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# On the Existence of Periodic Solutions of the Non-linear Differential Equation, $\ddot{x}+a(x)\cdot\dot{x}+\varphi(x)=p(t)$

By

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Many authors have investigated the conditions for the existence of the periodic solutions of the above differential equation; Nagumo[1], Furuya[2], Cartwright and Littlewood[3], Cartwright[4], and Reuter.[5] Now we prove it under weaker conditions by a simple method.

**Theorem.** The given differential equation, where $p(t)$ is periodic of period $\omega$, and $\int_0^\omega p(t)dt=0$, posseses at least one periodic solution of period $\omega$, if the following condtions are fulfilled:

a) $A(x)=\int_0^x a(x)dx \to \pm\infty$, for $x\to\pm\infty$ resp.

b) $\operatorname{sgn} x\cdot\varphi(x)\geqq 0$, for $|x|>q$

where $a(x)$, $\varphi(x)$, $\varphi'(x)$, $p(t)$ are continuous functions and $q$ is a positive number.

Proof. Put

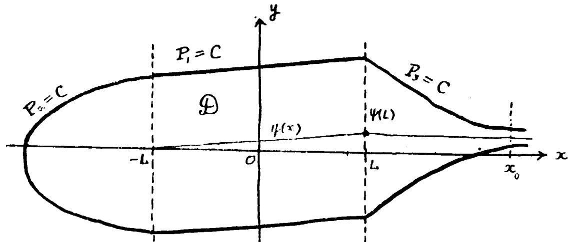


Fig. 1

$$\dot{x}=y-A(x)+\int_0^t p(t)dt=y-A(x)+P(t),\quad \dot{y}=-\varphi(x)$$

We consider, three functions

$P_1(x, y) = (y-\psi(x))^2/2$ for $|x| \leqq L$,

$P_2(x, y) = y^2/2 + \Phi(x) - \Phi(-L)$ for $x \leqq -L$, $\left(\Phi(x) = \int_0^x \varphi(x)dx\right)$,

$P_3(x, y) = (y-\psi(L))^2/2 + \Phi(x) - \Phi(L)$ for $x \geqq L$.

Here, $L$ and $\psi(x)$ are choosen as follows: Consider the derivative

$$\frac{dP_3(x(t), y(t))}{dt} = (y-\psi(L)) \cdot \{-\varphi(x)\} + \varphi(x) \cdot \{y - A(x) + P(t)\}$$
$$= -\varphi(x) \cdot \{A(x) - P(t) - \psi(L)\}$$

We choose $L(\geqq q)$ sufficiently large such that

$$A(x) > \operatorname*{Max.}_t P(t) \text{ for } x \geqq L,\ A(x) < \operatorname*{Min.}_t P(t) \text{ for } x \leqq -L,$$

and then, if we choose $\psi(L)$ $(> C)$ sufficiently small and define $\psi(x)$ as a linear function starting from $(-L, 0)$ and ending at $(L, \psi(L))$, we have (Fig 1)

$$\frac{dP_3(x, y)}{dt} \leqq 0 \quad \text{for } x \geqq L,$$
$$\frac{dP_2(x, y)}{dt} \leqq 0 \quad \text{for } x \leqq -L.$$

Now consider in $|x| \leqq L$, ($L$ is fixed as above),

$$\frac{dP_1(x, y)}{dt} = (y-\psi'x)) \cdot [-\varphi(x) - \psi'(x) \cdot \{y - A(x) + P(t)\}] - \psi'(x)$$
$$= -\psi'(x)(y-\psi(x))^2 - [\varphi(x) + \varphi'(x)\{\psi(x) - A(x) + P(t)\}](y-\psi(x)),$$

then if we choose $|y-\psi(x)|$ sufficiently large, we have also $\frac{dP_1(x, y)}{dt} < 0$, for $|x| \leqq L$, ($\psi' = \text{cnst.} > 0$).

Now consider $P_1(x, y) = (y-\psi(L))^2/2 = C$ for sufficiently large $C$, and then $P_2(x, y) = C$ for $x \leqq -L$, $P_3(x, y) = C$ for $x \geqq L$ ($C$ is the same!).

These curves enclose either

i) a bounded domain $\mathfrak{D}$ (it is the case $\Phi(x) \to \infty$ when $|x| \to \infty$) or ii) an unbounded domain $\mathfrak{D}$.

In case i), since $\frac{dP_i(x, y)}{dt} \leqq 0$ $(i = 1, 2, 3)$, the curve $(x(t), y(t))$

($t \geqq 0$) remains in $\mathfrak{D}$ if $(x(0), y(0)) \in \mathfrak{D}$ *).

In case ii), since $y$ is bounded for $(x, y) \in \mathfrak{D}$, $\dot{x}=y-A(x)+P(t)$ shows that, if we take $x_0$ sufficiently large, $\dot{x}<0$, for $x=x_0$, $x>0$ for $x=-x_0$ (Fig 1).

Then the same thing as i) is true for the domain $(x, y) \in \mathfrak{D}$, $|x| \leqq x_0$. q. e. d.

