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On the Special Riemann Spaces of Class Two

By

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In this paper we have presented the special Riemann spaces of class two and type one. In my previous papers[1] I defined the type number of space of class two and gave a necessary and sufficient condition that the space of type $\tau(\geqq 3)$ be of class two. But we have not yet any general research of spaces of type one and two.

Consider a n-dimensional variety V_n in such an $(n+p)$- dimensional euclidean space E_{n+p}, that the fundamental form

$$ds^2 = g_{ij} dx^i dx^j$$

is positive definite; and let $B^{\cdot\alpha}_{P}(P=I,\ldots\ldots, p;\ \alpha=1,\ldots, n+p)$ be a system of mutually orthogonal unite vectors normal to V_n and put $B^{\cdot\alpha}_{i}(i=1,\ldots, n;\ \alpha=1, \ldots, n+p)=\partial y^\alpha/\partial x^i$; where y^α are the cartesian coordinates in E_{n+p}. Then we obtain

$$\begin{aligned} dB^{\cdot\alpha}_{i} &= (\Gamma^k_{ij} B^{\cdot\alpha}_{k} + H^P_{ij} B^{\cdot\alpha}_{P}) dx^j, \\ dB^{\cdot\alpha}_{P} &= (-g^{kl} H^P_{lj} B^{\cdot\alpha}_{k} + H^Q_{Pj} B^{\cdot\alpha}_{Q}) dx^j, \end{aligned} \tag{0·1}$$

along a curve on V_n, where the functions H^P_{ij} are symmetric in the indices i, j and called the second fundamentel tensors for the normal $B^{\cdot\alpha}_{P}$; and the functions H^P_{Qi} are skew-symmetric in the indices P, Q. And those functions H^P_{ij} and H^P_{Qi} satisfy, moreover, the Gauss eqnation

$$R_{ijkl} = H^P_{ik} H^P_{jl} - H^P_{il} H^P_{jk}, \tag{0·2}$$

the Codazzi equation

$$H^P_{ij,k} - H^P_{ik,j} = H^Q_{ij} H^Q_{Pk} - H^Q_{ik} H^Q_{Pj}, \tag{0·3}$$

and finally the Ricci equation

$$H^P_{Qi,j} - H^P_{Qj,i} + (H^R_{Qi} H^P_{Rj} - H^R_{Qj} H^P_{Ri}) = g^{ab}(H^Q_{ai} H^P_{bj} - H^Q_{aj} H^P_{bi}). \tag{0·4}$$

In particular case for $p=1$ all the functions H^P_{Qi} vanish identically and in case for $p=2$ the expression in the parenthesis in the left member of (0.4) vanishes identically on account of the skew-symmetric property of H^P_{Qi}.

When we transform a system of normal vectors $B^{\cdot\alpha}_P$ to the another $\bar{B}^{\cdot\alpha}_P$, that is,

$$\bar{B}^{\cdot\alpha}_P = l^Q_P B^{\cdot\alpha}_Q\,; \qquad |\, l^Q_P \,| = \pm 1,$$

then the second fundamental tensors H^P_{ij} are transformed to $\bar{H}^P_{ij}$ defined by

$$\bar{H}^P_{ij} = l^Q_P H^Q_{ij}.$$

Throughout in this paper, by the space we shall mean the real Riemann space whose fundamental form is positive definite.

1. We consider such a n-dimensional variety V_n of an $(n+2)$-dimensional euclidean space, that, for example, the rank τ_{II} of the matrix $\|H^{II}_{ij}\|$ is less than two. Then we have from $(0\cdot 2)$

$$R_{ijkl} = H^{I}_{ik} H^{I}_{jl} - H^{I}_{il} H^{I}_{jk}. \tag{1·1}$$

If τ_{II} is equal to zero, we get from $(0\cdot 3)$

$$H^{I}_{ij,k} - H^{I}_{ik,j} = 0, \tag{1·2}$$

so that V_n can be imbedded in $(n+1)$-dimensional eucldean space. Consequently if V_n is of class two and $\tau_{II}<2$, we have $\tau_{II}=1$. Also, in this case, if the rank τ_I of the matrix $\|\, H^I_{ij} \,\|$ is more than three, the Codazzi equation $(1\cdot 2)$ is a result of the Gauss equation $(1\cdot 1)$ and hence V_n is of class one, which was proved by T. Y. Thomas.[2]

