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Note on the Surface Area and the Mapping of Bounded Variation

By

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The mapping of bounded variation has been considered by many authors since Banach has introduced it for the first time. Also in our country the late Prof. Okamura has defined the surface integral in his last paper "On the surface integral and Gauss-Green's theorem"[1] Where he introduced a concept of the mapping of bounded variation, and indicated that the surface area may be defined by it for the surface represented in the parametric form.

Recently T. Radó has developed a theory of the mapping of bounded variation from which he has defined the surface area $a(S)$ of the surface represented in the parametric form and proved that this area $a(S)$ is equal to Lebesgue's surface area $A(S)$ when $A(S)<\infty$. On the other hand, we are told that L. Cesari had obtained remarkable results[3] showing that the area $a(S)$ is equal to the area $A(S)$ even when $A(S)=\infty$, hence $a(S)$ is geometrically invariant. Yet this paper is not within our reach.

In the present paper, we intend to compare the mapping of bounded variation in Okamura's sense with that of Radó's and we shall proceed to prove that these two definitions are equivalent under certain conditions and they give the equal value of the surface areas.

I. Bounded variation in Radó's sense

After Radó's treatise "Length and area", we assume that a continuous mapping $\mathfrak{A}$ from the uv-plane (w-plane) to the xy-plane (z-plane) is given by the formulae of the form

$$\mathfrak{A}:\quad x=x(u,v),\ y=y(u,v) \qquad (u,v)\in\Delta \tag{1}$$

or briefly $\quad \mathfrak{A}:\quad z=\mathfrak{A}(w) \qquad w\in\Delta$

where $x(u,v)$, $y(u,v)$ are one valued continuous functions defined in the bounded closed region $\bar{\Delta}$ of the uv-plane.

i) The set $\varepsilon(\mathfrak{A}, \Delta)$

If E is a set on the xy-plane, $\mathfrak{A}^{-1}(E)$ denotes the set of those points $w\epsilon\Delta$ for which $\mathfrak{A}(w)\epsilon E$. If E reduces to a single point z, then $\mathfrak{A}^{-1}(z)$ is the set of those points $w\epsilon\Delta$ for which $\mathfrak{A}(w)=z$. Obviously $\mathfrak{A}(z)$ is a closed set relative to Δ. There are two sorts of components of $\mathfrak{A}(z)$, the one are continua (it may be a single point), and the other non-coutinua. The former of the components will be termed the maximal model continua for z under $\mathfrak{A}$ in Δ.

Let z be a point in the xy-plane and δ an open set in the uv-plane, we shall denote the degree of mapping of $\mathfrak{A}$ at the point z by $A_{\mathfrak{A}}(z, \delta)$ if $z\bar{\epsilon}\mathfrak{A}(\bar{\delta}-\delta)$ and put $A_{\mathfrak{A}}(z,\delta)=0$ if $z\epsilon\mathfrak{A}(\bar{\delta}-\delta)$. A region $O\epsilon\Delta$ will be called an indicator region for z when and only when $A_{\mathfrak{A}}(z, O)\fallingdotseq 0$. Then we shall say that a maximal model continum mentioned above is an essential maximal model continuum for z under $\mathfrak{A}$ in Δ if the following conditions are satisfied: $\alpha)\gamma\subset\Delta$, β) If O is any open set which contains γ, then there exists at least one indicator region for z in O. Next, for such a mapping $\mathfrak{A}$, we shall define a set $\varepsilon(\mathfrak{A}, \Delta)$ as follows:

A point $w\epsilon\Delta$ is contained in $\varepsilon(\mathfrak{A}, \Delta)$ when and only when w, taken by itself, is an essential maximal model for its image $z=\mathfrak{A}(w)$ under $\mathfrak{A}$ in Δ.

