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Riemann Spaces of Recurrent and Separated Curvature and their Imbedding.

By

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Introduction

T. Y. Thomas [4] showed that a hypersurface V_n of type more than two in a euclidean space is intrinsically rigid and the Codazzi equation of V_n is automatically satisfied as consequence of the Gauss equation, if V_n is of type more than three; and hence, in the case of type more than three, V_n is of class one if and only if the Gauss equation is satisfied, so the conditions (7.4), (8.4) and (8.10) of his paper is necessary and sufficient. Beside if V_n is of type three, the condition (10.2) of his paper must be imposed. Also he remarked that the discussion of space of type two requires essentially different methods than those of higher type number. Thus he did not discuss such spaces and that the present author does not know any research for such spaces after the paper of T. Y. Thomas.

The similar circumstances arise in the case of class two [1] and the author discussed a special type of such spaces of clas two and of lower type number [2].

In the first section of this paper we give a condition that there exists a solution H_{ij} satisfying the Gauss equation for V_n of type two and define a interesting class of space, which is called to be *of separated curvature.* The second section gives a number of necessary conditions, that the Codazzi equation is satisfied for V_n of type two. Moreover, in the rest two sections we deal with V_n of type two and class one from various points of views. At first, in the third section, we define a semi-covariantly-constant tensor and prove a interesting theorem for Ruse's space of recurrent curvature. Finally, in the fourth section, we define a semi-Codazzi tensor and give a explicit form of one of the conditions for space with projective connection to be of class one, which we dealed with in a recent paper [3].

1. Spaces of type two.

Let R_{ijkl} be the curvature tensor of n-dimensional Riemann space V_n of class one. V_n is of class one if and only if V_n does not flat and be imbedded in a $(n+1)$-dimensional euclidean space. Then there exists a system of functions $H_{ij}(=H_{ji})$ $(i,j=1, \cdots, n)$ satisfying the *Gauss equation*

$$R_{ijkl}=H_{ik}H_{jl}-H_{il}H_{jk}, \tag{1.1}$$

and the *Codazzi equation*

$$H_{ij,k}-H_{ik,j}=0. \tag{1.2}$$

We call H_{ij} usually the *second fundamental tensor* of V_n. It was shown by T. Y. Thomas that the rank τ of the matrix $\| H_{ij} \|$ is equal to that of the matrix

$$\left\| \begin{matrix} R_{abc1} & R_{abc2}\cdots\cdots\cdots R_{abcn} \\ \vdots & \vdots \qquad\qquad\quad \vdots \\ R_{pqr1} & R_{pqr2}\cdots\cdots\cdots R_{pqrn} \end{matrix} \right\| \tag{1.3}$$

if τ is more than one. This number is called by him the *type number* of V_n. In the case of τ being equal to zero or one we know well that V_n is flat.

Consider V_n of class one and type two, i. e.,

$$\begin{vmatrix} H_{ai} & H_{aj} & H_{ak} \\ H_{bi} & H_{bj} & H_{bk} \\ H_{ci} & H_{cj} & H_{ck} \end{vmatrix}=0. \tag{1.4}$$

The inverse determinant of (1.4) is accordingly of rank zero or one and hence, by means of (1.1), we have

$$\begin{vmatrix} R_{abij} & R_{abjk} \\ R_{bcij} & R_{bcjk} \end{vmatrix}=0.$$

From this it follows immediately that

$$\begin{vmatrix} R_{abij} & R_{abkl} \\ R_{cdij} & R_{cdkl} \end{vmatrix}=0. \tag{1.5}$$

This means that the matrix $\| R_{abij} \|$ $(a, b$: row, i, j: column) is of rank zero or one. Hence if V_n is of class one, the condition

$$\sum_{a,b,c,d}(R_{abcd})^2>0 \tag{1.6}$$

must be imposed, say, the matrix $\|R_{abij}\|$ has exactly rank one.