Now let us find a necessary and sufficient condition that a space V_n is of class two, τ_{II} is equal to one and τ_I is equal to three for a particular choice of a system of normal vectors $B^{\cdot\alpha}_P$. It has been shown by T. Y. Thomas,[3] that a necessary and sufficient condition for the Gauss equation $(1\cdot 1)$ having such a solution H^I_{ij}, that the rank τ_I of the matrix $\|\, H^I_{ij} \,\|$ is equal to three, be the matrix conditions

(I) rank of $\left\| \begin{matrix} R_{1abc} & R_{2abc} \cdots\cdots R_{nabc} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ R_{1ijk} & R_{2ijk} \cdots\cdots R_{nijk} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ R_{1pqr} & R_{2pqr} \cdots\cdots R_{npqr} \end{matrix} \right\| = 3,$

(II) $$R(a,\ b,\ c\ ;\ i,\ j,\ k) \geqq 0,$$
$$\sum_{a,b,c,i,j,k} R(a,\ b,\ c\ ;\ i,\ j,\ k) > 0,$$

and finally

(III) $$R_n(R) = 0\ ;$$

where $R(a,\ b,\ c\ ;\ i, j, k)$ is the determinant

$$\begin{vmatrix} R_{abij} & R_{abjk} & R_{abki} \\ R_{bcij} & R_{bcjk} & R_{bcki} \\ R_{caij} & R_{cajk} & R_{caki} \end{vmatrix},$$

and $R_n(R)$ is the resultant system of a system of homogeneous equations

$$t^2 R_{ijkl} = H^I_{ik} H^I_{jl} - H^I_{il} H^I_{jk},$$
$$H^I_{lm} R_{hijk} - H^I_{lk} R_{hijm} + H^I_{ji} R_{lkmh} - H^I_{jh} R_{lkmi} = 0,$$

as the above system of equations having a non-trivial solution $(t,\ H^I_{ij})$. Then the solution H^I_{ij} of the Gauss equation (1·1) is real and uniquely determined to within algebraic sign.

Further, for V_n of class two, there must exist two systems of functions $H^{II}_{ij} (= H^{II}_{ji})$ and $H_i (i, j, = 1,\ \ldots\ldots,\ n)$ satisfying the Codazzi equation

$$D^I_{aij} = -H^{II}_{ai} H_j + H^{II}_{aj} H_i, \tag{1·3}$$

$$D^{II}_{aij} = H^I_{ai} H_j - H^I_{aj} H_i, \tag{1·4}$$

the Ricci equation

$$H_{i,\,j} - H_{j,\,i} = g^{ab} (H^{II}_{ai} H^I_{bj} - H^{II}_{aj} H^I_{bi}), \tag{1·5}$$

and finally from $\pi_{II} = 0$

$$H^{II}_{ai} H^{II}_{bj} - H^{II}_{aj} H^{II}_{bi} = 0\ ; \tag{1·6}$$

where D^{p}_{aij} is defined by

$$D^{p}_{aij}=H^{p}_{ai,j}-H^{p}_{aj,i}.$$

From (1·3) and (1·6) we obtain

$$H^{II}_{kr}D^{I}_{jpq}-H^{II}_{jr}D^{I}_{kpq}=0, \tag{1·7}$$

and further easily

$$\text{(IV)} \qquad D^{I}_{aij}D^{I}_{bkl}-D^{I}_{akl}D^{I}_{bij}=0,$$

that is a necessary condition.

For the purpose that we obtain the equations determining the functions H^{II}_{ij}, differentiating (1·7) covariantly with respect to x^{t} and summing three equations obtained by permuting the indices i, j, k cyclically, we have

$$H_{(i}D^{I}_{jk)|rpq}+H^{II}_{r(i}D^{I}_{jk)|pq}=0^{(4)}, \tag{1·8}$$

on account of (1·4) ; where $D^{I}_{ik|rpq}$ and $D^{I}_{jk|pq}$ are defined by

$$D^{I}_{jk|rpq}=H^{I}_{kr}D^{I}_{jpq}-H^{I}_{jr}D^{I}_{kpq}, \tag{1·9}$$

$$D^{I}_{jk|pq}=D^{I}_{jpq,k}-D^{I}_{kpq,j}. \tag{1·10}$$

Multiplying (1·8) by H^{II}_{hd} and making use of (1·3) we obtain

$$D^{I}_{hd(i}D^{I}_{jk)|rpq}-H^{II}_{hd}H^{II}_{r(i}D^{I}_{jk)|pq}=H_{d}H^{II}_{h(i}D^{I}_{jk)|rpq}.$$

Further multiplying by D^{I}_{abc} and making use of (1·7), (1·9), (1·10), (IV) we obtain finally

$$D^{I}_{abc}D^{I}_{hpq}D^{I}_{d(ij}H^{I}_{k)r}=H^{II}_{hd}H^{II}_{ra}D^{I}_{(i|bc|}D^{I}_{jk)|pq}. \tag{1·11}$$

In this equation (1·11) the quantities D^{I}_{ijk} and H^{I}_{ij} are already known.