It is shown in Radó's treatise that $\varepsilon(\mathfrak{A}, \Delta)$ is a Borel set. (see Radó loc. cit. p. 295, IV. 1. 58)

ii) Mapping of bounded variation in Rado's sense

Consider a set B in the uv-plane for a continuous mapping $\mathfrak{A}$ defined above and we shall say that this set is a base set if B satisfies the following conditions:

α) B is a measurable set, β) For any oriented rectangle R ($R^{\circ}\epsilon\Delta$: R° is a set of all interior points of R), $\mathfrak{A}(R^{\circ}\cdot B)$ being the image of $R^{\circ}\cdot B$ under the mapping $\mathfrak{A}$ is measurable in the z-plane.

We shall define a function of rectangle $G(R)$ as follows:

$G(R)=m[\mathfrak{A}(R^{\circ}\cdot B)]$ ($m[\ ]$ shows the Lebesgue measure)

where R is an oriented rectangle which satisfies $R^{\circ}\subset\Delta$. Then we shall say that the mapping $\mathfrak{A}$ is a mapping of bounded variation with respect to B if for any sequence of closed oriented non-overlapping squares $s_i(s_i^{\circ}\subset\Delta;\ i=1, 2,\ldots\ldots, n)$, there is a constant M which satisfies the following inequality

$$\sum_{i=1}^{n} G(s_i) < M$$

We take the set $\varepsilon(\mathfrak{A}, \varDelta)=\varepsilon$ as a base set. ε satisfies the base set conditions since ε is a Borel set and $\mathfrak{A}(\varepsilon \cdot R)$ is also a Borel set. (see Kuratowski's Topologie I, Warsaw, 1933) we shall call the mapping $\mathfrak{A}$ "a mapping of bounded variation of Radó's sense" and we shall say briefly that $\mathfrak{A}$ is $BV(R)$.

iii) $N_{\mathfrak{A}}(z, \varDelta \cdot B)$

Let z be a point on the z-plane and $N_{\mathfrak{A}}(z, \varDelta \cdot B)$ denotes the number of points of $\mathfrak{A}^{-1}(z)$ contained in $\varDelta \cdot B$. ($N_{\mathfrak{A}}(z, \varDelta \cdot B)$ may be infinite)

THEOREM I. In order that the mapping $\mathfrak{A}$ is a mapping of bounded variation with respect to B, it is necessary and sufficient that $N_{\mathfrak{A}}(z, \varDelta \cdot B)$ is summable on the z-plane.

(The proof of this theorem is given by Radó loc. cit. p. 311 IV, 2. 13)

II. Bounded variation in Okamura's sense

Now we assume that a continuous mapping $\mathfrak{A}$ is given by the same fomulae as in the preceding section, then we shall define the variation $V_{\mathfrak{A}}(\delta)$ of the mapping $\mathfrak{A}$ on the open set δ in the w-plane, as follows:

$$V_{\mathfrak{A}}(\delta) = \int A_{\mathfrak{A}}(z, \delta)\, \mathrm{dm}(z),$$

where the integral means the Lebesgue integral taken over the whole z-plane and m signifies the Lebesgu measure. Using this notation, we shall say that the mapping $\mathfrak{A}$ is a mapping of bounded variation in Okamura's sense when and only when $\mathfrak{A}$ satisfies the following two conditions A) and B). We shall say briefly that this mapping $\mathfrak{A}$ is $BV(\mathrm{O})$.

A) We may divide $\bar{\varDelta}$ into the sum of a finite number of closed regions $\bar{\delta}_i (i=1, 2, \ldots\ldots, n)$ (*i.e.*, $\bar{\varDelta} = \bigvee_{i=1}^{n} \bar{\delta}_i$, if $i \neq j$ $\delta_i \delta_j = 0$), which may be however small, such that $m[\bigvee_{i=1}^{n} \mathfrak{A}(\bar{\delta}_i - \delta_i)] = 0$ is satisfied. denote this subdivision by (σ).

B) For every such subdivision (σ), $V_{\mathfrak{A}}(\delta_i)$ is determinate, and $\sum_{i=1}^{n} |V_{\mathfrak{A}}(\delta_i)|$ is bounded. We call the upper bound $\sup_{(\sigma)} \sum_{i=1}^{n} | V_{\mathfrak{A}}$

$(\delta_i)|$, the total variation on Δ of the mapping $\mathfrak{A}$.

