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On the imbedding problem of abstract varieties in projective varieties

By

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Since the notion of abstract varieties was introduced by Weil [3], it was asked whether every abstract variety can be imbedded (biregularly) in a projective variety or not. Though the writer hoped to solve the question affirmatively, he found unfortunately a counter example against the question. Indeed, there exists a non-singular abstract variety V (which is not complete) which has two different points P and P' such that if a function f on V is well defined at both P and P', then $f(P)=f(P')$.

After some preliminaries in § 1, we shall show the example in § 2. In § 3, we shall give a condition for an abstract variety to be imbedded in a projective variety and then we shall show in § 4 that even when a monoidal transform of a non-singular abstract variety V can be imbedded in a projective variety, V may not be imbedded in any projective variety. By the way, we shall give some remarks in § 5.

Terminology. Since the notion of abstract varieties corresponds to the notion of models in the sense of Nagata [1], we shall explain in terminology on models as in Nagata [1].

Results assumed to be known: Besides some basic results on rings and models, we shall make use of the criterion of simplicity by Jacobian matrix. Further, in §§ 4–5, we shall make use of some basic results on monoidal transformations and quadratic transformations (see [1, IV]).

§ 1. Some preliminary lemmas

LEMMA 1. *Let M and M' be models of the same function field. Then $M \cup M'$ is again a model if and only if the join $J(M, M')$ of M and M' is contained in $M \cap M'$.*

Proof. If $M \cup M'$ is a model, then $J(M, M') = M \cap M'$. Conversely, if $J(M, M')$ is contained in $M \cap M'$, then any spot P in $M \cup M'$ does not correspond to any other spot in $M \cup M'$ and we see that $M \cup M'$ is a model.

COROLLARY. *Let $\mathfrak{o}$ and $\mathfrak{o}'$ be affine rings over the same ground ring I and let A and A' be affine models defined by $\mathfrak{o}$ and $\mathfrak{o}'$ respectively. If there are elements $x \in \mathfrak{o}$ and $y \in \mathfrak{o}'$ such that $\mathfrak{o}[\mathfrak{o}'] = \mathfrak{o}[1/x] = \mathfrak{o}'[1/y]$, then $A \cup A'$ is a model.*[1)]

LEMMA 2. *Let $\mathfrak{o}$ be a normal ring and let x be an element of the field of quotients of $\mathfrak{o}$. Then all the relations of x over $\mathfrak{o}$ are generated by those of degree 1, that is, let $\{a_\sigma\}$ be a set of generators of $\{a;\ a \in \mathfrak{o},\ ax \in \mathfrak{o}\}$ and set $b_\sigma = a_\sigma x$, then $\mathfrak{o}[x] = \mathfrak{o}[X]/\mathfrak{a}$ with the ideal generated all $a_\sigma X - b_\sigma$.*

Proof. Assume that $c_0 x^n + c_1 x^{n-1} + \cdots + c_n = 0$ $(c_i \in \mathfrak{o})$. Let v be an arbitrary valuation whose valuation ring contains $\mathfrak{o}$. Then $v(c_0 x) \geq 0$. For, if $v(x) \geq 0$, then there is nothing to prove. Assume that $v(x) < 0$. Then $v(c_0 x) = v(c_1 + c_2 x^{-1} + \cdots + c_n x^{-n+1}) \geq 0$. Since $\mathfrak{o}$ is a normal ring, $\mathfrak{o}$ is the intersection of valuation rings and we see that $c_0 x \in \mathfrak{o}$. Now the assertion follows easily.

REMARK. *If $\mathfrak{o}$ is a Noetherian normal ring and if x is not in $\mathfrak{o}$, then $\{a;\ a \in \mathfrak{o},\ ax \in \mathfrak{o}\}$ is an ideal of rank 1,* because $\mathfrak{o}$ is the intersection of discrete valuation rings $\mathfrak{o}_{\mathfrak{p}}$ with prime ideals $\mathfrak{p}$ of rank 1.

§ 2. The construction of an example

Let K be a field and let x, y, z, w be algebraically independent elements over K. Set $u = xy + zw$, $x' = y/u^2$, $y' = u^2 x$, $z' = w/u^2$, $w' = u^2 z$. Then

LEMMA A. The operation $'$ defines an involution of the field $K(x, y, z, w)$ and $u' = u$.

Proof. $u = x'y' + z'w'$ and $x = y'/u^2$, $y = u^2 x'$, $z = w'/u^2$, $w = u^2 z'$ and we see the assertion.

From now on, we shall use $'$ in the sense of this involution.

Set $\mathfrak{o} = K[x, y, z, w]$ $(\mathfrak{o}' = K[x', y', z', w'])$ and let A and A' be the affine models defined by $\mathfrak{o}$ and $\mathfrak{o}'$ respectively.

