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A theorem for hypersurfaces of conformally flat space

By

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In the following we shall prove a theorem for especial hypersurfaces of conformally flat Riemannian space as follows.

Theorem. *If P and only $P(0<P\leqq n)$ of conformal principal radii of a hypersurface V^n of conformally flat space are equal identically in a neighborhood U of V^n, then U contains $\infty^{n-P}V^P$, such that V^P is umbilical in V^n and the conformal curvature tensor of V^P vanishes.*

Hence V^P is conformally flat if $P\geqq 4$. The conformal principal radii σ_a are defined in terms of principal radii ρ_a as follows.

$$\sigma_a=\rho_a-\frac{1}{n}\sum_b \rho_b.$$

It is clear that the theorem holds equally well, if conformal principal radii are replaced by principal radii.

We consider a variety V^n of coördinates x^i immersed in a Riemannian space V^m of coördinates y^α. Let $B^\alpha_P\,(P=n+1, \cdots, m)$ be mutually orthogonal unit vectors normal to V^n and $B^\alpha_i=\partial y^\alpha/\partial x^i$. Then there exist quantities H^P_{ij} and $H^P_{Qi}\,(P, Q=n+1, \cdots, m)$, such that

$$(1)\qquad \begin{aligned} B^\alpha_{i;j}&=\sum_P e_P H^P_{ij}B^\alpha_P, \\ B^\alpha_{P;j}&=-g^{hi}H^P_{hj}B^\alpha_i+\sum_Q e^Q H^Q_{Pj}B^\alpha_Q, \end{aligned}\qquad (e_P=\pm 1)$$

where g^{hi} are components of the fundamental tensor of V^n.

If V^m is conformal to a flat space and we put

$$(2)\qquad M^P_{ij}=H^P_{ij}-\frac{1}{n}g^{hk}H^P_{hk}g_{ij},$$

$M^{Pi}_{\ \ j}=g^{ik}M^P_{kj}$ and $N_{ij}=\sum_P e_P M^{Pk}_{\ \ i}M^P_{kj}$, then we have from the Gauss and Codazzi equations

$$(3)\quad C_{hijk}=\sum_P e_P(M^P_{hj}M^P_{ik}-M^P_{hk}M^P_{ij})+\frac{1}{n-2}(g_{hj}N_{ik}-g_{hk}N_{ij}$$
$$+g_{ik}N_{hj}-g_{ij}N_{hk})-\frac{g^{lm}N_{lm}}{(n-1)(n-2)}(g_{hj}g_{ik}-g_{hk}g_{ij}),$$

$$(4)\quad M^P_{ij;k}-M^P_{ik;j}+\sum_Q e_Q(M^Q_{ij}H^P_{Qk}-M^Q_{ik}H^P_{Qj})-\frac{1}{n-1}\{g_{ij}(M^{Pl}_{k;l}$$
$$+\sum_Q e_Q M^{Ql}_k H^P_{Ql})-g_{ik}(M^{Pl}_{j;l}+\sum_Q e_Q M^{Ql}_j H^P_{Ql})\}=0,$$

$$(5)\quad H^Q_{Pj;k}-H^Q_{Pk;j}-(M^{Pl}_jM^Q_{lk}-M^{Pl}_kM^Q_{lj})$$
$$+\sum_R e_R(H^R_{Pj}H^Q_{Rk}-H^R_{Pk}H^Q_{Rj})=0,$$

where C_{hijk} are components of the conformal curvature tensor of V^n.

K. Yano[2] shaw that the quantities $M^P_{ij}B^\alpha_P$ are invariant under a conformal transformation of V^m. Also K. Yano and Y. Muto[3] proved that a Riemannian space V^n is immersed in a conformally flat space, if and only if there exist M^P_{ij} and H^P_{Qi} satisfying the equations (3), (4) and (5). It should be remarked here that, though they gave further conditions for such a space, those conditions are obtained as consequences of (3), (4) and (5). If V^n is a hypersurface of $V^m(m=n+1)$, then (3) and (4) are respectively expressible in the following.

$$(6)\quad C_{hijk}=e(M_{hj}M_{ik}-M_{hk}M_{ij})+\frac{e}{n-2}(g_{hj}M^l_iM_{lk}-g_{hk}M^l_iM_{lj}$$
$$+g_{ik}M^l_hM_{lj}-g_{ij}M^l_hM_{lk})-\frac{eM^l_mM^m_l}{(n-1)(n-2)}(g_{hj}g_{ik}-g_{hk}g_{ij}),$$

$$(7)\quad M_{ij;k}-M_{ik;j}+\frac{1}{n-1}(g_{ij}M^l_{k;l}-g_{ik}M^l_{j;l})=0,$$

And (5) is satisfied identically.