THEOREM II. For the mapping $\mathfrak{A}$, let (σ) be a subdivision satisfying the condition A) and let a point z be given on the z-plane. If $z \in \sum_{i=1}^{n} \mathfrak{A}(\bar{\delta}_i - \delta_i)$, we put $u_\sigma(z) = \sum_{i=1}^{n} |A_{\mathfrak{A}}(z, \delta_i)|$. Otherwise $u_\sigma(z) = 0$. And we shall define $u(z, \mathfrak{A}, \Delta) = \sup_{(\sigma)} u_\sigma(z)$. Then the condition B) is equivalent to the fact that $u(z, \mathfrak{A}, \Delta)$ is summable on the z-plane and moreover $\int u(z, \mathfrak{A}, \Delta) dm(z) =$ total variation on Δ. (For the proof see Okamura loc. cit. p. 7)

III. The relation between $BV(R)$ and $BV(O)$

Using the terminology of the preceding sections and assumming that the mapping shall satisfy the condition A), we may conclude that $N_{\mathfrak{A}}(z, \Delta \cdot \varepsilon)$ is equal to $u(z, \mathfrak{A}, \Delta)$ almost everywhere on the z-plane.

We may suppose by the assumption A) that the set of z for which $\mathfrak{A}^{-1}(z)$ is a continuum, but not a single point, is the set of measure zero on the xy-plane. This set of measure zero will be denoted by N and we shall suppose that z does not belong to the set N.

1° From $N_{\mathfrak{A}}(z, \Delta \cdot \varepsilon) \neq \infty$, it follows $N_{\mathfrak{A}}(z, \Delta \cdot \varepsilon) = u(z, \mathfrak{A}, \Delta)$ for almost everywhere on the z-plane.

Before entering into its proof, we shall introduce an important notion. In this case the set $\mathfrak{A}^{-1}(z) \cdot \varepsilon$ consists of a finite number of single points. Hence if we take a point $w \in \mathfrak{A}^{-1}(z)$ and an arbitrary open set δ containing w, then $A_{\mathfrak{A}}(z, \delta)$ is constant for any δ having sufficiently small diameter $d(\delta)$ (by the property of the degree of mapping). We write

$$\lim_{d(\delta) \to 0} A_{\mathfrak{A}}(z, \delta) = J(w).$$

We call $J(w)$ after Radó, the essential local index of w. It is shown by Radó and Reichelderfer[1] that the set of w for which $|J(w)| \geq 2$ is countable and the set K of z's which are images of these w's under $\mathfrak{A}$ is of measure zero.

Now we shall proceed to prove the proposition 1°, assumming $z \in K + N$ $(m[K] = m[N] = 0)$. When we make the division (σ) sufficiently small, $u_\sigma(z, \Delta)$ tends to $u(z, \mathfrak{A}, \Delta)$, whereas the following inequality holds

$$(2) \qquad u_\sigma(z, \Delta) = \sum_{i=1}^{n} |A_{\mathfrak{A}}(z, \delta_i)| = \sum_{i=1}^{n'} |J(w_i)| \leq N_{\mathfrak{A}}(z, \Delta \cdot \varepsilon).$$

where $w_i (i=1, 2, \ldots\ldots\ldots\ldots, n')$ are the points of $\mathfrak{A}(z)\cdot\varepsilon$. We can make (σ) so small as we like; whence clearly

$$a(z, \mathfrak{A}, \Delta) \leq N_a(z, \Delta\cdot\varepsilon) \quad z\bar{\epsilon}K+N.$$

On the other hand, we take a succession of subdivisions (σ_ν) $(\nu=1, 2, \ldots\ldots)$ such that the diameters of divided parts tend to zero with $\frac{1}{\nu}$. Thus by the definition of (σ), if we put $\overset{\infty}{\underset{\nu=1}{V}}\overset{n}{\underset{i=1}{V}}(\bar{\delta}_i^\nu - \delta_i^\nu) = L$, then $m[L]=0$.