LEMMA B. $M = A \cup A'$ is a non-singular model.

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1) The converse of this corollary is not true. $A \cup A'$ is a model if and only if there are ideals $\mathfrak{a}$ and $\mathfrak{a}'$ in $\mathfrak{o}$ and $\mathfrak{o}'$ respectively and a natural number n such that $\mathfrak{o}[\mathfrak{o}'] = \mathfrak{o}[\mathfrak{a}^{-n}] = \mathfrak{o}'[\mathfrak{a}'^{-n}] = \mathfrak{a}\mathfrak{o}[\mathfrak{o}'] = \mathfrak{a}'\mathfrak{o}[\mathfrak{o}']$ by virtue of a result in [2].

Proof. Since $xx'+zz'=1/u$, $\mathfrak{o}[\mathfrak{o}']$ contains $1/u$ and we see that $\mathfrak{o}[\mathfrak{o}']=\mathfrak{o}[1/u]=\mathfrak{o}'[1/u]$. Since u is in both $\mathfrak{o}$ and $\mathfrak{o}'$, M is a model by the corollary to Lemma 1. Since A and A' are models of the affine 4-spaces, they are non-singular and M is also non-singular.

This model M is the required example as will be proved.

Set $a=xy(=x'y')$ $(a'=a)$, $b=zw(=z'w')$ $(b'=b)$, $c=yz(=x'w')$ $(c'=y'z'=xw)$. Then $u=a+b$ and $y, y', w, w', a, b, c, c'$ are in $\mathfrak{o}\cap\mathfrak{o}'$. Set $\mathfrak{o}^*=K[y, y', w, w', a, b, c, c']$. Then

LEMMA C. $\mathfrak{o}^*$ is contained in $\mathfrak{o}\cap\mathfrak{o}'$ and is closed under $'$.

Among y, y', w, w', a, b, c and c', there are relations as follows:

$$(a+b)^2a=yy',\ (a+b)^2b=ww',\ (a+b)^2c=yw',$$
$$(a+b)^2c'=y'w,\ aw=c'y,\ aw'=cy',\ cw=by,$$
$$c'w'=by',\ cc'=ab.$$

Therefore

$$x=a/y=c'/w=y'/(a+b)^2,\ z=c/y=b/w=w'/(a+b)^2,$$
$$x'=a/y'=c/w'=y/(a+b)^2,\ z'=c'/y'=b/w'=w/(a+b)^2.$$

LEMMA D. Set $\mathfrak{p}=y\mathfrak{o}^*+w\mathfrak{o}^*+(a+b)\mathfrak{o}^*$. Then $\mathfrak{p}$ is a prime ideal of rank 1 and $\mathfrak{o}^*_{\mathfrak{p}}$ is a discrete valuation ring.

Proof. Let $\mathfrak{p}^*$ be a minimal prime divisor of $(a+b)\mathfrak{o}^*$. Then $\mathfrak{p}^*$ contains yy', ww', yw', $y'w$. Therefore $\mathfrak{p}^*$ contains either y, w or y', w', which proves that $\mathfrak{p}$ or $\mathfrak{p}'$ is of rank 1, and we see that both $\mathfrak{p}$ and $\mathfrak{p}'$ are of rank 1. Since $\mathfrak{p}$ contains $a+b$, y, w, $\mathfrak{o}^*/\mathfrak{p}$ is a homomorphic image of the ring $K[Y, W, A, C, C']/(AW-CY, C'W+AY, CC'+A^2)$ (Y, W, A, C, C' are indeterminates which correspond to y', w', a (or $-b$), c, c' respectively. Obviously $K[Y, A, C, C']/(CC'+A^2)$ is a normal ring and the element CY/A is not integral over this residue class ring. Since $CY/A=-AY/C'$ and since A, C' generate a prime ideal, $AW-CY$ and $C'W+AY$ generate all the relations of the element $W=CY/A$ over $K[Y, A, C, C']/(CC'+A^2)$ and the ideal $(AW-CY, C'W+AY, CC'+A^2)$ is a prime ideal. Since $\mathfrak{p}$ is of rank 1, we see that $\mathfrak{o}^*/\mathfrak{p}$ is isomorphic to $K[Y, W, A, C, C']/(AW-CY, C'W+AY, CC'+A^2)$ and $\mathfrak{p}$ is a prime ideal. In order to prove that $\mathfrak{o}^*_{\mathfrak{p}}$ is a discrete valuation ring, we consider the Jacobian matrix derived from the relations among $y, y', w, w', a, b, c, c'$: The matrix modulo $\mathfrak{p}$ is expressed as follows:

$$\begin{pmatrix} y' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w' & 0 & 0 & 0 & 0 & 0 \\ w' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y' & 0 & 0 & 0 & 0 & 0 \\ c' & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & -a & -w' & 0 & y' & 0 \\ b & 0 & -c & 0 & 0 & 0 & 0 & 0 \\ 0 & b' & 0 & -c' & 0 & y' & 0 & -w' \\ 0 & 0 & 0 & 0 & b & a & -c' & -c \end{pmatrix}$$