Hereafter we assume that all of the principal radii ρ_a of V^n are real and none of the principal directions are null vectors. Such a hypersurface was called to be *proper* by A. Fialkow[1]. Then there exists an orthogonal ennuple in V^n, the unit vectors $\xi^i_{a)}$ of which are tangent to the lines of curvature, and the fundamental tensor g_{ij} and H_{ij} are expressible in terms of $\xi^i_{a)}$ and ρ_a as follows:

$$g_{ij}=\sum_a e_a\xi_{a)i}\xi_{a)j},\qquad H_{ij}=\sum_a e_a\rho_a\xi_{a)i}\xi_{a)j}\qquad (e=\pm 1).$$

Hence the tensor M_{ij} are written in the similar form:

$$M_{ij}=\sum_a e_a\sigma_a\xi_{a)i}\xi_{a)j}, \tag{8}$$

where σ_a are conformal principal radii of V^n. The coefficients γ_{abc} of rotation of the ennuple $\xi^i_{a)}$ are defined by

$$\gamma_{abc}=\xi_{a)i;j}\xi^i_{b)}\xi^j_{c)}.$$

Making use of them, the equation (7) are equivalent to the system of equations

$$(\sigma_a-\sigma_b)\gamma_{abc}=(\sigma_a-\sigma_c)\gamma_{acb}\qquad (a,b,c\neq), \tag{9}$$

$$\frac{\partial\sigma_a}{\partial s_b}+\frac{1}{n-1}\frac{\partial\sigma_b}{\partial s_b}+e_a(\sigma_a-\sigma_b)\gamma_{baa} \tag{10}$$
$$+\frac{1}{n-1}\sum_c e_c(\sigma_b-\sigma_c)\gamma_{bcc}=0\qquad (a\neq b).$$

Now we suppose first that all σ's are equal to σ in a neighborhood U of a point O. We have $\sigma=0$ by means of $\sum_a\sigma_a=0$, so that (8) gives $M_{ij}=0$ and hence $C_{hijk}=0$ from (6).

Next we consider the case where $\sigma_1=\cdots=\sigma_P=\sigma\neq\sigma_\lambda\,(0<P<n;\ \lambda=P+1,\cdots,n)$ in U. It follows from (9) that

$$\gamma_{p\lambda q}=0\qquad (p,q=1,\cdots,P;\,p\neq q;\,\lambda=P+1,\cdots,n). \tag{11}$$

Therefore $n-P$ vectors $\xi^i_{\lambda)}\,(\lambda=P+1,\cdots,n)$ are normal to a P-dimensional variety V^P, contained in U, and $\xi^i_{p)}\,(p=1,\cdots,P)$ constitute an orthogonal ennuple of V^P. Let u^p be coördinates of V^P and put $\partial x^i/\partial u^p=B^i_p$. The components $\eta^q_{p)}$ of vectors $\xi^i_{p)}$ in V^P are given by $\xi^i_{p)}=\eta^q_{p)}B^i_q$ and the second fundamental tensors H^λ_{pq} of V^P are defined by

$$B^i_{p;q}=\sum_\lambda e_\lambda H^\lambda_{pq}\xi^i_{\lambda)}. \tag{12}$$

Making use of the above equation we have

$$\gamma_{p\lambda q}=\xi^i_{p);j}\xi_{\lambda)i}\xi^j_{q)}=(\eta^s_{p);j}B^i_s+\eta^s_{p)}B^i_{s;j})\xi_{\lambda)i}\eta^r_{q)}B^j_r$$
$$=\eta^s_{p)}B^i_{s;r}\xi_{\lambda)i}\eta^r_{q)}=\sum_\mu e_\mu H^\mu_{sr}\xi^i_{\mu)}\xi_{\lambda)i}\eta^s_{p)}\eta^r_{q)}=H^\lambda_{rs}\eta^s_{p)}\eta^r_{q)}.$$