(A). Assume that all the coefficients $D^{I}_{(i|bc|}D^{I}_{jk)|pq}$ of $H^{II}_{hd}H^{II}_{ra}$ in (1·11) vanish. As all D^{I}_{ijk} can not be zero, since otherwise V_{n} be of class one, we have

$$D^{I}_{d(ij}H^{I}_{k)r}=0. \tag{1·12}$$

We can refer to such a system of coordinates (H'), that at the origin the matrix $\| H^{I}_{ij} \|$ has the form

$$\|H^I_{ij}\| = \left\| \begin{array}{c:c} H^I_3 & 0 \\ \hdashline 0 & 0 \end{array} \right\|, \quad |H^I_3| = \begin{vmatrix} H^I_{11} & H^I_{12} & H^I_{13} \\ H^I_{21} & H^I_{22} & H^I_{23} \\ H^I_{31} & H^I_{32} & H^I_{33} \end{vmatrix} \fallingdotseq 0,$$

because the matrix $\|H^I_{ij}\|$ is of the rank three. Making the conjugate H^{ij}_I of $H^I_{ij}(i,\ j=1,\ 2,\ 3)$ in $|H^I_3|$ and contracting (1·12) for the indices $r,\ i,\ j,\ k=1,\ 2,\ 3$; $d=1,\ldots\ldots,\ n$ by H^{kr}_I, we obtain $D^I_{dij}=0$ $(i,\ j=1,\ 2,\ 3;\ d=1,\ldots,\ n)$. Next taking the indices $r,\ i,\ j=1,$ $2,\ 3;\ k>3;\ d=1,\ldots,\ n$ in (1·12) we have

$$H^I_{ir}\, D^I_{djk} + H^I_{jr}\, D^I_{dki} = 0,$$

and contracting by H^{ir}_I we have $D^I_{djk}=0(j=1,2,3;\ k>3;\ d=1,\ldots,n)$. Finally taking the indices $i,r=1,2,3;\ j,k>3;\ d=1,\ldots,n$ in (1·12) we have $D^I_{djk}=0(j,k>3;\ d=1,\ldots n)$. Thus all D^I_{aij} are vanishing contrary to hypothesis. Consequently we get, as a necessary condition

$$\text{(V)} \qquad \sum_{i,j,k,p,q,b,c} (D^I_{(i|bc|}D^I_{jk)|pq})^2 > 0.$$

(B′). Next let us prove that a solution H^{II}_{ij} of the equation (1·11) is uniquely determined to within algebraic sign. In fact, let H^{II}_{ij} and $\bar{H}^{II}_{ij}$ be the two solutions of (1·11) and put

$$\bar{H}^{II}_{ij} = H^{II}_{ij} + h_{ij}. \tag{1·13}$$

Substituting (1·13) in (1·11) and making use of (V) we have

$$h_{hd}h_{ra} + h_{hd}H^{II}_{ra} + H^{II}_{hd}h_{ra} = 0. \tag{1·14}$$

Assume that the rank of the marix $\|h_{ij}\|$ is equal to $\tau(\fallingdotseq 0)$, and refer to a system of coordinates that at the origin the matrix $\|h_{ij}\|$ has the form

$$\|h_{ij}\| = \left\| \begin{array}{c:c} h_\tau & 0 \\ \hdashline 0 & 0 \end{array} \right\|, \quad |h_\tau| = \begin{vmatrix} h_{11} & \cdots\cdots & h_{1\tau} \\ \cdots & \cdots & \cdots \\ h_{\tau 1} & \cdots\cdots & h_{\tau\tau} \end{vmatrix} \fallingdotseq 0,$$

and make the conjugate h^{ij} of $h_{ij}(i,\ j=1,\ \ldots,\ \tau)$ in $|h_\tau|$. Contracting (1·14) for the indices $h,\ d,\ r,\ a=1,\ldots,\ \tau$ by h^{hd} gives

$$(\tau + h^{hd}H^{II}_{hd})\, h_{ra} + \tau H^{II}_{ra} = 0. \tag{1·15}$$

Further contracting by h^{ra} gives $h^{hd}H''_{hd}=-\tau/2$, so that from (1·15) we have $h_{ra}=-2H''_{ra}$, and then from (1·13) $\bar{H}''_{ij}=-H''_{ij}$ $(i, j=1, \dots, \tau)$. Next taking the indices $h, d, a=1, \dots, \tau$; $r>\tau$ in (1·14) we have $\bar{H}''_{ra}=0(r>\tau;\ a=1\ \dots, \tau)$ and taking the indices $h, d=1, \dots, \tau$; $r, a>\tau$ we have $H''_{ra}=0(r, a>\tau)$, so that $\bar{H}''_{ra}=0$ $(a=1, \dots, n;\ r>\tau)$. Hence we see $\bar{H}''_{ij}=-H''_{ij}(i, j=1, \dots n)$ for $\tau\neq 0$ and the above statement is proved.