By this Jacobian matrix, we see that the spot $\mathfrak{o}^*_{\mathfrak{p}}$ is simple, i.e., $\mathfrak{o}^*_{\mathfrak{p}}$ is a discrete valuation ring (because $\mathfrak{p}$ is of rank 1). Thus Lemma D is proved completely.

REMARK. We have proved also that $\mathfrak{o}^*/\mathfrak{p}$ is represented by $K[y', w', a, c, c']$ and the relations modulo $\mathfrak{p}$ are generated by $aw'-cy'$, $c'w'+ay'$, $cc'+a^2$.

Let v be the normalized valuation whose valuation ring is $\mathfrak{o}^*_{\mathfrak{p}}$. Then

$$v(y')=v(w')=v(a)=v(b)=v(c)=v(c')=0\,.$$

Since $(a+b)^2a=yy'$, we have $v(y)=2\cdot v(a+b)$. Similarly $v(w)=2\cdot v(a+b)$ and

$$v(a+b)=1,\ v(y)=v(w)=2\,.$$

Therefore we have

$$v(x)=v(z)=-2\,,\quad v(x')=v(z')=0\,.$$

LEMMA E. If $f\in\mathfrak{o}'$, then $v(f)\geq 0$.

Proof. $\mathfrak{o}'=K[x', y, z', w']$ and $v(x')$, $v(y')$, $v(z')$, $v(w')$ are non-negative, and we see the assertion.

Next we consider elements of $\mathfrak{o}$. Since $\mathfrak{o}=\mathfrak{o}^*[x, z]$, every element of $\mathfrak{o}$ is expressed in the form:

$$\sum f_{ij}x^iz^j\ \ (f_{ij}\in\mathfrak{o}^*).$$

We shall define a " reduced expression " so that the value by v can be obtained by the formal caluculation:

First we observe the relations as follows:

yz, wz, $(a+b)^2z$, yx, wx, $(a+b)^2x$ are in $\mathfrak{o}^*$,

$az=cx$, $c'z=bx$, $y'z=w'x$.

Now, let d be the degree (with respect to x and z) of an expression $\sum f_{ij}x^iz^j$ of an element f of $\mathfrak{o}$. The following procedure is applied first to f_{0d}, then to $f_{1,d-1}$, $f_{2,d-2}$ and so on:

f_{ij} is first expressed as a polynomial in y, y', w, w', $a+b$, a, c, c'. If there are terms which are divisible by y or w or $(a+b)^2$ and if $i+j>0$, then using the fact that yx, wx, $(a+b)^2x$, yz, wx, $(a+b)^2z$ are in $\mathfrak{o}^*$, the related terms (multiplied by x^iz^j) are changed to terms of lower degrees. Furthermore, for $j>0$, if there are terms which are divisible by a or c' or y', then the related terms are changed to terms of lower degree with respect to z (the total degree is not changed) making use of the relation $az=cx$, $c'z=bx$, $y'z=w'x$. Thus we may assume that

i) If $j>0$, then $f_{ij}=f_{ij0}+f_{ij1}(a+b)$ with $f_{ijk}\in K[w', c]$,

ii) If $i>0$, then $f_{i0}=f_{i00}+f_{i01}(a+b)$ with $f_{i0k}\in K[y', w', a, c, c']$; here if f_{i0k} is in $\mathfrak{p}$, then $f_{i0k}=0$. (For, if $f_{i0k}\in\mathfrak{p}$, then f_{i0k} can be expressed as $(aw'-cy')g_0+(c'w'+ay')g_1+(cc'+a^2)g_2$ by the remark after Lemma D. But $aw'=cy'$, $c'w'=by'$ and $cc'=ab$. Therefore $f_{i0k}=(a+b)y'g_1+(a+b)ag_2$. Thus f_{i0k} is divisible by $a+b$. Therefore if $k=0$, then f_{i00} can be writen in the form $f_{i01}(a+b)$; if $k=1$, then $f_{i01}(a+b)$ is divisible by $(a+b)^2$ and the term can be changed to a term of lower degree.)

