x=b,\ y=0,\ |u|\leqq d,\ |v|\leqq d)$. By the above theorem we have

Theorem 6. If we restrict solutions such as $|y'|\leqq d$ and

$|y''| \leqq d$, in order that the solution of (7), for which $y=0$ at $x=0$, $x=a$ and $x=b$, shall be unique, it is necessary and sufficient that there exist four functions, $\varphi_1(x, y, u, v)$, $\varphi_2(x, y, u, v)$, $\psi_1(x, y, u, v)$ and $\psi_2(x, y, u, v)$ such as follows: At first let

$$G_1 : 0 \leqq x \leqq a, \ |y| \leqq c, \ |u| \leqq d, \ |v| \leqq d,$$

$$G_2 : a \leqq x \leqq b, \ |y| \leqq c, \ |u| \leqq d, \ |v| \leqq d.$$

Then φ_j and ψ_j are continuous functions defined in $G_j (j=1, 2)$ and always

$$\varphi_1 \geqq 0, \quad \psi_1 \geqq 0, \quad \varphi_2 \geqq 0, \quad \psi_2 \geqq 0,$$

and for $|u| \leqq d$ and $|v| \leqq d$

$$\varphi_1(0, 0, u, v) = 0$$

$$\psi_2(b, 0, u, v) = 0,$$

and, $\varphi_2(a, 0, u, v) = 0$ for u and v such as $\varphi_1(a, 0, u, v) = 0$, and, $\psi_1(a, 0, u, v) = 0$ for u and v such as $\psi_2(a, 0, u, v) = 0$, and for $0 \leqq x \leqq a$

$$\varphi_1(x, 0, 0, 0) + \psi_1(x, 0, 0, 0) = 0,$$

$$\varphi_1(x, y, u, v) + \psi_1(x, y, u, v) > 0 \quad \text{provided } |y| + |u| + |v| \neq 0,$$

for $a \leqq x \leqq b$

$$\varphi_2(x, 0, 0, 0) + \psi_2(x, 0, 0, 0) = 0,$$

$$\varphi_2(x, y, u, v) + \psi_2(x, y, u, v) > 0 \quad \text{provided } |y| + |u| + |v| \neq 0.$$

$\varphi_1, \varphi_2, \psi_1$ and ψ_2 satisfy the Lipschitz condition with regard to (y, u, v), and for all points (x, y, u, v) in G_j, we have

$$\overline{\lim_{t \to 0}} \frac{1}{t} \{ \varphi_j(x+t, y+tu, u+tv, v+tf) - \varphi_j(x, y, u, v) \} \leqq 0,$$

$$\underline{\lim_{t \to 0}} \frac{1}{t} \{ \psi_j(x+t, y+tu, u+tv, v+tf) - \psi_j(x, y, u, v) \} \geqq 0, \text{(3)}$$

$$(j=1, 2)$$

Theorems analogous to the theorems 3 and 4 may easily be concluded. For Fukuhara's problem such as $y_1(0) = y_2(a) = y_3(b) = 0$ $(0 < a < b)$ of the system

$$\begin{cases} \dfrac{dy_1}{dx}=f_1(x, y_1, y_2, y_3), \\ \dfrac{dy_2}{dx}=f_2(x, y_1, y_2, y_3), \\ \dfrac{dy_3}{dx}=f_3(x, y_1, y_2, y_3), \\ G: 0 \leqq x \leqq b,\ b_i \leqq y_i \leqq b_i' \ (i=1,2,3) \ (b_i \leqq 0,\ b_i' \geqq 0), \end{cases}$$

where $f_i(x, 0, 0, 0)=0$ $(i=1, 2, 3,)$, S_O, S_A and S_B are $(x=0,\ y_1=0,\ b_2 \leqq y_2 \leqq b_2',\ b_3 \leqq y_3 \leqq b_3')$, $(x=a,\ b_1 \leqq y_1 \leqq b_1',\ y_2=0,\ b_3 \leqq y_3 \leqq b_3')$ and $(x=b,\ b_1 \leqq y_1 \leqq b_1',\ b_2 \leqq y_2 \leqq b_2',\ y_3=0)$ respectively.

Remark 2. The Okamura's work (1942)[5] can be generalized.