We will suppose $z\bar{\epsilon}L$. Then by $N_{\mathfrak{A}}(z, \Delta\cdot\varepsilon)=\infty$, we may take n so large that each partial region $\delta_i^{(n)}$ contains at most one point belonging to $\mathfrak{A}^{-1}(z)\cdot\varepsilon$. For such partial region $\delta_i^{(n)}$ which contains a point of $\mathfrak{A}^{-1}(z)\cdot\varepsilon$, we have $|A_{\mathfrak{A}}(z, \delta_i^{(n)})| \neq 0$. Therefore

$$(3) \qquad N_{\mathfrak{A}}(z, \varepsilon\cdot\Delta) \leq \sum_{i=1}^{m_n} |A_{\mathfrak{A}}(z, \delta_i)| = a_{\sigma_n}(z, \Delta) \leq a(z, \mathfrak{A}, \Delta)$$

holds and (2) and (3) give

$$N_{\mathfrak{A}}(z, \varepsilon\cdot\Delta) = a(z, \mathfrak{A}, \Delta), \quad z\bar{\epsilon}L+K+N, \ m[L]=m[K]=m[N]=0$$

$$2^\circ \qquad N_{\mathfrak{A}}(z, \varepsilon\cdot\Delta)=\infty \text{ yields } a(z, \mathfrak{A}, \Delta)=\infty$$

Because $\mathfrak{A}^{-1}(z)\cdot\varepsilon$ consists of an infinite numbers of points, we can select a sequence of points $w_1, w_2, \ldots\ldots\ldots$ Then we shall denote by $(\sigma_1), (\sigma_2), \ldots\ldots\ldots$ the succession of divisions which are again successively subdivided, and this sequence may be supposed to satisfy the following condition. Let the partial regions of (σ_ν) be denoted by $\delta_{\nu j}(j=1, 2, \ldots\ldots, n,)$ and the number of $\delta_{\nu j}$ Which contains at least one w_i by N_ν then $N_\nu \to \infty$ for $\nu \to \infty$

Take $i' > i$ so large that we have

$$N_i \leq \sum_{j=1}^{n_{i'}} |A_{\mathfrak{A}}(z, \delta_{i'j}) \leq a_{\sigma_{i'}}(z, \Delta) < a(z, \mathfrak{A}, \Delta);$$

here we shall make $i\to\infty$, hence $i'\to\infty$, $N_i\to\infty$, then $a(z, \mathfrak{A}, \Delta)=\infty$. Thus we may conclude by 1° and 2° that $a(z, \mathfrak{A}, \Delta)$ is summable on the z-plane when and only when $N_{\mathfrak{A}}(z, \varepsilon\cdot\Delta)$ is summable on the z-plane. Q.E.D

IV. Surface area.

In the present section we shall apply the property of the mapping of bounded variation described above to the Jordan surface and we will define the surface area.

Let the Jordan surface S be represented by the following expressions

$$S:\ x=x(u,\ v),\ y=y(u,\ v),\ z=z(u,\ v)\ \ (u,\ v)\in\Delta$$

where x, y, z are space rectangular coordinates. Let the three mappings $\begin{cases}x=x(u,\ v)\\ y=y(u,\ v),\end{cases}\begin{cases}y=y(u,\ v)\\ z=z(u,\ v)\end{cases}\begin{cases}x=x(u,\ v)\\ z=z(u,\ v)\end{cases}$ be denoted respectively by $x*y$, $y*z$, $z*x$. When all these mappings satisfy the assumption A), then we can define two kinds of the area corresponding to the concepts of $BV(O)$ and $BV(R)$.