Conversely, we shall show that the conditions (1.5) and (1.6) are sufficient that there exists a system of functions H_{ij} satisfying (1.1). At first we have as a particular case of (1.5)

$$R_{abab}R_{ijij}-(R_{abij})^2=0.$$

Hence, if all R_{abab} vanish, the curvature tensor is evidently equal to zero, contradicting to (1.6). Thus, say R_{1212} does not vanish and then we define S_{12} and $e(=\pm1)$ as follows:

$$R_{1212}=e(S_{12})^2,$$

and next define the other S_{ij} as follows:

$$R_{12ij}=eS_{12}S_{ij}.$$

We have immediately, by means of (1.5)

$$R_{ijkl}=e\,S_{ij}\,S_{kl}. \tag{1.7}$$

It is verified easily that these S_{ij} are determined uniquely to within algebraic sign. Moreover the skew-symmetric matrix $\|S_{ij}\|$ is of rank two. In fact we have from (1.7)

$$S_{ij}S_{kl}+S_{ik}S_{lj}+S_{il}S_{jk}=0, \tag{1.8}$$

because of a property of the curvature tensor. From (1.1), (1.4) and (1.7) we have

$$H_{ai}S_{jk}+H_{aj}S_{ki}+H_{ak}S_{ij}=0. \tag{1.9}$$

Now, say, if R_{1212} does not vanish, we choose three functions H_{11}, $H_{12}(=H_{21})$ and H_{22} arbitrarily, except that these must satisfy

$$R_{1212}=H_{11}H_{22}-(H_{12})^2,$$

and let us define the other H_{ij} by

$$\begin{aligned} H_{1k}S_{12}&=-H_{11}S_{2k}-H_{12}S_{k1},\\ H_{2k}S_{12}&=-H_{21}S_{2k}-H_{22}S_{k1}, \quad (j,k=3,\cdots,n),\\ H_{jk}S_{12}&=-H_{j1}S_{2k}-H_{j2}S_{k1},\end{aligned}$$

remembering (1.9). Then it is easily proved by substitution that these quantities $H_{ij}(i,j=1,\cdots,n)$ satisfy the equation (1.1). Consequently we have the

Theorem 1. *If V_n is of type two, there exists a system of functions $H_{ij}(i, j=1,\cdots, n)$ satisfying the Gauss equation (1·1) if and only if the rank of matrix $\|R_{abij}\|$ $(a, b:$ row, $i, j:$ column) is equal to one.*

Further we must choose H_{11}, H_{12} and H_{22} satisfying the Codazzi equation (1.2) for V_n to be of class one.

Now we have a type of space V_n, such that the curvature tensor satisfies (1.7), whenever V_n is of class one or not. We call such a space a *space of separated curvature* and S_{ij} defined by (1.7) the *separated curvature* of V_n. For example, a simple K^*-space, a kind of Ruse's spaces of recurrent curvature, dealed with H. S. Ruse and A. G. Walker [5], is of separated curvature. We shall return to such spaces in the third section.

2. Further conditions for V_n of type two and class one

Covariant differentiation of (1.9) with respect to x^l and subtraction the equations obtained by interchanging the index l and i, l and j, l and k, give

$$H_{ai}S_{jkl}-H_{aj}S_{ikl}-H_{ak}S_{jil}-H_{al}S_{jki}=0, \tag{2.1}$$

in consequence of (1.2) ; where we put

$$S_{ijk}=S_{ij,k}+S_{jk,i}+S_{ki,j}.$$

Multiplying (2.1) by H_{bh} and subtracting from this the equation obtained by interchanging a and b, we have

$$S_{hi}S_{jkl}-S_{hj}S_{ikl}-S_{hk}S_{jil}-S_{hl}S_{jki}=0, \tag{2.2}$$

on account of (1.1) and (1.7). This is necessary for V_n to be of class one. Next, differentiating (1.9) covariantly with respect to x^b and subtracting from this the equation obtained by interchanging a and b, we have

$$H_{a(i}S_{jk),b}-H_{b(i}S_{jk),a}=0. \tag{2.3}$$

Moreover, multiplying (2.3) by S_{cd} and summing the equations obtained by cyclic permutation of, b, c and d, we have

$$H_{a(i}S_{jk),b}S_{cd}+H_{a(i}S_{jk),c}S_{db}+H_{a(i}S_{jk),d}S_{bc}=0,$$

making use of (1.9) ; and finally, the equation gives

$$S_{h(i}S_{jk),b}S_{cd}+S_{h(i}S_{jk),c}S_{db}+S_{h(i}S_{jk),d}S_{bc}=0, \tag{2.4}$$

by the similar process as we have (2.2) from (2.1). This is also necessary for V_n of type twe being of class one.