An expression satisfying the above condition is called a *reduced expression* of the element f. Then

LEMMA F. If $f\in\mathfrak{o}$, then $v(f)$ is obtained by the formal calculation from a reduced expression of f. Namely, assume that $\sum f_{ij}x^iz^j$ is a reduced expression of f and let d be the degree of the expression with respect to x and z. If $d=0$, then obviously $v(f)=v(f_{00})$. If $d>0$, then $v(f)=-2d+\sigma$ with $\sigma=0$ or 1 according to i) there exists at least one $f_{d-j,j}$ which is not divisible by $(a+b)$ or ii) all the $f_{d-j,j}$ are divisible by $(a+b)$.

Proof. Assume that $d>0$. Use the notation f_{ijk} as before and set $g_1=\sum_{i+j<d}f_{ij}x^iz^j$, $g_2=\sum_j f_{d-j,j,0}x^{d-j}z^j$, $g_3=\sum_j f_{d-j,j,1}x^{d-j}z^j$. If $g_2\neq 0$, then since $v(g_1+(a+b)g_3)>-2d$, we may assume that $f=g_2$; if $g_2=0$, then since $v(g_1)>-2d+1$ and since $f=g_1+(a+b)g_3$ in this case, we may assume that $f=(a+b)g_3$. Then, in the last case we have only to prove that $v(g_3)=-2d$. Thus we have only to prove the case where $f=g_2$; it is sufficient to show that $v(f/x^d)=0$. It is obvious that $v(f)\geq -2d$ and $v(f/x^d)\geq 0$. Assume that $v(f/x^d)>0$. For a moment, we shall denote by $\bar{\ }$ the residue class in the field $\mathfrak{o}^*_{\mathfrak{p}}/\mathfrak{p}\mathfrak{o}^*_{\mathfrak{p}}$. Then that $v(f/x^d)>0$ means that $\sum_j \bar{f}_{d-j,j,0}\overline{(z/x)}^j=0$. Observe first that $\bar{a}$, $\bar{c}'$ and $\bar{y}'$ generate a prime ideal of rank 1 in $\mathfrak{o}^*/\mathfrak{p}$ (by the remark after Lemma D). Let v^* be a valuation whose valuation ring dominates $\mathfrak{o}^*_{(\mathfrak{p},a,c',y')}/(\mathfrak{p})$. Since $z/x=c/a=b/c'=w'/y'$, $v^*\overline{(z/x)}$ is negative. Now let e be the largest number such that $\bar{f}_{d-e,e,0}\neq 0$. If $e=0$, then it is a contradiction. Assume that $e>0$. Then $v^*(\bar{f}_{d-e,e})$ must be positive (see Lemma 3). But by the property of reduced expressions, $f_{d-e,e,0}\varepsilon K[w', c]$ and w' and c remains algebraically independent modulo $(\mathfrak{p}, a, c', y')\mathfrak{o}^*$, hence $v^*(\bar{f}_{d-e,e,0})$ cannot be positive. Thus we have a contradiction also in this case. Therefore $v(f)=-2d+\sigma$.

LEMMA G. An element f of $\mathfrak{o}$ is in $\mathfrak{o}^*$ if and only if $v(f)\geq 0$.

Proof. The only if part is obvious. If $v(f)\geq 0$, then the degree of a reduced expression of f must be zero and $f\varepsilon\mathfrak{o}^*$.

COROLLARY. $\mathfrak{o}^*=\mathfrak{o}\cap\mathfrak{o}'$.

Proof. This follows from Lemmas *E* and *G*.

LEMMA H. If an element f of $\mathfrak{o}$ is not in $\mathfrak{o}'$, then for every integer r such that $u^r f\varepsilon\mathfrak{o}'$, $u^r f$ is in the ideal $y'\mathfrak{o}'+w'\mathfrak{o}'$.

Proof. By Lemma E, $v(u^r f)\geq 0$ and $r\geq -v(f)$ and we have only to prove the case where $r=-v(f)$. Let $\sum f_{ij}x^i z^j$ be a reduced expression of f and let d be the degree of the expression. Since $f\notin\mathfrak{o}'$, d is positive and $-v(f)=2d-\sigma$ with the same σ as in Lemma F. Then obviously $u^r(\sum_{i+j<d} f_{ij}x^i z^j)\varepsilon u\mathfrak{o}'=(a+b)\mathfrak{o}'=(x'y'+z'w')\mathfrak{o}'\subseteq y'\mathfrak{o}'+w'\mathfrak{o}'$. If $\sigma=1$, then every $f_{d-j,j}$ is divisible by $(a+b)=u$. Therefore $u^r(\sum_{i+j=d} f_{ij}x^i z^j)=u^{2d}(\sum f_j^* x^{d-j}z^j)$ with $f^*\varepsilon\mathfrak{o}^*$; this last holds also in the case where $\sigma=0$. $u^{2d}(\sum f_j^* x^{d-j}z^j)=\sum f_j^*(u^2x)^{d-j}(u^2z)^j=\sum f_j^* y'^{d-j}w'^j\varepsilon y'\mathfrak{o}'+w'\mathfrak{o}'$. Thus Lemma H is proved.