Remark 3. φ and ψ etc. may be modified as follows:
They become continuous in the domain (closed) and their partial derivatives continuous in the domain (open) and the remaining properties are the same as the original ones. For instance, the condition

$$(8) \quad \overline{\lim_{t\to 0}} \frac{1}{t}\{\varphi(x+t, y_1+tf_1, \ldots\ldots, y_n+tf_n)-\varphi(x, y_1, \ldots\ldots, y_n)\} \leqq 0$$

will be replaced by the condition

$$(9) \quad \frac{\partial\varphi}{\partial x}+\sum_{i=1}^{n}\frac{\partial\varphi}{\partial y_i}f_i \leqq 0.$$

In (8) put

$$\varphi(x, y_1, y_2, \ldots\ldots, y_n)e^{-x}=\bar{\varphi}(x, y_1, y_2, \ldots\ldots, y_n),$$

then

$$\overline{\lim_{t\to 0}} \frac{1}{t}\{\bar{\varphi}(x+t, y_1+tf_1 \ldots, y_n+tf_n)-\bar{\varphi}(x, y_1, \ldots, y_n)\} \leqq -\bar{\varphi}(x, y_1, \ldots, y_n)$$

where $\bar{\varphi}(x, y_1, y_2, \ldots\ldots, y_n)$ is positive or zero with $\varphi(x, y_1, \ldots\ldots, y_n)$. Therefore we have to prove the following theorem in which we represent for convenience $(y_1, y_2, \ldots\ldots, y_n)$ etc. by vector-symbol $\mathfrak{y}$ etc..

Theorem 7. i) Let $\varphi(x, \mathfrak{y})$ be continuous and positive or zero in the closed domain $G\,[0 \leqq x \leqq a,\ b_i \leqq y_i \leqq b_i'\ (i=1, 2, \ldots\ldots, n)]$ and it satisfies Lipschitz condition, i.e., there exists a positive constant K such as in G

$$|\varphi(x', \mathfrak{y}')-\varphi(x, \mathfrak{y})| \leqq K(|x'-x|+|\mathfrak{y}'-\mathfrak{y}|)$$

ii) At each point $(x, \mathfrak{y})$ in G, let $\varphi(x, \mathfrak{y})$ satisfy the inequality

$$\overline{\lim_{t\to 0}}\frac{1}{t}\{\varphi(x+t, y+ft)-\varphi(x, y)\} \leqq -\varphi(x, y) \quad (\leqq 0),$$

where $f_i(x, y)$ $(i=1, 2, \ldots\ldots, n)$ are continuous in G.

Then there exists a continuous function $\tilde{\varphi}(x, y)$ in G such as follows: According as $\varphi(x, y)$ is positive or zero, $\tilde{\varphi}(x, y)$ is positive or zero. It has the bounded continuous partial derivatives $\frac{\partial\tilde{\varphi}}{\partial x}, \frac{\partial\tilde{\varphi}}{\partial y_1}, \ldots\ldots, \frac{\partial\tilde{\varphi}}{\partial y_n}$ in the inside of G which satisfy

$$\frac{\partial\tilde{\varphi}(x,y)}{\partial x}+\sum_{i=1}^{n}\frac{\partial\tilde{\varphi}(x,y)}{\partial y_i}f_i(x,y) \leqq 0.$$

This remark we owe to Prof. Nagumo. The proof is omitted. And also it is the same with ψ.

Thus we have succeeded in developing the profound idea of the late Prof. Hiroshi Okamura much regretted by his early death. At the end we express heartily thanks to Prof. Toshizô Matsumoto, to whom we owe a great debt for his guidance in our researches.

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References

1) The contents have been already published in "Sugaku" (The Mathematics) Vol. 2, No. 1. (1949).

2) Okamura, Mem. Coll. Sci. Kyoto Univ. A. 23 (1941), pp. 225—231.

3) By the remark of Prof. M. Nagumo φ and ψ can be replaced with the continuous functions whose patial derivatives $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y_1}, \ldots\ldots, \frac{\partial\varphi}{\partial y_n}, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y_1}, \ldots\ldots, \frac{\partial\psi}{\partial y_n}$ are continuous in the inside of G (except the boundary). Cf. Remark. 3.

4) Nagumo, Proc. of Phys-Math. Soc. of Japan. Series 3, vol. 25, p. 221.

5) Okamura, Mem. Coll. Sci. Kyoto Univ. A. 24 (1942), p. 22.