Thus we have necessary conditions (2.2) and (2.4) for the Codazzi equation (1.2) to be satisfied, but the author is certain that these conditions do not sufficient. In fact, we have immediately from (2.3) a system of linear homogeneous equations in terms of only three quantities H_{11}, H_{12} and H_{22}, and hence we have certain conditions, under which these equations are compatible. From this, in general, we have the ratio $H_{11}:H_{12}:H_{22}$ and then these quantities are given themselves uniquely from the equation $R_{1212}=H_{11}H_{22}-(H_{12})^2$. But, the discussion is very complicated in details. We return to this by the different point of view in the end of this paper.

3. Spaces of recurrent and separated curvature

[**A**]. Let a tensor X_{ij} of second order be given. If there exists such a function $\sigma(x)$ ($\neq$ constant) that $Y_{ij}=\sigma\cdot X_{ij}$ satisfies the equation

$$Y_{ij,k}=0, \tag{3.1}$$

but not X_{ij} itself, then X_{ij} is called a *semi-covariantly-constant tensor* (for brevity by *scc* we show). For example, the Ricci curvature tensor R_{ij} of Einstein space is *scc* and the factor $\sigma(x)$ is equal to n/R, if the scalar curvature R does not vanish. We give a condition for a given X_{ij} to be *scc*. (3.1) is written in the form

$$\frac{\partial\sigma}{\partial x^j}X_{ab}+\sigma X_{ab,j}=0,$$

and if we put

$$\frac{1}{\sigma}\frac{\partial\sigma}{\partial x^j}=\rho_j, \tag{3.2}$$

we have the fundamental equation

$$\rho_j X_{ab}+X_{ab,j}=0. \tag{3.3}$$

First, we find a condition that algebraic equation (3.3), in which ρ_j is unknown, has a solution. It follows evidently from (3.3)

$$X_{ab}X_{cd,j}-X_{cd}X_{ab,j}=0. \tag{3.4}$$

Conversely if (3.4) is satisfied, we define $\rho_j(j=1, \cdots, n)$ by the equations

$$\rho_j X_{pq}=-X_{pq,j} \quad (j=1, \ldots, n),$$

for suitable choice of non-vanishing X_{pq} and we see easily that these ρ_j satisfy (3.3). Since X_{ij} itself does not satisfy (3.1), ρ_j above determined is not all vanishing. And also, we see that a solution of (3.3) is uniquely determined.

Next, we find a condition of integrability of (3.2), i.e.

$$\rho_{jk}\equiv\rho_{j,k}-\rho_{k,j}=0.$$

According to above discussion, since ρ_j is expressible in terms of X_{ij} and its derivatives of first order, the above equation contains these quantities alone and hence is the implicit form of condition of integrability. However, it is preferable to write this condition explicitly and we can do so. In fact, differentiating (3.3) covariantly with respect to x^k and subtracting from this the equation obtained by interchanging j and k, we have

$$\rho_j X_{ab,k}-\rho_k X_{ab,j}+X_{ab,jk}-X_{ab,kj}=0. \tag{3.5}$$

This must be satisfied by ρ_j above determined as the solution of (3.3). Multiplying (3.5) by X_{hi} and making use of (3.3) we have

$$X_{ab,j}X_{cd,k}-X_{ab,k}X_{cd,j}+X_{cd}(X_{ab,jk}-X_{ab,kj})=0. \tag{3.6}$$

This is equivalent to

$$X_{ab,jk}-X_{ab,kj}=0, \tag{3.7$'$}$$

because of multiplying (3.6) by X_{hi} and making use of (3.4). Moreover (3.7′) is written in the form

$$X_{ib}R_{a\cdot jk}^{\cdot i}+X_{ai}R_{b\cdot jk}^{\cdot i}=0. \tag{3.7}$$

Conversely, if (3.4) and (3.7) are satisfied, we have (3.6) and hence (3.5) is satisfied by ρ_j, being the solution of (3.3). Consequently we have all $\rho_{jk}=0$. Thus we prove the