Set $P=\mathfrak{o}_{(y,w)}$, $P'=\mathfrak{o}'_{(y',w')}$ and $Q=\mathfrak{o}^*_{(y,y',w,w',a,b,c,c')}$. Then

LEMMA I. P and P' dominate Q.

Proof. $y, y', w, w', a, b, c, c'$ are in $y\mathfrak{o}+w\mathfrak{o}$. Since $(y, y', w, w', a, b, c, c')\mathfrak{o}^*$ is a maximal ideal, we see that $(y\mathfrak{o}+w\mathfrak{o})\cap\mathfrak{o}^*=(y, y', w, w', a, b, c, c')\mathfrak{o}^*$ and Q is dominated by P and therefore also by P' because $Q'=Q$.

LEMMA J. Let D be a divisorial closed set of the model M such that $D\cap A'$ has no component defined by $u\mathfrak{o}'$. Let f be an element of $\mathfrak{o}$ such that $f\mathfrak{o}$ defines $D\cap A$ and let r be the smallest integer such that $u^r f\varepsilon\mathfrak{o}'$ (r may be negative). Then $u^r f\mathfrak{o}'$ defines $D\cap A'$.

Proof. Since $\mathfrak{o}[\mathfrak{o}']=\mathfrak{o}[1/u]=\mathfrak{o}'[1/u]$, it follows that the closed set D' of A' defined by $u^r f\mathfrak{o}'$ coincides with $D\cap A'$ up to component defined by $u\mathfrak{o}'$. Since $u^r f\notin u\mathfrak{o}'$ by our assumption on r, D' has no component defined by $u\mathfrak{o}'$ (observe that $u\mathfrak{o}'$ is a prime ideal) and $D'=D\cap A'$.

LEMMA K. If a divisorial closed set D of M does not contain any of the spots P and P', then every element $f\in\mathfrak{o}$ such that $f\mathfrak{o}$ defines $D\cap A$ is in $\mathfrak{o}^*$ and $D\cap A'$ is defines by $f\mathfrak{o}'$.

Proof. Assume first that $f\notin\mathfrak{o}^*$. Then $f\notin\mathfrak{o}'$ by the corollary to Lemma G. Let r be the smallest integer such that $u^r f\in\mathfrak{o}'$. Then $u^r f\mathfrak{o}'$ defines $D\cap A'$ by Lemma J. By Lemma H, $u^r f\in y\mathfrak{o}'+w\mathfrak{o}'$, which shows that $P'\in D\cap A'$ and this is a contradiction. Thus $f\in\mathfrak{o}^*$. Since $P\notin D$, $f(P)\neq 0$. Since P dominate Q and since $f\in\mathfrak{o}^*$, $f(Q)=f(P)$. Similarly, $f(Q)=f(P')$. Thus $f(P')\neq 0$. Therefore $f\notin u\mathfrak{o}'$ and $f\mathfrak{o}'$ defines $D\cap A'$ by Lemma J.

Now we prove.

PROPOSITION 1. $Q=P\cap P'$.

Proof. Since Q is dominated by P and P' by Lemma I, Q is contained in $P\cap P'$. We prove the converse inclusion. Let f be an element of $P\cap P'$. Let D be the divisorial closed set of M which is defined as the pole of f. Then P and P' are not in D. Therefore there exists an element $g\in\mathfrak{o}^*$ which defines D by Lemma K. Then $g(P)=g(Q)=g(P')\neq 0$ and there exists a natural number r such that $g^r f$ has no pole on M. Then $g^r f\in\mathfrak{o}\cap\mathfrak{o}'=\mathfrak{o}^*$ and therefore $f=(g^r f)/g^r$ is in Q. Thus $Q=P\cap P'$.

COROLLARY. *Set $F=M(P)$ and $F'=M(P')$ (that is, F and F' are loci of the spots P and P' in M). If $P^*\in F$ and $P^{**}\in F'$, then $P^*\cap P^{**}=Q$. In particular, if a function f on M is well defined at P^* and P^{**}, then $f(P^*)=f(P^{**})$ and therefore M is not a subset of any projective model.*

Proof. P^* and P^{**} contains Q. On the other hand, $P^*\subseteq P$, $P^{**}\subseteq P'$ and therefore $Q=P^*\cap P^{**}$.