(**C.**) Next, let us find a necessary and sufficient condition that the solution H''_{ij} is real. Taking the indices r and a equal to h and d respectively in (1·11) gives

$$D'_{abc}\,D'_{hpq}\,D'_{d(ij}H'_{k)h}=(H''_{hd})^2 D'_{(i\,|\,bc\,|}D'_{jk)\,|\,pq}\,,$$

and hence we have

(VI) $$D'_{abc}D'_{hpq}D'_{d(ij}H'_{k)h}\cdot D'_{(i\,|\,bc\,|}D'_{jk)\,|\,pq}\geqq 0,$$

that is a necessary condition. Conversely we see easily that if (VI) is satisfied, neither of these salution H''_{ij} can be pure imaginary.

Now we put

$$H''_{ij}=p_{ij}+\sqrt{-1}\ q_{ij}\,; \tag{1·16}$$

where the p's and q's are all real. Substituting from (1·16) in (1·11) and equating the imaginary part to zero we obtain

$$p_{hd}q_{ra}+q_{hd}p_{ra}=0. \tag{1·17}$$

By the similar process which was used in (B), it follows from (1·17) that if the rank of the matrix $\|q_{ij}\|$ does not vanish, all the p_{ij} are equal to zero, so that the solution H''_{ij} is real or pure imaginary. Consequently if (VI) is satisfied, the solution is real.

(**D.**) Finally we see easily that the necessary and sufficient condition for the system of equation (1·11) to have a solution H''_{ij}, is

(VII) $$\begin{vmatrix} D'_{abc}\,D'_{hpq}\,D'_{d(ij}H'_{k)r} & D'_{(i\,|\,bc\,|}D'_{jk)\,|\,pq} \\ D'_{aef}\,D'_{hst}\,D'_{d(xy}H'_{z)r} & D'_{(x\,|\,ef\,|}D'_{yz)\,|\,st} \end{vmatrix}=0$$

that is the resultant system of the system of homogeneous equations

$$t^2D'_{abc}\,D'_{hpq}\,D'_{d(ij}H'_{k)r}-H''_{hd}H''_{ra}D'_{(i\,|\,bc\,|}D'_{jk)\,|\,pq}=0,$$

having a non-trivial solution $(t,\ H''_{ij})$.

Conversely if the conditions (V), (VI), (VII) are satisfied, it is easily seen that (1·18) has a non-trivial solution $(t,\ H''_{ij})$; where $t \neq 0$, and thus H''_{ij}/t is a solution of (1·11) and also is real. Hence the functions H''_{ij} are uniquely determined to within algebraic sign; for example, when

$$D'_{1bc}\, D'_{1pq}\, D'_{1(ij}H'_{k)|1} \cdot D'_{(i|bc|}D'_{jk)|pq} > 0,$$

we have

$$H''_{11} = +\sqrt{\frac{D'_{1bc}D'_{1pq}D'_{1(ij}H'_{k)1}}{D'_{(i|bc|}D'_{jk)pq}}}\ , \tag{1·19}$$

and for another H''_{hd} we have

$$H''_{hd} = \frac{D'_{1bc}D'_{hpq}D'_{d(ij}H'_{k)1}}{H''_{11}D'_{(i|bc|}D'_{jk)|pq}}. \tag{1·20}$$

(**E**). From (1·11) we have by means of (IV)

$$H''_{hd}H''_{ra} - H''_{ha}H''_{rd} = 0,$$

so that the condition (1·6) is satisfied.