Lemma. *A given tensor X_{ij} is scc if and only if the equations* (3·4) *and* (3·7) *are satisfied.*

[B]. We treat again with V_n of separated curvature and assume that the separated curvature S_{ij} is *scc*. It is distinguishing property that this tensor S_{ij} satisfies always (3.7). In fact, we have

$$S_{ib}R_{a\cdot jk}^{\;i}+S_{ai}R_{b\cdot jk}^{\;i}=S_{jk}(S_{ib}S_{ac}+S_{ai}S_{bc})g^{ci}=-S_{jk}S_{ab}S_{ei}g^{ci}=0,$$

by means of (1.8). Accordingly the assumption for S_{ij} to be *scc* imposes upon this tensor a condition (3.4) alone, i. e.

$$S_{kl}S_{ij,m}-S_{ij}S_{kl,m}=0. \tag{3.8}$$

Covariant differentiation of (1.7) gives

$$R_{ijkl,m}=e(S_{ij,m}S_{kl}+S_{ij}S_{kl,m}),$$

and it follows from (3.8)

$$R_{ijkl,m}=2e\,S_{ij}S_{kl,m}.$$

Then, owing to (3.3), that is $-\rho_m S_{kl}=S_{kl,m}$, we have finally

$$R_{ijkl,m}=R_{ijkl}\cdot K_m \quad (K_m=-2e\,\rho_m). \tag{3.9}$$

Consequently V_n is of recurrent curvature. Conversely if V_n of separated curvature is of recurrent curvature, we see

$$R_{ijkl,m}=R_{ijkl}\cdot K_m=e\,S_{ij}S_{kl}\cdot K_m=e(S_{ij}S_{kl,m}+S_{ij,m}S_{kl}).$$

From this it follows

$$S_{ij}\Big(S_{kl,m}-\frac{1}{2}K_mS_{kl}\Big)+S_{kl}\Big(S_{ij,m}-\frac{1}{2}K_mS_{ij}\Big)=0.$$

Since the rank of matrix $\|S_{ij}\|$ is equal to two, there exists such a coördinate that at the origin only one component S_{12} of separated curvature does not vanish. Referring to such a coördinate, (3.10) with $i=k=1$, $j=l=2$ gives

$$S_{12,m}-\frac{1}{2}K_mS_{12}=0 \quad (m=1,\cdots,n).$$

Putting $i=1$, $j=2$, and $(k,l)\neq(1,2)$ in (3.10) we now have

$$S_{kl,m}-\frac{1}{2}K_mS_{kl}=0 \quad (m=1,\cdots,n).$$

Consequently there exist quantities ρ_j satisfying

$$\rho_jS_{kl}+S_{kl,j}=0 \quad (j,k,l=1,\cdots,n),$$

and that these ρ_j satisfies $\rho_{j,k}-\rho_{k,j}=0$, since (3.7) is always satisfied. Hence S_{ij} is *scc* and thus we have the interesting

Theorem 2. *A space V_n of separated curvature is of recurrent curvature if and only if the separated curvature tensor is scc.*