§ 3. A condition for a model to be a subset of projective model

THEOREM 1.[2)] *Let M be a model over a ground ring I. Then M is a subset of a projective model if and only if there exist affine models $A_1, \cdots, A_n$ which are defined by affine rings $\mathfrak{o}_1, \cdots, \mathfrak{o}_n$ re-*

2) The present theorem was established under co-operation with Mr. Y. Nakai.

spectively and a system of elements $a_{ij}(i, j=1, \cdots, n)$ *such that* 1) M *is the union of the* A_i*'s*, 2) $a_{ij} \varepsilon \mathfrak{o}_j$, 3) $\mathfrak{o}_i[\mathfrak{o}_j]=\mathfrak{o}[a_{ij}]$, 4) $a_{ii}=1$ *and* 5) $a_{ij}a_{jk}=a_{ik}$.

Proof. Assume first there are affine models and elements as above. Let x_{ij}'s be elements of $\mathfrak{o}_j$ such that $\mathfrak{o}_j=I[x_{1j}, \cdots, x_{mj}]$. Since $a_{ij} \varepsilon \mathfrak{o}_j$ and $\mathfrak{o}_j[\mathfrak{o}_i]=\mathfrak{o}_j[a_{ji}]=\mathfrak{o}_j[1/a_{ij}]$, $x_{ki}a_{ij}^r \varepsilon \mathfrak{o}_j$ for sufficiently large r's (for every k) and there exists such an r independently on i and j. Then considering a_{ij}^r instead of a_{ij}, we may asuume that $x_{ki}a_{ij} \varepsilon \mathfrak{o}_j$ for every (i, j, k). Let V be the projective model defined by the homogeneous coordinates $(a_{11}, a_{21}, \cdots, a_{n1}, x_{11}a_{11}, \cdots, x_{m1}a_{11}, x_{12}a_{21}, \cdots, x_{m2}a_{21}, \cdots, x_{1n}a_{n1}, \cdots, x_{mn}a_{n1})$. Then the affine representative of V defined by the i-th coordinate $\neq 0$ $(i=1, \cdots, n)$ coincides with A_i and M is a subset of V. Conversely, assume that M is a subset of a projective model V. Since M is a model, M is an open set of V and $F=V-M$ is a closed set of V. Therefore there exists a homogeneous coordinates $(t_0, \cdots, t_n)$ which defines V such that F is defined by $t_0=t_1=\cdots=t_r=0$. Then M is the union of affine models $A_0, \cdots, A_r$ defined by the affine rings $\mathfrak{o}_0=I[t_0/t_0, t_1/t_0, \cdots, t_n/t_0], \cdots, \mathfrak{o}_r=I[t_0/t_r, t_1/t_r, \cdots, t_n/t_r]$ respectively. With these affine models and elements $a_{ij}=t_i/t_j$, the condition in Theorem 2 is satisfied.

§ 4. A monoidal transform of the example

We recall the model M defined in § 3: M is the union of the following affine models A and A'.

A is defined by $\mathfrak{o}=K[x, y, z, w]$ and A' is defined by $\mathfrak{o}'=K[x', y', z', w']$, where x, y, z, w are algebraically independent elements and setting $u=xy+zw$, $x'=y/u^2$, $y'=u^2x$, $z'=w/u^2$, $w'=u^2z$.

We set again $P'=\mathfrak{o}'_{(y',w')}$. Then

PROPOSITION 2. *The monoidal transform* M^* *of* M *with the center* P' *is a subset of a projective model.*

Proof. Since $M(P') \cap A$ is empty, M^* is the union of the affine model A and the following affine models A_1 and A_2:

A_1 is defined by the affine ring $\mathfrak{o}_1=K[x', y', z', w'/y']$,

A_2 is defined by the affine ring $\mathfrak{o}_2=K[x', z', w', y'/w']$.

Obviously

$$u^2x^3=y'^3/u^4=y'^3/(x'y'+z'w')^4=1/(x'+z'(w'/y'))^4y',$$
$$u^2z^3=w'^3/u^4=1/(x'(y'/w')+z')^4w'.$$

Set $a_{01}=(x'+(z'w'/y'))^4y'$, $a_{02}=((x'y'/w')+z')^4w'$, $a_{00}=a_{11}=a_{22}=1$, $a_{10}=1/a_{01}$, $a_{20}=1/a_{02}$, $a_{12}=a_{10}a_{02}$, $a_{21}=1/a_{12}$. Then $a_{10}=u^2x^3$, $a_{20}=u^2z^3$, $a_{12}=u^2x^3/u^2z^3=y'^3/w'^3$, $a_{21}=w'^3/y'^3$. Denoting $\mathfrak{o}$ by $\mathfrak{o}_0$, we see that $a_{ij}\in\mathfrak{o}_j$ and $a_{ii}=1$, $a_{ij}a_{jk}=a_{ik}$ for every $i, j, k=0, 1, 2$. In order to prove that M^* is a subset of a projective model, it is sufficient to prove that $\mathfrak{o}_i[\mathfrak{o}_j]=\mathfrak{o}_i[a_{ij}]$ for every (i, j). For $i=j$, the assertion is obvious and we prove the case where $i\neq j$. Since $a_{ij}\in\mathfrak{o}_j$ we have only to prove that $\mathfrak{o}_i[\mathfrak{o}_j]\subseteq\mathfrak{o}_i[a_{ij}]$.