Hence, in virtue of (11), H^λ_{pq} are expressed in the form

(13) $$H^{\lambda}_{pq}=-\sum_{r}\gamma_{\lambda rr}\eta_{r)p}\eta_{r)q}.$$

On the other hand, if we take $a=r$ and $b=\lambda$ in (10), then

$$\frac{\partial\sigma}{\partial s_{\lambda}}+\frac{1}{n-1}\frac{\partial\sigma_{\lambda}}{\partial s_{\lambda}}+(\sigma-\sigma_{\lambda})e_{r}\gamma_{\lambda rr}$$

$$+\frac{1}{n-1}\sum_{c}e_{c}(\sigma_{\lambda}-\sigma_{c})\gamma_{\lambda cc}=0\,,$$

from which we find $e_{1}\gamma_{\lambda 11}=\cdots=e_{P}\gamma_{\lambda PP}$. If we denote by $-\gamma_{\lambda}/P$ these quantities $e_{P}\gamma_{\lambda pp}$, then (13) are written in the form

(14) $$H^{\lambda}_{pq}=\gamma_{\lambda}g_{pq},$$

where g_{pq} are components of the fundamental tensor of V^{P}. It follows that V^{P} is umbilical in V^{n}.

The space V^{P} may be looked upon as a subspace of the enveloping space V^{n+1}. The $n-P+1$ normals of V^{P} in V^{n+1} are B^{α} and $\zeta^{\alpha}_{\lambda)}=\xi^{i}_{\lambda)}B^{\alpha}_{i}$. The second fundamental tensors $\overline{H}_{pq}$ and $\overline{H}^{\lambda}_{pq}$ of V^{P} are given by the equations

$$B^{\alpha}_{p;q}=e\overline{H}_{pq}B^{\alpha}+\sum_{\lambda}e_{\lambda}\overline{H}^{\lambda}_{pq}\zeta^{\alpha}_{\lambda)}\quad\left(B^{\alpha}_{p}=\frac{\partial y^{\alpha}}{\partial u^{p}}\right).$$

Besides, we have from (1) and (12)

$$B^{\alpha}_{p;q}=(B^{\alpha}_{i}B^{i}_{p})_{;q}=eH_{ij}B^{i}_{p}B^{j}_{q}B^{\alpha}+\sum_{\lambda}e_{\lambda}H^{\lambda}_{pq}\zeta^{\alpha}_{\lambda)},$$

so that the following relations are obtained.

$$\overline{H}_{pq}=H_{ij}B^{i}_{p}B^{j}_{q},\qquad \overline{H}^{\lambda}_{pq}=H^{\lambda}_{pq},$$

from wich we have in consequences of (14) and the definition of ρ_{a}

$$\overline{H}_{pq}=\sum_{a}e_{a}\rho_{a}\xi_{a)i}\xi_{a)j}B^{i}_{p}B^{j}_{q}=\sum_{r=1}^{P}e_{r}\rho_{r}\xi_{r)i}\xi_{r)j}B^{i}_{p}B^{j}_{q}=\sum_{r}e_{r}\rho_{r}\eta_{r)p}\eta_{r)q},$$

$$\overline{H}^{\lambda}_{pq}=\gamma_{\lambda}g_{pq}.$$

Substituting from these expressions in

$$\overline{M}_{pq}=\overline{H}_{pq}-\frac{1}{P}g^{rs}\overline{H}_{rs}g_{pq},\qquad \overline{M}^{\lambda}_{pq}=\overline{H}^{\lambda}_{pq}-\frac{1}{P}g^{rs}\overline{H}^{\lambda}_{rs}g_{pq},$$

we have immediately $\overline{M}_{pq}=0$, $\overline{M}^{\lambda}_{pq}=0$, and hence the conformal curvature tensor C_{pqrs} of V^{P} vanishes by means of (3)

Finally the theorem has been established.

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