[**C**]. From (3.3) we have

$$\rho X_{ab} = -X_{ab,j}\rho^j ;$$

where ρ^j is a vector and $\rho=\rho_j\rho^j$. We shall prove that there is a similar property for the separated curvature, if V_n is of class one, independent that S_{ij} is *scc* or not, that is to say, V_n is of recurrent curvature or not. In fact, we take a non-trivial solution σ^a of equation

$$H_{ia}.\sigma^a=0 \quad (i, a=1,\cdots, n). \tag{3.11}$$

Contraction (2.3) by σ^b we have

$$H_{ai}T_{jk}+H_{aj}T_{ki}+H_{ak}T_{ij}=0 ; \tag{3.12}$$

where we put $T_{jk}=S_{jk,b}\cdot\sigma^b$. From (3.12) it follows that the skew-symmetric $\| T_{ij} \|$ is of rank two or zero. Multiplying (3.12) by H_{bl} and subtracting the equation obtained by interchanging a and b we have

$$S_{li}T_{jk}+S_{lj}T_{ki}+S_{lk}T_{ij}=0, \tag{3.13}$$

by means of (1.1) and (1.7). Refer to a similar coördinate with respect to T_{ij} as in the above paragraph of Theorem 2, if T_{ij} has the rank two. It is easily verified from (3.13) that only one component T_{12} does not equal to zero. Consequently there exists a quantity $\varkappa(\neq 0)$ satisfying

$$\varkappa S_{ij}=T_{ij}\equiv S_{ij,k}\cdot\sigma^k, \tag{3.14}$$

or otherwise we have

$$T_{ij}\equiv S_{ij,k}\cdot\sigma^k=0. \tag{3.15}$$

Thus we obtain the

Thorem 3. *If V_n of separated curvature is of class one, we have* (3·14) *or* (3·15) ; *where σ^i is a non-trivial solution of* (3·11).

There are $(n-2)$ linearly independent solutions of (3.11), for everyone of which the equations (3.14) or (3.15) are always satisfied.

4. The semi-Codazzi tensors

We remember the Codazzi equation (1.2) and know that the second fundamental tensor of hypersurface in a euclidean space satisfies a system of differential equations of the following type:

$$Y_{ai,j}-Y_{aj,i}=0. \tag{4.1}$$

Let us generalize such a property and define a certain class of tensors. If a tensor X_{ij} is given and there exists such a function $\sigma(x)$ ($\neq$ constant) that $Y_{ij}=\sigma\cdot X_{ij}$ satisfies (4.1), then we call X_{ij} a *semi-Codazzi tensor.* The present author met with such a tensor, when he discussed the *imbedding problem of space with projective connection* [3], and at that time a condition of integrability was given in a implicit form, in the sense that we noted in [A] of the third section. We just now discuss this problem throughoutly and give a explicit form of the condition. Also a similar circumstance arises as we discuss a Riemannian V_n of type two and class one; that is, we have three independent solutions of (1.9) and hence H_{ij} satisfying (1.2) must be determined as a linear combination of them. Our problem is eventually reduced to that of finding three coefficients of such a combination.

From (4.1) we have

$$\sigma_{,j}X_{ai}-\sigma_{,i}X_{aj}+\sigma X_{aij}=0. \quad (X_{aij}=X_{ai,j}-X_{aj,i}).$$

Defining quantities ρ_i as (3.2) this equation is reducible to

$$\rho_j X_{ai}-\rho_i X_{aj}+X_{aij}=0. \tag{4.2}$$

Suppose that the rank of the matrix $\| X_{ij}\|$ is more than one in the following. We see easily that a solution ρ_j of (4.2) is uniquely determined, if (4.2) admits a solution. At first we give a condition for (4.2) having a solution. Multiplying (4.2) by X_{bk} and making use of (4.2) give

$$\rho_k X_{abij}=X_{abijk}; \tag{4.3}$$

where we put

$$X_{abij}=X_{ai}X_{bj}-X_{aj}X_{bi},$$

$$X_{abijk}=X_{ai}X_{bkj}-X_{aj}X_{bki}-X_{bk}X_{aij}.$$

It follows from (4.3) that we have as a necessary condition

$$X_{abij}X_{cdlmk}-X_{cdlm}X_{abijk}=0, \tag{4.4}$$

since (4.2) and (4.3) must be compatible. Conversely if (4.4) is satisfied, then we have ρ_j, a solution of (4.2). In fact, since there exists $X_{abij} \neq 0$, say X_{1212}, by means of our hypothesis on the rank of $\| X_{ij} \|$, we define $\rho_k (k=1,\cdots,n)$ by