i) When $i=0$, $j=1$: Since $a_{01}=1/u^2x^3$, $\mathfrak{o}_0[a_{01}]=\mathfrak{o}_0[1/u, 1/x]$, which contains $\mathfrak{o}_1$ because $x'=y/u^2$, $y'=u^2x$, $z'=w/u^2$, $w'/y'=z/x$.

ii) The case where $i=0$, $j=2$ can be proved similarly.

iii) When $i=1$, $j=2$: Since $a_{12}=1/(w'/y')^3$, $\mathfrak{o}_1[a_{12}]=\mathfrak{o}_1[1/(w'/y')]=\mathfrak{o}_1[y'/w']$, which contains $\mathfrak{o}_2$ obviously.

iv) The case where $i=2$, $j=1$ can be proved similarly.

v) When $i=1$, $j=0$: Since $a_{10}=1/(x'+(z'w'/y'))^4y'$, $\mathfrak{o}_1[a_{10}]=\mathfrak{o}_1[1/y', 1/(x'+(z'w'/y'))]=\mathfrak{o}_1[1/y', 1/u]$. Since $\mathfrak{o}_1[1/u]$ contains $\mathfrak{o}_0$, $\mathfrak{o}_1[a_{10}]$ contains $\mathfrak{o}_0$.

vi) The case where $i=2$, $j=0$ can be proved similarly.

Thus we see that M^* is a subset of a projective model.

§ 5. Some remarks and a related question

I) *Affine models containing given spots.*

THEOREM 2. *Let $P_1, \cdots, P_n$ be spots of a function field. Then there exists an affine model which contains P_i's if and only if every P_i is a ring of quotients of the intersection $\mathfrak{d}$ of the spots P_i's.*

Proof. The only if part is obvious. Assume that every P_i is a ring of quotients of $\mathfrak{d}$. Let $\mathfrak{o}_i$ be an affine ring which has a prime ideal $\mathfrak{p}_i$ such that $P_i=(\mathfrak{o}_i)_{\mathfrak{p}_i}$ ane let $x_{1i}, \cdots, x_{mi}$ be elements of $\mathfrak{o}_i$ which generate $\mathfrak{o}_i$. Let S_i be the set of elements of $\mathfrak{d}$ which are units in P_i. Since $P_i=\mathfrak{d}_{S_i}$, $\mathfrak{o}_i\subseteq\mathfrak{d}_{S_i}$ and there exists an element $s_i\in S_i$ such that $x_{ji}s_i\in\mathfrak{d}$. Let $\mathfrak{o}^*$ be the affine ring generated by all the s_i's and all the $x_{ji}s_i$'s. Then the affine model A^* defined by $\mathfrak{o}^*$ contains the spots P_i.

REMARK. *If spots $P_1, \cdots, P_n$ are in a model M and if there exists an affine model A which contains $P_1, \cdots, P_n$, then there exists an affine model A^* which contains the P_i's and is contained in M.*

Proof. $M\cap A$ is a model containing the spots P_i and $F=A-(M\cap A)$ is a closed set of the affine model A which does not contain any of the P_i's. Therefore there exists a hypersurface H

of A which contains F and which does not contain any of the P_i's. Then $A-H$ is an affine model contained in M and $A-H$ contains all the P_i's.

II) *Two existence theorems of models.*

THEOREM 3. *Let M be a model and let P be a spot. Then there exists a model M' which contains M and P if and only if the following condition is satisfied: If P is a specialization of a spot Q and if Q corresponds to a spot Q' in M, then $Q=Q'$.*

Proof. The only if part is obvious because if P is a specialization of a spot Q, then Q is in every model which contains P. Assume that the condition is satisfied. Let A be an affine model which contains P. Let C be the set of spots $P' \varepsilon A$ such that P' correspond to some spots in M which is different from P'. Then C is a constructive set. By our assumption, the closure $\overline{C}$ of C does not contain P. Then $A-\overline{C}$ is a model and $M \cup (A-\overline{C})$ is a model which contains M and P.