Firstly in the case of $BV(O)$, the surface area $A_o(S)$ is defined, for all divisions (σ) mentioned above, by the following formula

$$A_o(S)=\sup_{(\sigma)}\sum_{i=1}^{n}\sqrt{[V_{x*y}(\delta_i)]^2+[V_{z*x}(\delta_i)]^2+[V_{z*x}(\delta_i)]^2}$$

For brevity we sahll write it $A_o(S)=\sup\limits_{(\sigma)}\sum\limits_{i=1}^{n}\sqrt{\sum\limits_{x*y}[V_{x*y}(\delta_i)]^2}$

Secondly in the case of $BV(R)$. we define the area $A_R(S)$ as follows:

$$A_R(S)=\sup_{(\sigma)}\sum_{i=1}^{n}\sqrt{[\int N_{x*y}(p,\ \delta_i)\mathrm{dm}(p)]^2*}$$
$$\overline{*+[\int N_{y*z}(p,\ \delta_i)\mathrm{dm}(p)]^2+[\int N_{z*x}(p,\delta_i)\mathrm{dm}(p)]^2}$$

For brevity, we write

$$A_R(S)=\sup_{(\sigma)}\sum_{i=1}^{n}\sqrt{\sum_{x*y}[\int N_{x*y}(p,\ \delta_i)\mathrm{dm}(p)]^2}$$

Then we have, by the results of the preceding section,

$$A_R(S)=\sup_{(\sigma)}\sum_{i=1}^{n}\sqrt{\sum_{x*y}[\int a(p,\ x*y,\ \delta_i)\mathrm{dm}(p)]^2}.$$

Since $V_{x*y}(\delta_i)\leq\int a(p,\ x*y,\ \delta_i)\mathrm{dm}(p)$, it is obvious that

$$A_R(S)\geq A_o(S)$$

By the definition of $A_R(S)$, there exists such a division (σ) that for any $\varepsilon > 0$, the following inequality holds

$$(4) \qquad A_R(S) - \frac{\varepsilon}{2} \leq \sum_{i=1}^{n} \sqrt{\sum_{x*y} [\int a(p, x*y, \delta_i) \mathrm{dm}(p)]^2}$$

Now we subdivide this subdivision again (and the subdivided regions being denoted by δ_{ij}), so that the following inequality holds

$$(5) \qquad \sum_{i=1}^{n} \sqrt{\sum_{x*y} [\int a(p, x*y, \delta_i) \mathrm{dm}(p)]^2} - \frac{\varepsilon}{2} \leq \sum_{i=1}^{n} \sqrt{\sum_{x*y} [\sum_{j=1}^{n_i} | \int A_{x*y}(p, \delta_{ij}) \mathrm{dm}(p)]^2}$$

Next by Minkowski's inequality, we have

$$(6) \qquad \sum_{i=1}^{n} \{ \sum_{x*y} [\sum_{j=1}^{n_i} | \int A_{x*y}(p, \delta_{ij}) \mathrm{dm}(p) |]^2 \}^{\frac{1}{2}} \leq \sum_{i=1}^{n} \sum_{j=1}^{n_i} \{ \sum_{x*y} | \int A_{x*y}(p, \delta_{ij}) \mathrm{dm}(p) |^2 \}^{\frac{1}{2}} \leq A_o(S)$$

By (4), (5), (6), we have

$$A_R(S) - \varepsilon \leq A_o(S).$$

Since ε is arbitrary, $A_R(S) \leq A_o(S)$, hence $A_R(S) = A_o(S)$. For example, when the Lebesgue area $A(S)$ is finite, then the assumption A) is fulfilled[5] and therefore $A_R(S) = A_o(S) = A(S)$.

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2) T. Radó "Length and area" Ameri. Math. Coll. Publication 1948 Chapt. V, 1.1 p.425

3) L. Cesari "Carratterizazione analitica delle superficie continue di area finita second Lebegue" Annali di R. Scoula Norm. Sup. Pisa ser. 2 Vol. 10 1941 pp. 253-294 and Vol. 11 1942 pp. 106-117.

4) T. Radó loc. cit. p. 303 IV, 1.74

5) H. Lebesgue "Integral, logueur, aire" Annali. Math. Appli. Vol. 7 1902 pp. 231-359.