Remark. (1) This theorem is also stated as follows:
For each solution we can choose a numbe $B$, (generally depending on each solution) such that $|x(t)|<B$, $|\dot{x}(t)|<B$.

(2). If moreover the conditions hold:
$\Phi(x) \rightarrow \infty\ (|x| \rightarrow \infty)$, $\operatorname{sgn} x \cdot \varphi(x)>0$ for $|x|>L$,
then we can choose a number $B$ (not depending on each solution) such that $|x(t)|<B$, $|\dot{x}(t)|<B$, $t \geqq t_0$ ($t_0$ depending on each solution).

**Remark** **) Prof. G. E. H. Reuter has communicated us that he published in Proc. Cambridge Phil. Soc. 47 (1951) the result similar to ours by a rather complicated way, and that he has obtained the following result which will be published in Journal of the London Math. Soc. 27 (1952):

The differential equation $\ddot{x}+f(\dot{x})+\varphi(x)=p(t)$ posseses a periodic solution, if

i) $\operatorname{sgn} y \cdot f(y) \rightarrow \infty\ (y \rightarrow \pm \infty)$,
ii) $\varphi(x) \rightarrow \pm \infty\ (x \rightarrow \pm \infty)$,
iii) $p(t)$ is a continuous periodic function of period $\omega$,

where $f(y), \varphi(x)$, are continuous with their derivatives (or fulfil the Lipschitz condition).

As we cannot yet see his proofs and since our method is also applicable to this case, we shall show in the following our proof. Put

$$\dot{x}=y$$
$$\dot{y}=-f(y)-\varphi(x)+p(t),$$

and consider the quantity analogous to the energy $P(x, y)=y^2/2+\Phi(x)$, $\left(\Phi(x)=\int_0^x \varphi(x)\,dx\right)$, we see that along any trajectory this

*) The detailed proof would be nesessary; while since $\frac{dP_t}{dt} \leqq 0$, (equal sign take place), we have only to show that the curve $(x(t), y(t))$ never goes out of at the joint points of the curves $P_t=C$.
If $\varphi(x) \equiv 0$, in $L-\varepsilon \leqq x \leqq L+\varepsilon$, it can easily be proved. If $\varphi(x) \not\equiv 0$, it is enough to take $L$ so as $\varphi(L)>0$. For $\varphi(-L)$ we consider in the same way.

**) Added Feb 15, 1952,

quantity might increase for sometime, but in the sequel it decreases up to the value less than its initial value provided that initially it is sufficiently large.

In fact, $\dfrac{dP}{dt}=y\{-f(y)-\varphi(x)+p(t)\}-\varphi(x)\cdot y=y\{-f(y)+p(t)\}.$

By the assumption we can take $q_0$ such that sgn $y\cdot f(y)\geqq \text{Max.}\ |p(t)|+\varepsilon$, $(\varepsilon>0)$ for $|y|\geqq q_0$. Then we have two lines $y=q_0$, $y=-q_0$.

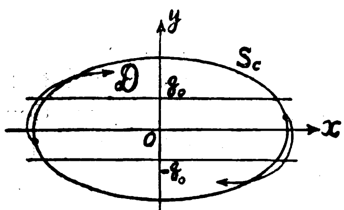


Fig. 2

Now consider the trajectory whose initial point is on $S_c$. ($S_c$ is a simple closed curve $P(x,y)=C$, and we denote the inner domain by $\mathfrak{D}$ (see Fig 2).

For $|y|\geqq q_0$, we have $\dfrac{dP}{dt}<0$. For $|y|<q_0$ $\dfrac{dP}{dt}$ may be positive; therefore it may be possible the variation $\delta P$ along the trajectory whose initial point is such that $|y|<q_0$ is positive. But when the trajectory passes through the narrow range $|y|\leqq q_0$,

$$\delta P=\int\frac{dP}{dt}dt=\int\frac{dP}{dt}/\dot{y}\;dy=\int\frac{y\{-f(y)+p(t)\}}{-f(y)-\varphi(x)-p(t)}dy,$$

is sufficiently small when $C$ is sufficiently large by the mercy of the boundedness of $y$, $f(y)$, $p(t)$.

When the point $(x(t),\ y(t))$ arrives at the line $|y|=q_0$, the point moves monotonously with respect to $x$ in the range $|y|>q_0$, and the variation is

$$\delta P=\int\frac{dP}{dt}/\dot{x}\,dx=\int\{-f(y)+P(t)\}dx\leqq-\varepsilon\,|\,\delta x\,|,$$

where $\delta x$ is the variation of the $x$-component of the trajectory. Therefore the curve $(x(t),y(t))$ must enter into $\mathfrak{D}$. (Fig 2). Therefore the trajectory $(x(t),y(t))$ which issues from $(x(0),\ y(0))\in\mathfrak{D}$ must remain bounded for $t\geqq 0$.

We see moreover that there exists a sufficiently large constant $C_0$ such that every solution $(x(t),\ y(t))$ ultimately enters into the inner domain of $S_{c_0}$.

By the above analysis we see easily that we can describe a simple closed curve $S$ such that if the initial point $(x(0), y(0))$ lies in it, the trajectory $(x(t), y(t))$ never goes out from the inner domain of $S$.

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