THEOREM 4. *Let P and Q be spots. Then the following threee conditions are equivalent to each other:*

1) *There exists a model which contains P and Q.*

2) *If P and Q are specializations of spots P' and Q' respectively and if P' corresponds to Q', then $P'=Q'$.*

3) *$P[Q]$ is a ring of quotients of both P and Q.*

Proof. It is obvious that 2) follows from 1). Assume that 2) holds. Let A and A' be affine models which contain P and Q respectively. Since the function fields of P and Q correspond to each other, P and Q are spots of the same function field. Therefore $A \cap A'$ is a model. Let D be the union of irreducible components of $A-(A \cap A')$ which do not contain P and set $M' = A-D$. Then M' is a model which contains P. Further, if P' is a specialization of a spot Q' and if Q' corresponds to a spot $Q \varepsilon M'$, then $Q=Q'$ and therefore there exists a model M which contains P and Q by theorem 3. Thus 1) and 2) are equivalent to each other. Next, we assume again 2) holds. Let $\mathfrak{P}$ be the set of prime ideals $\mathfrak{p}$ of P such that Q is a specialization of $P_{\mathfrak{p}}$ and let S be the intersection of the complements of $\mathfrak{p} \varepsilon \mathfrak{P}$ in P. If an element $s \varepsilon S$ is a non-unit in $P[Q]$, then there exists a place $\mathfrak{v}$ which dominates $P[Q]$ such that s is a non-unit in $\mathfrak{v}$. Let $\mathfrak{n}$ be the maximal ideal of $\mathfrak{v}$ and set $P'=P_{(\mathfrak{n} \cap P)}$, $Q'=Q_{(\mathfrak{n} \cap P)}$. Then by our assumption $P'=Q'$, but $s \varepsilon \mathfrak{n} \cap P$, which is a contradiction.

Thus every element of S is a unit in $P[Q]$ and $P[Q]=P_S$. Similarly, $P[Q]$ is a ring of quotients of Q and 3) holds. Conversely, assume that 3) holds, then obviously 2) holds and therefore 2) and 3) are equivalent to each other. Thus 1), 2) and 3) are equivalent to each other.

REMARK. *There exists a model which contains given spots $P_1, \cdots, P_n$ if (and only if) there exists a model M_{ij} which contains P_i and P_j for every pair (i, j).*

Proof. We prove the assertion by induction on n. When $n=2$, the assertion is obvious. Assume that the assertion is true for $n-1$ spots. Then there exists a model M which contains $P_1, \cdots, P_{n-1}$. Let A be an affine model which contains P_n and set $D=M-(A \cap M)$. D is a closed set of M. Let D' be the union of irreducible components of D which do not contain any of $P_1, \cdots, P_{n-1}$. Then $M''=M-D'$ is a model which contains $P_1, \cdots, P_{n-1}$. If P_n is a specialization of a spot Q and if Q corresponds to a spot Q' in M'', then $Q=Q'$ and therefore there exists a model M^* which contains M'' and P_n by Theorem 3.

III) *A remark on imbedding in a complete model.*

THEOREM 5. *Let P be a spot in a model M. If the induced model $\phi_P(M)$ of M defined by the spot P is a complete model and if the monoidal transform M^* of M with the center P is a subset of a complete model, then M is a subset of a complete model.*

Proof. Let D^* be the set of spots in M^* which correspond to spots in the locus $M(P)$ of P in M. We shall prove that $(M^*-D^*) \cup M$ is a complete model. Since M^*-D^* is a model, $(M^*-D^*) \cup M$ is the union of a finite number of affine models. Let $\mathfrak{v}$ be a place of the function field of M. Since M^* is a complete model, $\mathfrak{v}$ has a unique center P^* in M^*. If $P^* \in D^*$, then the projection P' of P^* in M is the only one spot in $(M^*-D^*) \cup M$ which is dominated by $\mathfrak{v}$; if $P^* \notin D^*$, then P^* is the only one spot in $(M^*-D^*) \cup M$ which is dominated by $\mathfrak{v}$. By the uniqueness of spots dominated by places, we see first that $(M^*-D^*) \cup M$ is a model and by the existence of such spots, we see that $(M^*-D^*) \cup M$ is a complete model.

COROLLARY. *Let P be a spot of dimension zero in a model M. Let M^* be the quadratic transform of M with the center P. Then M is a subset of a complete model if and only if so is M^*. When M is a normal model, M is a subset of a complete model if and* *only if the derived normal model of M^* is a subset of a complete model.*

Proof. The first half is an immediate consequence of theorem 5. The last half can be proved by the same way as in the proof of theorem 5.

We want to ask the following question:

PROBLEM. *Is every normal complete model a projective model?*

If the answer of this question is affirmative, then 1) our example in § 2 is a model which is not a subset of any complete model and 2) it holds that if a quadratic transform of a normal model is a subset of a projective model, then so is the original model (by the corollary to Theoren 5).

Observe that "quadratic transform" in the corollary to Theorem 5 cannot be replaced by " monoidal transform " without any additional condition (by our example) and the same to 2) just above.

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