$$\rho_k X_{1212} = X_{1212k}.$$

And it follows from (4.4) that these ρ_k satisfy all equations of (4.3). We now put

$$D_{aij} = \rho_j X_{ai} - \rho_i X_{aj} + X_{ai'j},$$

that is a left-hand member of (4.2). By the similar method as we get (4.3), we obtain

$$X_{ai} D_{bkj} - X_{aj} D_{bki} - X_{bk} D_{aij} = 0. \tag{4.5}$$

At first suppose $| X_{ij} | \neq 0$. Contraction (4.5) with respect to X^{ak} gives immediately $D_{bij}=0 (b, i, j=1,\cdots,n)$. Next suppose $| X_{ij} |=0$ and then the rank of $\| X_{ij} \|$ is equal to $\tau (n > \tau \geqq 2)$. Refer to such a coordinate that at the origin the matrix $\| X_{ij} \|$ has the form

$$\left\| \begin{array}{ccc|c} X_{11} & \cdots & X_{1\tau} & \\ \vdots & & \vdots & 0 \\ X_{\tau 1} & \cdots & X_{\tau\tau} & \\ \hline & 0 & & 0 \end{array} \right\|, \qquad \begin{vmatrix} X_{11} & \cdots & X_{1\tau} \\ \vdots & & \vdots \\ X_{\tau 1} & \cdots & X_{\tau\tau} \end{vmatrix} \neq 0.$$

Making ure of the same process as in the case of $| X_{ij} | \neq 0$ we have $D_{bij}=0 (b, i, j=1,\cdots,\tau)$. (4.5) with $a > \tau ; b, k \leqq \tau$ gives $D_{aij}=0$ $(a > \tau ; i, j=1,\cdots,n)$ and with $a, b, k \leqq \tau ; i, j > \tau$ gives $D_{aij}=0 (a \leqq \tau ;$ $i, j > \tau)$. And finally, putting $a, b, i, j \leqq \tau ; k > \tau$ we have

$$X_{ai} D_{bkj} - X_{aj} D_{bki} = 0,$$

from which we obtain easily $D_{aik}=0 (a, i \leqq \tau ; k > \tau)$. Thus we conclude that all $D_{aij}=0$ and consequently we proved above statement.

The functions ρ_j so determined do not all vanish; since otherwise we should have all $X_{aij}=0$ from (4.2), contradicting to our hypothesis.

Moreover a condition must be impose that we have a function $\sigma(x)$. This is equivalent to a condition of integrability

$$\rho_{ij} \equiv \rho_{i,j} - \rho_{j,i} = 0 ;$$

where ρ_i is the solution of (4.2). For this purpose we differentiate (4.2) covariantly with respect to x^k and sum the equations obtained by cyclic permutting of i, j, k and it follows that

$$\rho_i X_{ajk}+\rho_j X_{aki}+\rho_k X_{aij}-X_{a(ij,k)}=0. \tag{4.6}$$

On the other hand we multiply (4.2) by ρ_k and sum the equations obtained by cyclic permutting of i, j, k. Then we get

$$\rho_i X_{ajk}+\rho_j X_{aki}+\rho_k X_{aij}=0. \tag{4.7}$$

Hence it follows from (4.6) $X_{a(ij,k)}=0$. That is equivalent to

$$X_{bi}R_{a\cdot jk}^{\cdot b}+X_{bj}R_{a\cdot ki}^{\cdot b}+X_{bk}R_{a\cdot ij}^{\cdot b}=0 \tag{4.8}$$

making use of Ricci identity. Since (4.7) is satisfied by the solution ρ_j of (4.2), we have

$$X_{bclmi}X_{ajk}+X_{bclmj}X_{aki}+X_{bclmk}X_{aij}=0, \tag{4.9}$$

multiplying (4.7) by X_{bclm} and making use of (4.3). Thus we get necessary conditions (4.8) and (4.9). Conversely if these conditions are satisfied and that $\| X_{ij} \|$ is of rank more than two, then we conclude that the solution ρ_j of (4.2) satisfies the equation $\rho_{ij}=0$. In fact we have

