

Note on the Intersection of an Algebraic Variety with the Generic Hyperplane.

By

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Let V ($r > 1$) be an algebraic variety immersed in the projective space L^N , k a field of definition for V , and H^{N-1} a generic hyperplane with respect to k in L . Then it is well known that the intersection $H \cdot V$ is defined and is an irreducible variety¹⁾ W^{r-1} . In this case the question arises that the variety may get a new singular point other than that of V . When the characteristic p of the universal domain is 0, we know that such a case cannot take place by the second theorem of Bertini. In this note we shall show that it is also true even when the case $p \neq 0$.

To discuss this problem it is sufficient to consider the case when V is lying in the affine space S^N .

Let

$$F(X) = \sum_{i=1}^N u_i X_i - v = 0.$$

be the defining equation for H , where (u_i, v) are $(N+1)$ -independent variables over k and X_i denote the indeterminates of S . Let $P = (x_1, \dots, x_N)$ be any point on W , which is simple on V . Then we can choose $(N-r)$ polynomials $G_\nu(X)$ ($\nu = 1, \dots, N-r$) from the defining ideal of V over k such that the rank of the matrix

$$A = \left(\frac{\partial G_\nu}{\partial x_i} \right) \begin{matrix} (\nu = 1, \dots, N-r) \\ (i = 1, \dots, N) \end{matrix}$$

is equal to $N-r$. Then the simplicity of P on W can be concluded if we can show that the rank of the matrix

$$B = \begin{pmatrix} \frac{\partial G_\nu}{\partial x_i} \\ u_i \end{pmatrix} \quad \left(\begin{array}{l} \nu=1, \dots, N-r \\ i=1, \dots, N \end{array} \right)$$

is equal to $N-r+1$.

Let $\dim_k(x) = s$, $\dim_{k(u, v)}(x) = t$, then $\dim_k(u, v, x) = \dim_k(u, x) = N+1+t$, hence $\dim_{k(r)}(u) = N+1+t-s$. Since the point $P = (x)$ lies on V , we have $s \leq r$, and $N+1+t-s \geq N+1+t-r \geq N-r+1$, *i. e.* there are at least $(N-r+1)$ independent variables over $k(x)$ among $u_i (i=1, \dots, N)$. Since the rank of the matrix A is $N-r$, without loss of generality we can assume that

$$\left| \frac{\partial G_\nu}{\partial x_i} \right| \neq 0 \quad \left(\begin{array}{l} \nu=1, \dots, N-r \\ i=1, \dots, N-r \end{array} \right).$$

Suppose that the rank of the matrix B is at most equal to $N-r$, then we have the relations²⁾

$$\begin{vmatrix} \frac{\partial G_1}{\partial x_1} & \dots & \frac{\partial G_1}{\partial x_{N-r}} & \frac{\partial G_1}{\partial x_k} \\ \dots & \dots & \dots & \dots \\ \frac{\partial G_{N-r}}{\partial x_1} & \dots & \frac{\partial G_{N-r}}{\partial x_{N-r}} & \frac{\partial G_{N-r}}{\partial x_k} \\ u_1 & \dots & u_{N-r} & u_k \end{vmatrix} = 0 \quad (k=N-r+1, \dots, N)$$

and $u_k (k=N-r+1, \dots, N)$ can be expressed as the linear combinations of $u_i (i=1, \dots, N-r)$ with the coefficients in $k(x)$. It contradicts to the preceding fact.

Thus we get the

THEOREM. *Let P be a point on the generic hyperplane section W^{r-1} of an algebraic variety $V^r (r > 1)$. Then if P is simple on V , P is also simple on W .*

Notes

1) This can be shown easily using the Lemma, due to Mr. T. Matsusaka, which is a generalization of Zariski's Lemma. O. Zariski. "Pencils on an algebraic variety and a new proof of a theorem of Bertini." Trans. A. M. Soc. 1941, p. 68, Lemma 5. T. Matsusaka. "The theorem of Bertini on linear systems in modular fields." Mem. of the College of Science, Univ. Kyôto Vol. XXVI, Math. No. 1, 1950, p. 55. th. 2. 4.

2) For the device of the following proof I thank to my friend Mr. Hajime Nishimura.

Added in the proof. I find in his letter to Mr. J. Igusa of Nov. 3, 1950, that Prof. Weil had already obtained this result as an important lemma in the general theory of the Picard variety. (Nov. 25, 1950.)