II. Now, for the purpose that we get the functions $H_i (i=1, \ldots, n)$ satisfying the equation (1·4), consider a system of homogeneous equations

$$t\, D''_{aij} - H'_{ai}H_j + H'_{aj}H_i = 0, \tag{2·1}$$

where $t,\ H_i (i=1, \ldots, n)$ are unknown. We see easily that all of H_i can not be zero for V_n of class two. Therefore the equation (2·1) must have a non-trivial solution $(t,\ H_i)$, so that we have a matrix condition;

rank of

$$\text{(VIII)}\qquad \left\| \begin{array}{cccccc} D''_{112} & H'_{12} & H'_{11} & 0 & 0 & \cdots\cdots\cdots\cdots\ 0 \\ D''_{212} & H'_{22} & H'_{n1} & 0 & 0 & \cdots\cdots\cdots\cdots\ 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots\ 0 \\ D''_{n12} & H'_{n2} & H'_{n1} & 0 & 0 & \cdots\cdots\cdots\cdots\ 0 \\ D''_{113} & H'_{13} & 0 & H'_{11} & 0 & \cdots\cdots\cdots\cdots\ 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots \end{array} \right\|$$

$$\left\| \begin{array}{cccccccc} D''_{n13} & H'_{n3} & 0 & H'_{n1} & 0 \ldots\ldots\ldots\ldots & & & 0 \\ \ldots\ldots & & & & & & & \\ \ldots\ldots & & & & & & & \\ D'_{1n-1n} & 0 & 0 & 0 & 0 \ldots\ldots\ldots & H'_{1n} & & H'_{1n-1} \\ \ldots\ldots & & & & & & & \\ D'_{nn-1n} & 0 & 0 & 0 & 0 \ldots\ldots\ldots & H'_{nn} & & H'_{nn-1} \end{array} \right\| \leqq n$$

as a necessary condition. Conversely, if the condition (VIII) is satisfied, the system of equations (2·1) has a non-trivial solution (t, H_i), where t is not vanishing; that is proved easily, putting $t=0$ in (2·1) *i, e.*

$$H'_{ai}H_j - H'_{aj}H_i = 0,$$

and refering to the system of coordinates (H'). Therefore we have H_i/t satisfying the system of equations (1·4). Also, in the similar manner, it is to be seen easily that the solutian H_i is uniquely determined and is real. Explicitly we have the form H_i, refering to the system of coordinates (H'), as follows,

$$\begin{aligned} &H_i = \frac{1}{2} H'^{ab}_{I} D''_{abi} \quad (a,\ b,\ i=1,\ 2,\ 3), \\ &H_i H'_{jk} = D''_{jki} \quad (i>3\,;\ j,\ k=1,\ 2,\ 3), \end{aligned} \tag{2·2}$$

where the indices $j,\ k$ of H'_{jk} are to be chosen for $H'_{jk} \neq 0$.

Thus obtained H'_{ij}, H''_{ij} and $H_i (i,\ j=1, =, n)$ must satisfy the equation (1·3), *i. e.*

$$\text{(IX)} \qquad D'_{aij} + H''_{ai}H_j - H''_{aj}H_i = 0,$$

that is a necessary condition.

Now we remark that those H'_{ij}, H''_{ij} and H_i satisfy the Ricci equation (1·5). In fact, differentiating (1·4) covariantly with respect to x^k and summing three equations obtained by cyclic permutation of the indices $i,\ j,\ k$ give by means of (IX) and (1·2)

$$H'_{a(i}D_{jk)} = 0, \tag{2·3}$$

where

$$D_{jk} = H_{j,k} - H_{k,j} - g^{ab}(H''_{aj}H'_{bk} - H''_{ak}H'_{bj}).$$

We have immediately $D_{jk}=0$, refering to the system of coordinates (H').

Consequently we obtain the

Theorem: ......*A Riemann space V_n of dimensionality $n(\geqq 3)$ is of class two and the matrices $\| H^{I}_{ij} \|$ and $\| H^{II}_{ij} \|$, whose elements are the second fundamental tensors of V_n, are equal to three and one respectively for a particular choice oj a system of normal vectors if, and only if, the inequalities* (II), (V), (VI) *and the equations* (III), (IV), (VII), (IX) *and finally the matrix conditions* (I), (VIII) *are satisfied.*

Finally it is to be noted that the curvature tensor R_{ijkl} of those V_n satisfies the equation

$$R_{a.i(j}^{\ \ b}R_{|b|.kl)}^{\ \ a}=0,$$

making use on (1·1) and therefore V_n is of *type one.*[5] But those V_n are not the general spaces of class two ane type one.

Referenccs

(1) Jour. Japan Math. So., vol. 2. nos. 1—2. Riemann spaces of class two and their algebraic characterization. pp. 67–92.

(2) Acta Mate., 61 (1936) p. 189

(3) l. c., p. 194

(4) for example

$$A_{(i|b|}B_{jk)}=A_{ib}B_{jk}+A_{jb}B_{ki}+A_{kb}B_{ij}$$

(5) M. Matsumoto, l. c., p. 69