$$X_{a(i}\rho_{jk)}+X_{a(ij}\rho_{k)}-X_{a(ij,k)}=0, \tag{4.10}$$

differentiating (4.2) covariantly. On the other hand it follows from (4.3), (4.8) and (4.9) $X_{a(ij}\rho_{k)}=0$ and $X_{a(ij,k)}=0$, so that (4.10) gives

$$X_{ai}\rho_{jk}+X_{aj}\rho_{ki}+X_{ak}\rho_{ij}=0.$$

From this it follows easily that, if the rank of $\| X_{ij} \|$ is more than than two, all ρ_{jk} is equal to zero and thus in this case we proved the above mentioned. But, in case of the rank two, we can not conclude this from the above relation.

In this particular case, we discuss directly as follows. That is, if we differentiate (4.3) covariantly with respect to x^l and subtract the equation obtained by interchanging k and l, then we have, in virture of $\rho_{jk}=0$

$$\rho_k X_{abij,l}-\rho_l X_{abij,k}=X_{abijk,l}-X_{abijl,k} \tag{4.11}$$

Since (4.11) must be satisfied by the solution ρ_j of (4.2), i. e., (4.3), we have easily

$$X_{cdhmk}X_{abij,l}-X_{cdhml}X_{abij,k}=X_{cdhm}(X_{abijk,l}-X_{abijl,k}). \tag{4.12}$$

It is evident that (4.12) is necessary and sufficient for $\rho_{ij}=0$ in case of the rank of matrix $\| X_{ij} \|$ being two.

Finally we obtain a factor $\sigma(x)$, integrating (3.2) and this $\sigma(x)$ does not equal to constant; since otherwise ρ_j would be all equal to zero. Thus we obtain a factor $\sigma(x)$ and then general factor has the form $c\sigma(x)$, where the quantity c is a constant of integration. Summarizing above discussions we have the

Theorem 4. *A necessary and sufficient condition that a given tensor X_{ij} is semi-Codazzi type, is as follows.*

(A). *If the rank of $\| X_{ij} \|$ is more than two, the equations* (4·4), (4·8) *and* (4·9) *are satisfied.*

(B). *If the rank of $\| X_{ij} \|$ is equal to two, the equations* (4·4), *and* (4·12) *are satisfied.*

Then the factor $\sigma(x)$ is determined uniquely to within constant coefficient.

It is, of course, evident that if the rank of $\| X_{ij} \|$ is more than one, the equations (4.4) and (4.12) are necessary and sufficient for X_{ij} to be semi-Codazzi. But the condition (4.12) contains derivatives of X_{ij} of second order, but not (4.8) and (4.9).

This (A) of Theorem 4 can be applied to the problem for space with projective connection, which we remarked at the beginning of this section.

Now we discuss a generalized case of the above problem. Let us determine N function $\sigma_P (p=1,\cdots,N)$, such that $Y_{ij}=\sum\sigma_P X^P_{ij}$, where X^P_{ij} are given, satisfies the Codazzi equation. In this case the equation. by which the factor are determined, is the following

$$\sigma_{P,j}X^P_{ai}-\sigma_{P,i}X^P_{aj}+\sigma_P X^P_{aij}=0. \tag{4.13}$$

It follows by the same method as we have (4.6)

$$\sigma_P (X^P_{bi}R_{a\cdot jk}^{\ \ b}+X^P_{bj}R_{a\cdot ki}^{\ \ b}+X^P_{bk}R_{a\cdot ij}^{\ \ b})=0 \tag{4.14}$$

By means of this equation, if there exists at least one of $X^P_{b(i} R_{|a|\cdot jk)}^{\ \ b}$ not vanishing, then a number of σ_P is determined by the remaining σ_P and hence Y_{ij} is written as the form

$$Y_{ij}=\lambda_Q \bar{X}^Q_{ij} \quad (Q=1,\cdots,\bar{N}<N)\ ;$$

where $\bar{X}^Q_{ij}$ is linear combination in terms of X^P_{ij} and its coefficients is already known. Proceeding in this way we get in general Y_{ij}

$=\sigma\cdot Z_{ij}$ finally, where Z_{ij} is linear combination of X^P_{ij} and its coefficients are already known. Thus we may apply the above Theorm in general. But, unfortunately it happens that this method cannot be applied to our main problem of V_n of type two to be of class one, because of from (1.9)

$$H_{b(i}R_{|a|\cdot jk)}^{\ \ b}=H_{b(i}S_{jk)}S_{a\cdot}^{b}=0.$$

Hence the equation as (4.14) imposes nothing upon σ_P.

Finally we shall touch on a particular type of semi-Coazzi tensor X_{ij}, such that the rank of the matrix $\|X_{ij}\|$ is equal to one. It follows from (4.2) that we obtain easily the following conditions

$$\begin{aligned} &X_{ak}X_{bij}-X_{bk}X_{aij}=0,\\ &X_{ai}X_{bjk}+X_{aj}X_{bki}+X_{ak}X_{bij}=0. \end{aligned} \tag{4.15}$$

Conversely if these conditions are satisfied, we obtain ρ_j, solution of (4.2), as follows:

$$\rho_j X_{11}=\rho_1 X_{1j}-X_{11j} \quad (j=2,\cdots,n)\,; \tag{4.16}$$

where ρ_1 is arbitrary function and we must choose $X_{11}\neq 0$. Thus we meet with a similar problem that we remarked at the below paragraph of Theorem I.

We know that independent equations of (4.2) is given by (4.16), so that we must determine a function ρ satisfying (4.16), a system of partial differential equations. We see from the theories of differential equation that (4.16) is equivalent to

$$U_jf\equiv X_{11}\frac{\partial f}{\partial x_j}-X_{1j}\frac{\partial f}{\partial x^1}-X_{11j}\frac{\partial f}{\partial \rho}=0 \quad (j=2,\cdots,n), \tag{4.17}$$

that is, if we find a solution of (4.17), we have immediately a solution of (4.16). Therefore if (4.2) has a solution not to be constant, (4.17) must constitute a *complete system*. We have from (4.17)

$$\begin{aligned} &(U_kU_j-U_jU_k)f\\ &=\frac{1}{X_{11}}\Big\{\Big(X_{11}\frac{\partial X_{11}}{\partial x^k}-X_{1k}\frac{\partial X_{11}}{\partial x^1}\Big)\frac{\partial f}{\partial x^j}-\Big(X_{11}\frac{\partial X_{11}}{\partial x^j}-X_{1j}\frac{\partial X_{11}}{\partial x^1}\Big)\frac{\partial f}{\partial x^k}\\ &-\Big(X_{11}\frac{\partial X_{1j}}{\partial x^k}-X_{11}\frac{\partial X_{1k}}{\partial x^j}+X_{1j}\frac{\partial X_{1k}}{\partial x^1}-X_{1k}\frac{\partial X_{1i}}{\partial x^1}\Big)\frac{\partial f}{\partial x^1} \end{aligned}$$

$$-\left(X_{11}\frac{\partial X_{11j}}{\partial x^k} - X_{11}\frac{\partial X_{11k}}{\partial x^j} + X_{1j}\frac{\partial X_{11j}}{\partial x^1} - X_{1k}\frac{\partial X_{11j}}{\partial x^1}\right)\frac{\partial f}{\partial \rho}\Big\}$$

$$(j, k=2, \cdots, n), \qquad (4.18)$$

And, substituting (4.17) and making use of (4.8), (4.15) and the equations obtained by differentiating the second of (4.15) covariantly, we have finally all $(U_kU_j-U_jU_k)f=0$. Hence (4.17) is Jacobi's complete system and so integrable. Consequently we have the

Theorem 5. *A necessary and sufficient condition that a given tensor X_{ij} is semi-Codazzi type, where the rank of the matrix $\| X_{ij} \|$ is equal to one, is that the equations* (4·15) *and* (4·8) *are satisfied.*

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